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NONDETERMINISTIC MECHANICS

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PREFACE

It has been recognized for several decades that uncertainty and absence of determinism play an important role in engineering sciences; to-date, three basic techniques have been developed to deal with various uncertainties, namely in variation and scatter in uncertain system parameters such as mechanical properties, geometric parameters, boundary conditions, in model uncertainties induced by modeling errors as well as actions such as the impact of earthquakes, wind loads, imperfect road profiles, or turbulence experienced by aircraft. These methodologies are (a) probabilistic or stochastic modeling; (b) fuzzy sets based analysis, and (c) anti-optimization of structures.

Spectacular advances have been recorded in stochastic mechanics based in the construction of stochastic models of uncertainties as soon as the probability theory can be used; likewise industrial applications have been developed worldwide in using fuzzy sets and logic for devising reliable machines and components, and a relatively recent new-old field has emerged, referred to as anti-optimization (also known as convex modeling of uncertainty, ellipsoidal modeling, guaranteed approach, maxmin, and worst case analysis)—that identifies uncertainty with boundedness. Interval analysis is the specific and simplest form of dealing with the best and worst scenarios under uncertainty. Imprecise probabilities combine both set and stochastic uncertainties, to obtain bounds of probabilities or expectations of variables.

However, the question which analysis is preferable for researchers and engineers is short of a consensus, as one can anticipate.

The aim of this book is to present the current state of the art of nondeterministic mechanics in its various forms. The topics range from stochastic problems to fuzzy sets; from linear to nonlinear problems; from specific methodologies to combinations of various techniques; from theoretical considerations to practical applications.

It is specially designed to illuminate the various aspects of above three techniques and deepen the discussion of their pros and cons.

The book is divided in three parts. Part 1 is devoted to stochastic analysis; it contains papers by Umberto Alibrandi and Giuseppe Ricciardi; Christian Soize; Isaac Elishakoff and Lova Andriamasy. Part 2 is devoted to nonstochastic analysis with papers by Thomas Haag and Michael Hanss; and by Alberto Bernardini and Fulvio Tonon.

Part 3 consists of one paper by Michael Oberguggenberger, dealing with combined methods.

If this volume stimulates further mutual and useful dialogue between proponents of differing methodologies, with demarcation of the area in which an approach ought to be preferred—our effort will be amply rewarded.

Isaac Elishakoff and Christian Soize, editors

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Part 1
Stochastic Models

Stochastic Methods in Nonlinear Structural Dynamics

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Abstract The uncertainties are inherent in any structural problem. Here attention is focused only on the uncertain nature of the dynamic actions and its consequences on the structural response. In the framework of stochastic dynamics, only three methods are the most used: the Moment Equation Method (MEM), the Stochastic Linearization (SL) and the Monte Carlo Simulation (MCS). The MEM in conjuction with a closure method (CM) allows to obtain the response statistical moments, but it increases in complexity as the problem dimension increases. The SL is easily applied to a large variety of engineering problems. Providing information limited to the first two moments of the system response, unfortunately it suffers of accuracy in the case of strongly nonlinear behavior. MCS is able to give additional information on the structural response, yielding estimates for the probability density function of the nonlinear response, but it is computationally expensive. In this paper some improvements of these three methods are presented, which allow to overcome the aforementioned drawbacks.

1 Introduction

In many cases of engineering interest, the structural systems are subject to dynamic excitations characterized by complex random fluctuations, that can not be adequately represented by deterministic models. Stochastic methods are able to represent adequately these type of excitations as stochastic processes, giving a characterization of the structural response in terms of statistical moments or in terms of the probability density function (pdf). This is a very complicated task to be accomplished, and in literature much research has been devoted to this topic in the last several decades, giving rise to methods based on: (i) the numerical solution of Fokker-Planck equation, such as path integration (Naess and Johnsen, 1993; Wehner and Wolfer, 1983), cell mapping (Hsu, 1987) and finite element method (Spencer and Bergman, 1993), (ii) the Moment Equation Method (MEM), (iii) the

Stochastic Linearization (SL), (iv) the Monte Carlo Simulation (MCS). The latter ones (MEM, SL and MCS) have gained wide popularity by their applicability to general nonlinear systems with many degrees of freedom (MDOF) and they will be described in detail in the following subsections.

1.1 Moment Equation Method

In the MEM approach the response statistical characterization is given by the response moments or by other quantities related to the former such as cumulants or quasi-moments (Stratonovich, 1967; Ibrahim, 1985). This method consists in writing differential equations for the response statistical moments of any order, taking advantage from the Fokker-Planck equation or from the Itï¿œ differential rule.

When dealing with nonlinear systems a serious problem arises in the MEM approach, which stems from the need of knowing the response pdf to close the hierarchy of the resulting equations. In order to overcome this difficulty, the so called closure methods (CM) are used. The key idea is to approximate the response pdf in a series form, truncating it at a certain term. The coefficients of the above mentioned series can be written as functions of the response moments or of the response cumulants or of the response quasi-moments. Thus, neglecting the terms beyond a given order is equivalent to make moments or cumulants or quasi-moments zero, which makes the moment equations solvable. In this way the moments of higher order are expressed in terms of moments of lower order by means of nonlinear relationships.

Cumulant neglect closure method has been independently developed by Wu and Lin (1984) and Ibrahim et al. (1985), which is certainly the most popular among the closure methods. Central moment and cumulant closure methods have been proposed by Bellman and Richardson (1968) and Sancho (1970a), (1970b) in a mathematical context. Bover (1978) and Sperling (1979) independently presented the quasi-moments closure method. Crandall (1980) presented the Hermite moment closure method. All these closure methods lead to a set of differential moment equations which are nonlinear. This nonlinearity is a major drawback in the case of systems with many degrees of freedom. This shortcoming has been overcome by Grigoriu (1999), which closed the moment equations taking advantage from MCS in order to estimate lower order moments, obtaining so a set of linear equations whose unknowns are the higher-order moments.

1.2 Stochastic Linearization

The SL method is the most versatile method for the analysis of general nonlinear structures under random excitations. In almost 60 years since its virtually simultaneous presentations by Booton (1953), Kazakov (1956) and Caughey (1959), SL has been widely applied in the study of various nonlinear structures. For example, the monographs by Roberts and Spanos (1991) and Socha (2008) and many review papers (Socha and Soong, 1991; Elishakoff, 2000; Falsone and Ricciardi, 2003) written in these last years confirm its success.

The basic idea of SL is to replace the original nonlinear system by an equivalent linear one, whose determination is performed by minimizing the difference between the two systems in some statistical sense. The SL exhibits different forms based on the adopted pdf for the evaluation of the coefficients that appear in the linearized system (Kozin, 1988).

The Gaussian Stochastic Linearization (GSL) is based on the hypothesis of Gaussianity of the response process and it allows approximating the second order moments of the response. In this way, the probabilistic characterization of the equivalent linear system response reduces to the evaluation of its Gaussian properties. Against relatively little numerical efforts, unfortunately the GSL method gives accurate results for weakly nonlinear systems only. This drawback is due to the inadequacy of the Gaussian assumption to represent the non-Gaussian characteristic of the response for systems that exhibit strong nonlinear behaviour.

Kozin (1988) introduced the concept of "true" linearization. He showed that, if the averages appearing in the expressions of the equivalent coefficients are evaluated by the exact probability density function of the response, the "true" stochastic linearization leads to the exact results in terms of response covariances. Starting from these observations, alternative approaches have been developed based on a more realistic representation of the non Gaussian probability density function of the response process. Beaman and Hedrick (1981) improved the accuracy of the GSL method by using the classical Gram-Charlier series expansion of the unknown probability density function of the response, which includes up to fourth order terms. The coefficients of the series expansion are approximately evaluated by solving the non-linear system of the moment equations up to fourth order. Pradlwarter (1991) proposed a numerical method based on non-Gaussian stochastic linearization and on the Fokker-Planck equation, requiring nonlinear transformations in order to consider the non-Gaussian properties of the stochastic response. In the non-Gaussian linearization method proposed by Chang (1992), the non-Gaussian density is built as the weighted sum of undetermined Gaussian densities. The undetermined Gaussian parameters

are then derived through solving a set of non-linear algebraic moment relations. Lee (1995) improved the stochastic linearization results for the Duffing oscillator by performing a non-Gaussian closure scheme, based on the abridged Edgeworth series expansion of the probability density of the response. Hurtado and Barbat (1996) proposed an improved non-Gaussian stochastic linearization for the Bouc-Wen-Baber hysteretic model by using mixed discrete-continuous Gaussian distributions. Recently, Crandall (2004) proposed the use of non-Gaussian distributions for the stochastic linearization of the power law and the double-well Duffing oscillators.

1.3 Monte Carlo Simulation

Whereas only biased estimates for the first two statistical moments of the response are obtainable by the SL method, MCS yields unbiased estimates for the probability density function of the nonlinear response. The first applications in structural dynamics are due to Shinozuka (1972) and Shinozuka and Wen (1972) and later extended by other authors (Grigoriu, 1993; Grigoriu, 1998; Spanos and Zeldin, 1998; Schueller and Spanos, 2000).

MCS is very robust and relatively easy to apply. A sample size of few hundred independent realizations is generally sufficient to obtain a suitable estimate for the first lower order moments and to provide information on the shape of the distribution. MCS becomes an increasingly attractive method as the problem dimension increases. Hence MCS is often the only feasible solution for real engineering problems with large dimensions. Besides being more efficient than analytical-based approaches, it has the advantages that the tools of deterministic analysis can be fully exploited. Against its efficiency, the computational costs required are often very high. Clearly, the computational efforts increase with the dimension and the complexity of the nonlinear structural models. Hence, efficient techniques have been proposed in order to reduce the computational burden, such as variance reduction procedures (Roberts, 1986), importance sampling (Tanaka, 1998; Au and Beck, 2001), controlled MCS (Harnpornchai et al., 1999).

1.4 Outline

The MEM is difficult to apply in the case of complex structures, increasing dramatically the number of nonlinear equations to be solved. The SL is less accurate in the case of structures with high nonlinearity. The MCS is in general time consuming, when higher order statistics or the probability densities are of interest. In this paper, the aforementioned methods are called in their classical form first. Then, some improved versions are proposed for overcoming these problems.

2 Stochastic Structural dynamics

2.1 Motion equations of the structural system

Linear structure. A general form of the equations of motion of an n d.o.f. linear system is as follows:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{C}\dot{\mathbf{u}}(t) + \mathbf{K}\mathbf{u}(t) = \mathbf{G}_{\mathrm{e}}\mathbf{F}_e(t) \tag{1}$$

where $\mathbf{u}(t)$ is the n_s-vector of generalized displacement, $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are the mass, damping and stiffness matrices, of order $n_s \times n_s$, $\mathbf{G}_{\mathrm{e}}$ is an influence matrix, of order $n_s \times n_e$, and $\mathbf{F}_{\mathrm{e}}(t)$ is the n_e-vector of external generalized forces.

Nonlinear structure. The response $\mathbf{u}(t)$ of a general dynamical system with nonlinear behavior subjected to an external force vector $\mathbf{F}_{\mathrm{e}}(t)$ satisfies the following nonlinear differential equations:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{h}[\mathbf{u}(t), \dot{\mathbf{u}}(t)] = \mathbf{G}_{\mathrm{e}}\mathbf{F}_e(t) \tag{2}$$

where $\mathbf{h}[\mathbf{u(t)}, \dot{\mathbf{u}}(t)]$ is the n_s-vector of non-linear restoring forces, which depends on the displacements and velocities.

External and parametric excitation. A more general form of a system of differential equations of motion can be given as:

$$\mathbf{M}\ddot{\mathbf{u}}(t) + \mathbf{h}[\mathbf{u}(t), \dot{\mathbf{u}}(t)] = \mathbf{G}_e\mathbf{F}_e(t) + \mathbf{G}_p[\mathbf{u}(t), \dot{\mathbf{u}}(t)]\mathbf{F}_p(t) \tag{3}$$

where $\mathbf{F}_{\mathrm{p}}(t)$ is the n_p-vector of parametric excitations, modulated by the matrix $\mathbf{G}_p[\mathbf{u(t)}, \dot{\mathbf{u}}(\mathbf{t})]$, of order $n_s \times n_p$, which depends on the system response.

The state vector approach. By introducing the vector of state variables $\mathbf{Z}(t) = [\ \mathbf{u}^T(t) \quad \dot{\mathbf{u}}^T(t)\ ]^T$, of order $2n_s$, eq.(3) can be written in the following first order form:

$$\dot{\mathbf{Z}}(t) = \mathbf{a}[\mathbf{Z}(t)] + \mathbf{V}[\mathbf{Z}(t)]\mathbf{F}(t) \tag{4}$$

where $\mathbf{F}(t) = [\ \mathbf{F}_e^T(t) \quad \mathbf{F}_p^T(t)\ ]^T$ is the vector of external and parametric excitations, of order $n_f = n_e + n_p$, while $\mathbf{a}[\mathbf{Z}(t)]$ and $\mathbf{V}[\mathbf{Z}(t)]$ are a $2n_s$-vector and a $2n_s \times n_f$ matrix, respectively, given as:

$$\mathbf{a}[\mathbf{Z}(t)] = \begin{bmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{h}[\mathbf{u}(t), \dot{\mathbf{u}}(t)] \end{bmatrix}, \tag{5}$$

$$\mathbf{V}[\mathbf{Z}(t)] = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{G}_e & \mathbf{M}^{-1}\mathbf{G}_p[\mathbf{u}(\mathbf{t}), \dot{\mathbf{u}}(\mathbf{t})] \end{bmatrix} \tag{6}$$

that are functions of the response vector of state variables $\mathbf{Z}(t)$.

2.2 Filtered excitations

The vector of random excitations is frequently assumed as a stationary Gaussian stochastic process, characterized by a Power Spectral Density (PSD) matrix $\mathbf{S_{FF}}(\omega)$ (or by its inverse Fourier transform, the correlation matrix $\mathbf{R_{FF}}(\tau)$). It can be considered as the response of a system of linear filter differential equations excited by m Gaussian white noises:

$$\dot{\mathbf{F}}(t) = \mathbf{A}_f\mathbf{F}(t) + \mathbf{V}_f\mathbf{W}(t) \tag{7}$$

The matrices $\mathbf{A}_f$ and $\mathbf{V}_f$ are constant. The random excitations $\mathbf{W}(t) = [\ W_1(t) \quad W_2(t) \quad \dots \quad W_m(t)\ ]^T$ are zero-mean Gaussian white noise processes, fully characterized by the second order cross-correlation functions $E[W_k(t_1)W_l(t_2)] = Q_{kl}\delta(t_1 - t_2)$, $E[\,\cdot\,]$being stochastic averaging, $\delta(\cdot)$ the Dirac's delta function, $Q_{kl} = 2\pi S_{kl}$ the constant strengths of the white noise processes, S_{kl} the cross-power spectral density of $W_k(t)$ and $W_l(t)$.

In general, the vector of random excitations can be considered as the response of a system of nonlinear filter differential equations excited by m white noises:

$$\dot{\mathbf{F}}(t) = \mathbf{a}_f[\mathbf{F}(t)] + \mathbf{V}_f[\mathbf{F}(t)]\mathbf{W}(t) \tag{8}$$

where the vector $\mathbf{a}_f[\mathbf{F}(t)]$ and the matrix $\mathbf{V}_f[\mathbf{F}(t)]$ depend on the excitation vector $\mathbf{F}(t)$.

An example: the earthquake excitation. As an example, the earthquake excitation on a structure is given as:

$$\mathbf{G}_e\mathbf{F}_e(t) = -\mathbf{M}\boldsymbol{\tau}\ddot{u}_g(t) \tag{9}$$

where $\ddot{u}_g(t)$ is the ground acceleration and τ is an influence vector.

In the model of the earthquake excitation given by Kanai-Tajimi (KT) (Kanai, 1957; Tajimi, 1960), the ground acceleration $\ddot{u}_g(t)$ is characterized by the following PSD:

$$\mathrm{S}_{KT}(\omega;\omega_g,\varsigma_g,S_w) = \frac{\mathrm{S}_w(\omega_g^4 + 4\varsigma_g^2\omega_g^2\omega^2)}{(\omega_g^2 - \omega^2)^2 + 4\varsigma_g^2\omega_g^2\omega^2} \tag{10}$$

where ω_g is the soil natural frequency, that determines the dominant range of input frequencies, while ς_g is the ground damping ratio, that indicates the sharpness of the power spectral density shape. The KT seismic model can be considered as a white noise excitation at bedrock level, with constant power spectrum S_w, filtered through the overlaying soil deposits. The seismic excitation $\ddot{u}_g(t)$ is a stationary stochastic filtered white noise process obtained as solution of the following set of linear filter equations:

$$\begin{cases} \ddot{u}_g(t) = \omega_g^2 q_g(t) + 2\varsigma_g \omega_g \dot{q}_g(t) \\ \ddot{q}_g(t) + 2\varsigma_g \omega_g \dot{q}_g(t) + \omega_g^2 q_g(t) = w(t) \end{cases} \tag{11}$$

The well known drawback of the KT model is that its PSD function approaches S_w when the frequency tends to zero, which is not consistent with the power spectrum of the real earthquake records.

The Clough-Penzien (CP) model overcomes this drawback (Clough and Penzien, 1975), by introducing a further filter with two additional parameters ω_p and ς_p ; the CP-PSD is given by:

$$\mathrm{S}_{CP}(\omega;\omega_p,\varsigma_p,\omega_g,\varsigma_g,\mathrm{S}_w) = \tilde{\mathrm{S}}_{CP}(\omega;\omega_p,\varsigma_p)\cdot \mathrm{S}_{KT}(\omega;\omega_g,\varsigma_g,\mathrm{S}_w) \tag{12}$$

$$\tilde{\mathrm{S}}_{CP}(\omega;\omega_p,\varsigma_p) = \frac{\omega^4}{(\omega_p^2-\omega^2)^2 + 4\varsigma_p^2\omega_p^2\omega^2} \tag{13}$$

The linear filter equations of the CP model are:

$$\begin{cases} \ddot{u}_g(t) = \omega_g^2 q_g(t) + 2\varsigma_g\omega_g\dot{q}_g(t) - \omega_p^2 q_p(t) - 2\varsigma_p\omega_p\dot{q}_p(t) \\ \ddot{q}_p(t) + 2\varsigma_p\omega_p\dot{q}_p(t) + \omega_p^2 q_p(t) = \omega_g^2 q_g(t) + 2\varsigma_g\omega_g\dot{q}_g(t) \\ \ddot{q}_g(t) + 2\varsigma_g\omega_g\dot{q}_g(t) + \omega_g^2 q_g(t) = w(t) \end{cases} \tag{14}$$

2.3 Dynamic equation of the whole system

The differential equations of the structure and of the filter, eqs.(4) and (8), can be cast in the following general form:

$$\dot{\mathbf{X}}(t) = \mathbf{f}[\mathbf{X}(t)] + \mathbf{G}[\mathbf{X}(t)]\mathbf{W}(t) \tag{15}$$

where $\mathbf{X}(t) = [\ \mathbf{Z}^T(t) \quad \mathbf{F}^T(t)\ ]^T$ is the vector of order $n = 2n_s + n_f$ collecting the displacements and velocities of the structure and the components of the filtered external and parametric excitations, while $\mathbf{f}[\mathbf{X}(t)]$ and $\mathbf{G}[\mathbf{X}(t)]$ are an n-vector and an $n \times m$ matrix, respectively, given as:

$$\mathbf{f}[\mathbf{X}(t)] = \begin{bmatrix} \mathbf{a}[\mathbf{Z}(t)] + \mathbf{V}[\mathbf{Z}(t)]\mathbf{F}(t) \\ \mathbf{a}_f[\mathbf{F}(t)] \end{bmatrix} \tag{16}$$

$$\mathbf{G}[\mathbf{X}(t)] = \begin{bmatrix} \mathbf{0} \\ \mathbf{V}_f[\mathbf{F}(t)] \end{bmatrix} \tag{17}$$

that, in general, are functions of the vector $\mathbf{X}(t)$. Note that in the case of linear filter equations, $\mathbf{G}[\mathbf{X}(t)] \equiv \mathbf{G} = \text{cost}$, then the whole system is subjected to white noises external excitations only.

2.4 Itï¿œ stochastic differential equations

Let us consider a non-linear system whose dynamic behavior, described by the time evolution of the n-vector of state variables $\mathbf{X} = \mathbf{X}(t)$, is governed by the system (15) of n first-order differential equations (in scalar form):

$$\frac{d}{dt}X_i(t) = f_i[\mathbf{X}(t)] + \sum_{k=1}^{m} G_{ik}[\mathbf{X}(t)]W_k(t) \tag{18}$$

where $X_i(t)$ (with $i = 1, 2, \ldots, n$) are the components of the response state vector $\mathbf{X}(t)$, while $W_k(t)$ (with $k = 1, 2, \ldots, m$) are m random excitations. Functions $f_i[\mathbf{X}(t)]$ and $G_{ik}[\mathbf{X}(t)]$ of the state vector $\mathbf{X}(t)$ are generally non-linear. If $G_{ik}[\mathbf{X}(t)]$ are constant, i.e. $G_{ik}[\mathbf{X}(t)]{=}G_{ik}$, then the system is said to be excited by additive excitations only; on the contrary, if the functions $G_{ik}[\mathbf{X}(t)]$ depend on the response process $\mathbf{X}(t)$ the system is said to be excited by multiplicative excitations.

The white noise processes $W_k(t)$ are not mean square Riemann integrable and, consequently, eq.(18) has no traditional mathematical meaning. The latter can be considered to be formally equivalent to the following Itï¿œ-type stochastic differential equations (Lin, 1967; Ibrahim, 1985):

$$dX_i(t) = m_i[\mathbf{X}(t)]dt + \sum_{k=1}^{m} G_{ik}[\mathbf{X}(t)]d\xi_k(t) \tag{19}$$

where $\xi_k(t)$ (with $k = 1, 2, \ldots, m$) are m Wiener processes whose increments are characterized by cross-correlation functions $E[d\xi_k(t_1)d\xi_l(t_2)] = Q_{kl}\delta(t_1 - t_2)dt_1dt_2$, while $m_i[\mathbf{X}(t)]$ are the drift coefficients related to the coefficients of the equations of motion (18) by the following expression:

$$m_i[\mathbf{X}(t)] = f_i[\mathbf{X}(t)] + z_i[\mathbf{X}(t)] \tag{20}$$

In particular, the second terms on the right-hand side of eq.(20), given by

$$z_i[\mathbf{X}(t)] = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} Q_{kl}G_{jl}[\mathbf{X}(t)]\frac{\partial}{\partial X_j}G_{ik}[\mathbf{X}(t)] \tag{21}$$

are the well-known Wong–Zakai (or Stratonovich) correction terms (Wong and Zakai, 1965; Stratonovich, 1967). From eq.(21) it is easy to note that the Wong–Zakai correction terms $z_i[\mathbf{X}(t)]$ vanish in the case of purely additive excitations only.

By introducing the standard Wiener processes $B_j(t)$ (with $j = 1, 2, \ldots, n$), whose increments $dB_j(t)$ are characterized by the cross-correlation functions $E[dB_j(t_1)dB_l(t_2)] = \delta(t_1 - t_2)dt_1dt_2$, eq.(19) can be rewritten in the following equivalent Itï¿œ form:

$$dX_i(t) = m_i[\mathbf{X}(t)]dt + \sum_{j=1}^{n}\sigma_{ij}[\mathbf{X}(t)]dB_j(t) \tag{22}$$

where $\sigma_{ij}[\mathbf{X}(t)]$ are the diffusion coefficients related to the coefficients of the equations of motion (18) by the following expressions:

$$\sigma_{ij}^2[\mathbf{X}(t)] = \sum_{k=1}^{m}\sum_{l=1}^{m} Q_{kl}G_{ik}[\mathbf{X}(t)]G_{jl}[\mathbf{X}(t)] \tag{23}$$

3 Moment equation method (MEM)

3.1 The moment equations of the response

Exact solutions for non-linear systems subjected to additive or both additive and multiplicative random excitations are difficult to obtain. In many cases, therefore, it is necessary to adopt approximate solutions. In the case in which the response of the system can be represented by a Markov process, a frequently used approximate solution is given by the moment equation method (MEM). This approach is based on the use of the Itï¿œ differential rule (Lin, 1967), given by:

$$d\varphi[\mathbf{X}(t)] = \sum_{i=1}^{n}\left\{\left[m_i[\mathbf{X}(t)]dt + \sum_{j=1}^{n}\sigma_{ij}[\mathbf{X}(t)]dB_j(t)\right]\frac{\partial\varphi[\mathbf{X}(t)]}{\partial X_i}\right\}$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[\sigma_{ij}^2[\mathbf{X}(t)]\frac{\partial^2\varphi[\mathbf{X}(t)]}{\partial X_i\partial X_j}dt\right] \tag{24}$$

where $\varphi[\mathbf{X}(t)]$ is a differentiable function of the response vector process $\mathbf{X}(t)$. By applying the stochastic average to both members of eq.(24) and dividing by dt, the following differential equation governing the evolution of the average of the function $\varphi(\mathbf{X}) = \varphi[\mathbf{X}(t)]$ is obtained:

$$\frac{d}{dt}E[\varphi(\mathbf{X})] = \sum_{i=1}^{n}\left[m_i(\mathbf{X})\frac{\partial\varphi(\mathbf{X})}{\partial X_i}\right] + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left[\sigma_{ij}^2(\mathbf{X})\frac{\partial^2\varphi(\mathbf{X})}{\partial X_i\partial X_j}\right] \quad (25)$$

Setting $\varphi(\mathbf{X}) = X_1^{k_1}X_2^{k_2}\dots X_n^{k_n}$, the differential equation (25) rule all the moments of order $k = k_1 + k_2 + \ldots + k_n$ of the response state vector $\mathbf{X}(t)$. For linear systems, the moment equations can be easily solved. Unfortunately, for non-linear systems, these equations constitute an infinite hierarchy; in fact the moment differential equations up to a given order contain moments of higher order and, consequently, this system is not solvable.

Two simple examples: the linear oscillator and the nonlinear Duffing oscillator. Let us consider the following linear oscillator subjected to a white noise:

$$\ddot{U}(t) + \beta\dot{U}(t) + kU(t) = \sqrt{2\beta}W(t) \quad (26)$$

where β and k are constants and $W(t)$ is a Gaussian white noise with intensity $q = 2\pi S_W$, S_W being the constant power spectral density. In terms of the state variables we have:

$$\dot{\mathbf{X}}(t) = \mathbf{D}\mathbf{X}(t) + \mathbf{G}W(t) \quad (27)$$

where

$$\mathbf{X}(t) = \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} U(t) \\ \dot{U}(t) \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 1 \\ -k & -\beta \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} 0 \\ \sqrt{2\beta} \end{bmatrix} \quad (28)$$

The stationary response is Gaussian and the complete probabilistic characterization is given by the second order moment of displacement and velocity. By applying eq.(25), the following second order moment equations result:

$$\frac{d}{dt}E[X_1^2] = 2E[X_1X_2] \quad (29)$$

$$\frac{d}{dt}E[X_1X_2] = -kE[X_1^2] + E[X_2^2] - \beta E[X_1X_2] \quad (30)$$

$$\frac{d}{dt}E[X_2^2] = -2\beta E[X_2^2] - 2kE[X_1X_2] + 2\beta q \tag{31}$$

Eqs.(29)-(31) constitute a closed set of equations whose unknowns are the second order moments $E[X_1^2]$, $E[X_1X_2]$ and $E[X_2^2]$. It can be solved by a numerical scheme in order to obtain the time evolution of the response moments. The stationary solution can be easily determined by solving the set of algebraic equations resulting from eqs. (29)-(31) where the left-hand sides are set to zero. The stationary second order moment are:

$$E[X_1^2] = \frac{q}{k}, \quad E[X_1X_2] = 0, \quad E[X_2^2] = q \tag{32}$$

Let us consider now the following nonlinear Duffing oscillator:

$$\ddot{U}(t) + \beta\dot{U}(t) + kU(t) + \epsilon kU^3(t) = \sqrt{2\beta}W(t) \tag{33}$$

where $\epsilon > 0$ is a nonlinear parameter. In this case the response is not Gaussian and, in principle, all the moments of every order are needed in order to characterize the response. Operating in a similar way, by using the Itï¿œ differential rule the moment equations of every order can be written. For example, the second order moment differential equations are:

$$\frac{d}{dt}E[X_1^2] = 2E[X_1X_2] \tag{34}$$

$$\frac{d}{dt}E[X_1X_2] = -kE[X_1^2] - \epsilon kE[X_1^4] + E[X_2^2] - \beta E[X_1X_2] \tag{35}$$

$$\frac{d}{dt}E[X_2^2] = -2\beta E[X_2^2] - 2kE[X_1X_2] - 2\epsilon kE[X_1^3X_2] + 2\beta q \tag{36}$$

These equations cannot be solved, because moments of fourth order appear. This circumstance is also evident in the moment equations of every order r, where the moments of order $r+2$ appear. Then the moment differential equations constitute an infinite hierarchy of equations, whose solution requires the use of a closure scheme in order to express the higher order moments in terms of the lower order ones.

3.2 Closure schemes

The moment equation method applied to nonlinear systems requires the use of a closure scheme in order to become solvable the hierarchy of the equations, by expressing in an approximate way the higher order moments in terms of lower order ones. This goal can be obtained by introducing a suitable approximation of the pdf of the response.

The Gaussian Closure (GC). The simplest closure method is given by the Gaussian approximation and the resulting scheme is called Gaussian Closure (GC). The Gaussian probability density function is given as:

$$p_{\mathbf{X}}^{G}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n Det(\mathbf{\Sigma_X})}} exp\left[-\frac{1}{2}(\mathbf{x}-\mu_{\mathbf{X}})^T \mathbf{\Sigma_X}^{-1}(\mathbf{x}-\mu_{\mathbf{X}})\right] \tag{37}$$

where $\mu_{\mathbf{X}} = E[\mathbf{X}]$ is the mean vector and $\mathbf{\Sigma_X} = E[(\mathbf{X}-\mu_{\mathbf{X}})(\mathbf{X}-\mu_{\mathbf{X}})^T]$ is the covariance matrix of the response. In this way, the moment of order grather than two can be expressed in term of the moment up to second order by means of nonlinear relationships. The moment equation method in conjuction with the GC consider the second order moment equations only.

An example: the Gaussian Closure applied to the nonlinear Duffing oscillator. As an illustration, the Duffing oscillator is considered. In the second order moment equations (34)-(36) the fourth order moments $E[X_1^4]$ and $E[X_1^3 X_2]$ appear. From the Gaussian hyphothesis, they can be expressed in an approximate way in terms of the second order moments as follows:

$$E[X_1^4] \approx E_G[X_1^4] = 3(E[X_1^2])^2 \tag{38}$$

$$E[X_1^3 X_2] \approx E_G[X_1^3 X_2] = 3E[X_1^2]E[X_1 X_2] \tag{39}$$

In the stationary state, the algebraic second order moment equations reduce to the following set of nonlinear equations:

$$E[X_1 X_2] = 0 \tag{40}$$

$$-kE[X_1^2] - 3\epsilon k(E[X_1^2])^2 + E[X_2^2] - \beta E[X_1 X_2] = 0 \tag{41}$$

$$-2\beta E[X_2^2] - 2kE[X_1 X_2] - 6\epsilon k E[X_1^2]E[X_1 X_2] + 2\beta q = 0 \tag{42}$$

whose solution is:

$$E[X_1^2] = \frac{1}{6\epsilon}\left(\sqrt{1+12\epsilon\frac{q}{k}} - 1\right), \quad E[X_1 X_2] = 0, \quad E[X_2^2] = q \tag{43}$$

that constitutes an approximation of the second order moments of the response.

Non Gaussian Closure (NGC) methods. For strongly nonlinear systems, non Gaussian Closure (NGC) methods are necessary to obtain an adequate approximation of the response. They are based on different representation of the non Gaussian characteristics of the response. Among all, the most used schemes are the cumulant closure (CC) method and the quasi-moment closure (QMC) method. The first closure method is based on the truncation of the series expansion of the log-characteristic function of the response process, whose coefficients are the cumulants; the second is based on the truncation of the A-type Gram-Charlier series expansion of the non-Gaussian probability density function of the response, whose coefficients are the quasi-moments (or, alternatively, the Hermite moments) of the response process. For both methods, the resulting equations are nonlinear.

3.3 The modified Quasi-Moment Closure (QMC) method

Recently (Muscolino et al., 2003), a modified version of the quasi-moment closure method has been proposed. The method takes advantage of the great accuracy of the Monte Carlo Simulation (MCS) in evaluating the first two moments of the response process by considering just few samples. The quasi-moment neglect closure is used to close the infinite hierarchy of the moment differential equations of the response process. Moreover, in order to determine the higher order statistical moments of the response, the second-order probabilistic information given by MCS in conjunction with the quasi-moment neglect closure leads to a set of linear differential equations. The method has been developed for a more general case of external non Gaussian Poisson white noise excitations (DiPaola and Falsone, 1993). The case of the Gaussian white noise input can be considered as a particularization of the non Gaussian case. As an illustration, the scalar case is proposed, the extension to MDOF systems being available in Muscolino et al. (2003), taking advantage from the kronecker algebra.

Poisson white noise excitation. Let us consider the differential equation describing a one-dimensional system subjected to a purely external Poisson white noise process, given by:

$$\frac{d}{dt}X(t) = f[X(t)] + GI(t) \tag{44}$$

where $I(t)$ is a scalar Poisson white noise process (Grigoriu, 1987; Ricciardi, 1994), that is a sequence of impulses with random amplitude and arriving at random times, given by

$$I(t) = \sum_{k=1}^{N(t)} Y_k \delta(t - t_k) \tag{45}$$

the amplitudes $\{Y_k\}$ being a family of random variables, mutually independent and independent of the time instants t_k , with prescribed distribution $p_Y(y)$, $\delta(t)$ the Dirac's delta function, t_k the random time arrivals. In eq.(45), $N(t)$ is a counting Poisson process with parameter λ. The probabilistic characterization of the Poisson white noise process $I(t)$ can be given in terms of its correlation functions (Stratonovich, 1963):

$$C_r[I(t_1), I(t_2), \ldots, I(t_r)] = \lambda E[Y^r]\delta(t_1 - t_2)\delta(t_1 - t_3)\ldots\delta(t_1 - t_r) \tag{46}$$

It follows that $I(t)$ is a non-normal delta-correlated process, with intensity coefficients related to the probabilistic characterization of the random variables $\{Y_k\}$ and of the counting Poisson process $N(t)$. The Poisson white noise process $I(t)$ is not mean square Riemann integrable and, consequently, eq.(44) has no traditional mathematical meaning. The latter can be considered to be formally equivalent to the following generalized Itï¿œ-type stochastic differential equation:

$$dX(t) = f[X(t)]dt + Gd\gamma(t) \tag{47}$$

where $d\gamma(t)$ is an increment of the compound Poisson process $\gamma(t)$, defined by

$$\gamma(t) = \sum_{k=1}^{N(t)} Y_k U(t - t_k) \tag{48}$$

$U(t)$ being the unit step function.

Moment equations. The Itï¿œ differential rule generalized to the case of non Gaussian delta correlated input process leads to the following moment differential equations (DiPaola and Falsone, 1993):

$$\dot{m}_r[X(t)] = rE[X^{r-1}f(X)] + \sum_{k=1}^{r} L_{r,k} m_{r-k}[X(t)]G^k \lambda E[Y^k] \tag{49}$$

where $\dot{m}_r[X(t)] = E[X^r(t)]$ is the moment of order r and $L_{r,k} = r!/[k!(r - k)!]$. Note that in the case of zero-mean Gaussian white noise input, the sum

in eq.(49) needs to be performed only for $k = 2$. In this case $q = \lambda E[Y^2]$ is usually defined as the strength of the white process.

For simplicity's sake, let us assume that the deterministic nonlinear function $f[X(t)]$ is a polynomial of p-th order:

$$f[X(t)] = \sum_{j=1}^{p} a_j X^j \tag{50}$$

By means of this equation, we can rewrite eq.(49) as follows:

$$\dot{m}_r[X(t)] = r \sum_{j=1}^{p} a_j m_{r+j-1}[X(t)] + \sum_{k=1}^{r} L_{r,k} m_{r-k}[X(t)] G^k \lambda E[Y^k] \tag{51}$$

As a consequence of the nonlinearity of the system, in the first summation of the right member of eq.(51) moments until the $(r + p-1)$-th order appear. It follows that the differential equation governing the evolution of statistics of order r will now include statistics of higher order than r. As an example, if cubic non-linearity is considered, the differential eq.(51) contains moments of $(r+2)$-th order. The numerical solution of eq.(51) can be obtained only if a Gaussian or NGC is performed (Muscolino, 1993). The most used closure techniques, such as the cumulant and the quasi-moment neglect closure, lead to a set of nonlinear equations. This fact represents the major drawback in the numerical applications, since the uniqueness of the solution is not guaranteed, as emphasized by Wojtkiewicz et al. (1996).

Non Gaussian Closure. In Muscolino et al. (2003), a new closure technique is proposed, based on the modification of the well-known quasi-moment neglect closure and taking advantage on the great accuracy of the MCS in the evaluation of the second-order statistics of the response. The new closure technique leads to a set of linear equations for the evaluation of higher order statistical moments. Indeed, the relationships between quasi-moments and statistical moments can be written as follows (Muscolino, 1993) (for $s > 2$):

$$b_s[X] = m_s[X] + (-1)^s \sum_{\substack{i = 0, 2 \ldots (s = even) \\ i = 3, 5 \ldots (s = odd)}}^{s-2} B_{s,i} \sigma_X^{s-i} m_i[X]$$

$$+(-1)^s \sum_{\substack{i=0,2\ldots(s=even)\\ i=3,5\ldots(s=odd)}}^{s} B_{s,i}\sigma_X^{s-i}\left\{\sum_{r=1}^{i}(-1)^r\frac{i!}{r!(i-r)!}m_{i-r}[X]\mu_X^r\right\} \tag{52}$$

in which $b_s[X]$ are the quasi-moments of the response process and

$$B_{s,s}=(-1)^s, \qquad B_{s,i}=(1+i)B_{s-1,i+1}-B_{s-1,i-1}. \tag{53}$$

If the mean μ_X and the standard deviation σ_X of the response process are approximated by MCS, that is $\mu_X = m_1 \cong \widetilde{\mu}_X$ and $\sigma_X = \sqrt{m_2 - m_1^2} \cong \widetilde{\sigma}_X$, and if a quasi-moments closure of order R is adopted, the statistical moments of order higher than R can be evaluated from eq.(52) assuming the quasi-moments of order higher than R equal to zero ($b_s[X] = 0,\ s > R$) and obtaining linear relationships. Then, we obtain a set of $R-2$ linear differential equations, whose unknowns are the statistical moments of the response process of order greater than two, up to order R. The unique solution of the set of linear differential equations can then be obtained by traditional numerical procedures. Moreover, since the number of realization required to approximate accurately the first two statistical moments of the response is relatively small, the method is very competitive to find additional probabilistic information in terms of moments of higher order than two.

An example: the nonlinear half-oscillator. As an example, let us consider as an application the following nonlinear half-oscillator (Muscolino et al., 2003):

$$\frac{d}{dt}X(t) = -\nu X(t) - \epsilon X^3(t) + I(t) \tag{54}$$

where ν and ϵ are positive constant and $I(t)$ is a Poisson white process, with the impulse occurrence defined by the parameter λ. The impulse amplitude is assumed as a standardized Gaussian random variable Y . Then, the moments of odd order are zero and the moments of even order are $E[Y^{2k}] = (2k-1)!!$. Taking into account that $f[X(t)] = -\nu X(t) - \epsilon X^3(t)$ is an odd function, the moments of odd order of the response process are zero and only the moments of even order must be evaluated. The stationary problem was solved for the following parameter set: $\nu = \epsilon = 1$, $\lambda = 30$. The approximate variance $\widetilde{\sigma}_X^2 = 2.289$ of the response process has been evaluated by MCS with 100 samples only and exploiting the ergodicity of the response process. The obtained value is a very good approximation

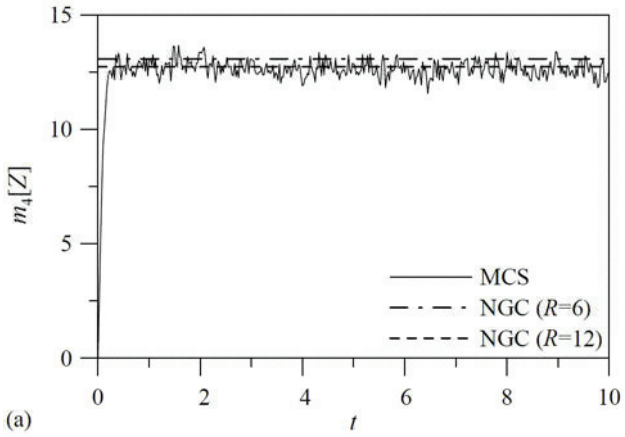

Figure 1. Comparison of the proposed method with the non-stationary MCS results for different order of NGC: fourth-order moment of the response process.

of the assumed exact value determined by MCS with 10^9 samples. The algebraic stationary moment equations of order $r = 4, 6, \ldots, R$ have been considered and the closure method has been performed, by setting to zero the quasi-moment of order $(R+2)$ and approximating the moment of $(R+2)$-th order. The resulting linear algebraic moment equations have been solved and the stationary values of the approximate moments have been compared with non-stationary MCS results with $10,000$ samples.

In Figs. 1 and 2 the fourth-order and the sixth-order approximate stationary moments are plotted for different values of R, revealing the good approximation of results compared with non-stationary solutions by simulation (with zero initial condition). In Table 1, the stationary moments of the response process up to $10-th$ order evaluated by the proposed method for different closure orders are compared with MCS results with 10^9 samples, assumed as exact solutions. This comparison reveals the good performances of the method and the accuracy increases as the closure order increases.

4 The Stochastic Linearization (SL)

The stochastic linearization (SL) is the most used approximate method for the analysis of nonlinear structural systems under random excitations (Roberts and Spanos, 1991; Socha and Soong, 1991; Elishakoff, 2000; Fal-

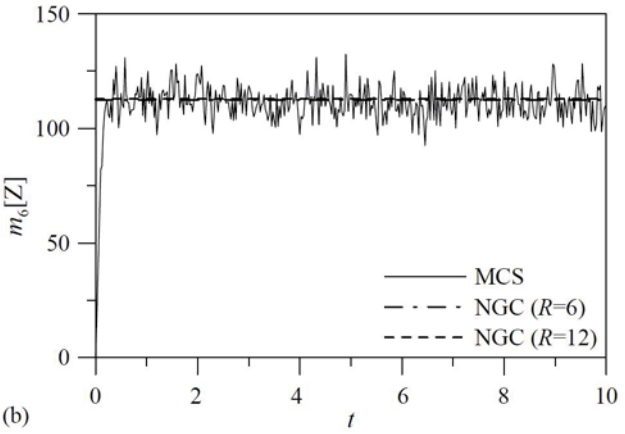

Figure 2. Comparison of the proposed method with the non-stationary MCS results for different order of NGC: sixth-order moment of the response process.

sone and Ricciardi, 2003). This success is due to the wide applicability of the SL method in solving real stochastic structural dynamic problems, often characterized by a large number of degrees of freedom and by complex mathematical models to represent adequately the non-linear structural behavior.

Booton Booton (1953), Kazakov (1956) and Caughey (1959) have independently introduced the method around to the half of the past century. The basic idea is to replace the original non-linear system by an equivalent linear one, whose determination is performed by minimizing the difference between the two systems in some statistical sense. The SL method exhibits different forms based on the adopted probability density function for the evaluation of the coefficients that appear in the linearized system (Kozin, 1988).

4.1 The Gaussian Stochastic Linearization (GSL)

In this section the basic concepts of the GSL method for the stochastic structural dynamics are given. Let us consider a nonlinear structural system excited by external excitations whose dynamic behavior is governed by the following system of first-order differential equations:

$$\dot{\mathbf{X}}(t) = \mathbf{f}[\mathbf{X}(t)] + \mathbf{G}\mathbf{W}(t) \tag{55}$$

Table 1. Stationary moments of the response evaluated by the proposed method for different order of NGC compared with assumed exact values by MCS (10^9 samples)

	$m_4[X]$	$m_6[X]$	$m_8[X]$	$m_{10}[X]$
$R = 4$	13.74	176.1	-	-
$e(\%)$	(8.96)	(58.50)		
$R = 6$	13.08	112.4	715	-
$e(\%)$	(3.73)	(1.17)	(48.34)	
$R = 8$	12.85	112.7	1441	30,591
$e(\%)$	(1.90)	(1.44)	(4.12)	(31.52)
$R = 10$	12.76	112.7	1435	24,460
$e(\%)$	(1.19)	(1.44)	(3.68)	(5.16)
$R = 12$	12.73	112.8	1432	24,440
$e(\%)$	(0.95)	(1.53)	(3.47)	(5.07)
Exact	**12.61**	**111.1**	**1384**	**23,260**
(MCS)				

where $\mathbf{W}(t) = [\ W_1(t)\quad W_2(t)\quad \ldots\quad W_m(t)\ ]^T$ are zero-mean Gaussian white noise processes, with $E[\mathbf{W}(t_1)\mathbf{W}^T(t_2)] = \mathbf{Q}\delta(t_1 - t_2)$, $\mathbf{Q}$ being the $(m \times m)$-matrix collecting the strengths of the white noise processes.

Following the basic idea of stochastic linearization, it is necessary to approximate the original nonlinear system by an appropriate linear one. By assuming that the nonlinearity is odd and the response is with zero mean, eq.(55) is replaced by the following linearized one:

$$\dot{\mathbf{X}}(t) = \mathbf{A}_{eq}\mathbf{X}(t) + \mathbf{G}\mathbf{W}(t) \tag{56}$$

where the matrix $\mathbf{A}_{eq}$ must be chosen in such a way that the difference between the systems given in eq.(55) and (56) is minimum in some statistical sense. By following the classical approach of SL, the difference is measured on the motion equation and the quantity to be minimised is the mean square

error, that is:

$$\mathbf{e} = \mathbf{f}[\mathbf{X}(t)] - \mathbf{A}_{eq}\mathbf{X}(t), \qquad E[\mathbf{e}^T\mathbf{e}] = \min \tag{57}$$

The minimization conditions impose that:

$$\nabla_{\mathbf{A}_{eq}} E[\mathbf{e}^T\mathbf{e}] = \mathbf{0} \tag{58}$$

where $\nabla_{\mathbf{A}_{eq}}$ is a $n \times n$ differential operator whose (i,j)-component is the partial derivative made with respect to the (i,j)-component of the matrix $\mathbf{A}_{eq}$. By replacing the first of eqs.(57) into eq. (58), after some algebra we obtained:

$$\mathbf{A}_{eq} = E[\mathbf{f}(\mathbf{X})\mathbf{X}^T]\left\{E[\mathbf{X}\mathbf{X}^T]\right\}^{-1} \tag{59}$$

that constitutes the expression of the equivalent matrix $\mathbf{A}_{eq}$ given by the SL method. The latter exhibits different forms based on the adopted probability density function for the approximation of the averages appearing in eq.(59).

In the classical GSL method the averages appearing in eq.(59) are evaluated taking into account the Gaussianity of the response process. In this way, as the response has been assumed to be Gaussian and with zero mean, the averages are approximated as:

$$E[\,\cdot\,] \cong E[\,\cdot\,]_G = \int_{\Re_n} (\,\cdot\,) p^G_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \tag{60}$$

where the Gaussian probability density function is given by eq.(37) with $\mu_{\mathbf{X}} = \mathbf{0}$. Then, the expression (59) of the equivalent matrix becomes:

$$\mathbf{A}^G_{eq}(\mathbf{\Sigma}^{\mathrm{G}}_{\mathbf{X}}) = E_G[\mathbf{f}(\mathbf{X})\mathbf{X}^T][\mathbf{\Sigma}^{\mathrm{G}}_{\mathbf{X}}]^{-1} \tag{61}$$

In this way, the equivalent matrix $\mathbf{A}_{eq}$ is expressed in terms of the statistical moments until the second order of the response only. These moments can be evaluated as solution of the following non-linear Lyapunov-type differential equation:

$$\dot{\mathbf{\Sigma}}^{\mathrm{G}}_{\mathbf{X}} = \mathbf{A}^G_{eq}\mathbf{\Sigma}^{\mathrm{G}}_{\mathbf{X}} + \mathbf{\Sigma}^{\mathrm{G}}_{\mathbf{X}}\mathbf{A}^G_{eq} + \mathbf{G}\mathbf{Q}\mathbf{G}^T \tag{62}$$

It is worth noticing that eq.(62) is nonlinear, because the matrix depends on the second order moments, that, in turn, depend on the probability density function adopted. Hence a numerical procedure for solving the nonlinear equations must be applied.

4.2 The Non Gaussian Stochastic Linearization (NGSL)

Against relatively little numerical efforts, unfortunately the GSL method gives accurate results for weakly non-linear systems only and, as recently highlighted, it suffers of unacceptable errors in the case of large structures (Micaletti et al., 1998). An improved approximation can be obtained if the relaxation of the Gaussian response assumption is introduced. In this way, the averages in eq.(59) are determined by adopting a better approximation of the probability density function of the response. The procedure proposed by Ricciardi (2007) uses as a probability density function a modified A-type Gram-Charlier series expansion, where the covariance matrix is not unknown, but it is assumed equal to the covariance matrix derived by the GSL method. In order to present the aforementioned method, a standardization procedure is needed.

Standardization. The following coordinate transformation is adopted:

$$\mathbf{X}(t) = \mathbf{\Gamma}_G \mathbf{Y}(t), \qquad \mathbf{\Gamma}_G = \mathbf{\Psi}_G \mathbf{\Lambda}_G^{1/2} \tag{63}$$

where the full matrix $\mathbf{\Psi}_G$ and the diagonal matrix $\mathbf{\Lambda}_G$ are evaluated by solving the following eigenproblem related to the covariance matrix $\mathbf{\Sigma}_{\mathbf{X}}^{\mathrm{G}}$:

$$\mathbf{\Sigma}_{\mathbf{X}}^{\mathrm{G}} \mathbf{\Psi}_G = \mathbf{\Psi}_G \mathbf{\Lambda}_G \tag{64}$$

with the normality condition $\mathbf{\Psi}_G^T \mathbf{\Psi}_G = \mathbf{I}_n$. In eq.(64), the covariance matrix $\mathbf{\Sigma}_{\mathbf{X}}^{\mathrm{G}}$ is supposed known, as solution of the GSL method.

It can be easily shown that the vector process $\mathbf{Y}(t)$ has approximated covariance matrix $\mathbf{\Sigma}_{\mathbf{Y}}^{\mathrm{G}} = \mathbf{I}_n$, derived by the GSL. Moreover, by adopting the coordinate transformation eq.(63), eq.(55) can be rewritten as follows:

$$\dot{\mathbf{Y}}(t) = \mathbf{r}[\mathbf{Y}(t)] + \mathbf{U}\mathbf{W}(t) \tag{65}$$

where

$$\mathbf{r}[\mathbf{Y}(t)] = \mathbf{\Gamma}_G^{-1} \mathbf{f}[\mathbf{\Gamma}_G \mathbf{Y}(t)], \qquad \mathbf{U} = \mathbf{\Gamma}_G^{-1} \mathbf{V} \tag{66}$$

By following the classical approach of the SL method, eq.(65) is replaced by the following linearized one:

$$\dot{\mathbf{Y}}(t) = \mathbf{B}_{eq} \mathbf{Y}(t) + \mathbf{U}\mathbf{W}(t) \tag{67}$$

and by minimizing the mean square error, we find the following form of the equivalent matrix $\mathbf{B}_{eq}$:

$$\mathbf{B}_{eq} = E[\mathbf{r}(\mathbf{Y})\mathbf{Y}^T] \left\{ E[\mathbf{Y}\mathbf{Y}^T] \right\}^{-1} \tag{68}$$

If GSL method is adopted, in order to evaluate the averages appearing in eq.(68), $E[\mathbf{Y}\mathbf{Y}^T] \cong E_G[\mathbf{Y}\mathbf{Y}^T] = \mathbf{I}_n$; then, the following expression of the equivalent matrix is obtained:

$$\mathbf{B}_{eq}^G = E_G[\mathbf{r}(\mathbf{Y})\mathbf{Y}^T] \tag{69}$$

It is emphasized that, by introducing the coordinate transformation, the problem is simplified, as the vector process $\mathbf{Y}(t)$ has uncorrelated components with unit variances.

Non Gaussian probability density function. In order to improve the GSL results, the NGSL method approximate the averages appearing in eq.(18) by using the A-type Gram-Charlier series expansion of the non-Gaussian probability density function, given as

$$p(\mathbf{y}) \cong p_{NG}(\mathbf{y}) = p_G(\mathbf{y}) \times$$

$$\times \left[1 + \sum_{\substack{j=2,4,\dots \\ i_1+i_2+\dots+i_n=j}}^{M} \frac{1}{i_1!i_2!\dots i_n!} C_{i_1,i_2,\dots,i_n} H_{i_1}(y_1) H_{i_2}(y_2) \dots H_{i_n}(y_n) \right] \tag{70}$$

where M is the truncation order and $p_G(\mathbf{y})$ is the joint Gaussian probability density function of the uncorrelated random processes $Y_\ell (\ell = 1, 2, \dots, n\ )$, defined as:

$$p_G(\mathbf{y}) = \prod_{\ell=1}^{n} p_G(y_\ell) = \prod_{\ell=1}^{n} \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{y_\ell^2}{2}\right] \tag{71}$$

In eq.(70), $H_k(y_\ell)$ is the one-dimensional Hermite polynomial defined as follows:

$$H_k(y_\ell) = (-1)^k \exp\left[\frac{1}{2}y_\ell^2\right] \frac{d^k}{dy_\ell^k} \left\{ \exp\left[-\frac{1}{2}y_\ell^2\right] \right\} \tag{72}$$

and $C_{i_1,i_2,\dots,i_n}$ is the modified Hermite moment, defined as:

$$C_{i_1,i_2,\dots,i_n} = E[H_{i_1}(Y_1) H_{i_2}(Y_2) \dots H_{i_n}(Y_n)] \tag{73}$$

Improved equivalent matrix and covariance matrix. The NGSL method takes into account the non-Gaussian character of the response process and concurs to estimate the expression of the equivalent matrix of the linearized system in a more accurate way. By adopting the non-Gaussian probability density function (70), $\mathbf{B}_{eq}$ assumes the following form:

$$\mathbf{B}_{eq}^{NG} = E_{NG}[\mathbf{r}(\mathbf{Y})\mathbf{Y}^T]\left\{E_{NG}[\mathbf{Y}\mathbf{Y}^T]\right\}^{-1} \tag{74}$$

After some algebra, the equivalent matrix assumes the following form:

$$\mathbf{B}_{eq}^{NG} = \mathbf{R}^{NG}[\mathbf{I}_n + \mathbf{S}]^{-1} \tag{75}$$

where the matrices $\mathbf{S}$ and $\mathbf{R}^{NG}$ are given as:

$$\mathbf{S} = \begin{bmatrix} C_{2,0,\ldots 0} & C_{1,1,\ldots 0} & C_{1,0,\ldots 1} \\ C_{1,1,\ldots 0} & C_{0,2,\ldots 0} & C_{0,1,\ldots 1} \\ & & \\ C_{1,0,\ldots 1} & C_{0,1,\ldots 1} & C_{0,0,\ldots 2} \end{bmatrix} \tag{76}$$

$$\mathbf{R}^{NG} = \mathbf{R}^{G} + \sum_{\substack{j=2,4,\ldots \\ i_1+i_2+\ldots+i_n=j}} \frac{1}{i_1!i_2!\ldots i_n!} C_{i_1,i_2,\ldots,i_n}\mathbf{R}_{i_1,i_2,\ldots,i_n} \tag{77}$$

The matrix $\mathbf{R}^G$ and the (j,ℓ)-th element of the matrix $\mathbf{R}_{i_1,i_2,\ldots,i_n}$ can be easily evaluated by performing the following averages:

$$\mathbf{R}^{G} = \hat{E}_G[\mathbf{r}(\mathbf{Y})\mathbf{Y}^T] \tag{78}$$

$$(\mathbf{R}_{i_1,i_2,\ldots,i_n})_{j\ell} = \hat{E}_G[r_j(\mathbf{Y})Y_\ell H_{i_1}(Y_1)H_{i_2}(Y_2)\ldots H_{i_n}(Y_n)] \tag{79}$$

where $\hat{E}_G[\,\cdot\,] = \int_{\Re_n}(\,\cdot\,)p_G(\mathbf{y})d\mathbf{y}$ that involves simple one-dimensional integrals, generally known in closed form. Note that if the non-linear term $r_j(\mathbf{Y})$ is a polynomial expression, the averages are a combination of double factorials, taking advantage of the particular structure of the $\mathbf{Y}$-space.

By using the coordinate transformation, we find the following expression of the equivalent matrix in the original space:

$$\mathbf{A}_{eq}^{NG} = \mathbf{\Gamma}_G\mathbf{R}^{NG}[\mathbf{I}_n + \mathbf{S}]^{-1}\mathbf{\Gamma}_G^{-1} \tag{80}$$

This expression represents a non-Gaussian improved version of the equivalent matrix determined by means of the GSL method. A better approximation of the covariance matrix $\mathbf{\Sigma}_{\mathbf{Y}}^{\mathrm{NG}}$ can be evaluated as follows:

$$\mathbf{\Sigma}_{\mathbf{Y}}^{\mathrm{NG}} = E[\mathbf{Y}\mathbf{Y}^T] \cong E_{NG}[\mathbf{Y}\mathbf{Y}^T] = [\mathbf{I}_n + \mathbf{S}] \tag{81}$$

and the expression of the covariance matrix $\mathbf{\Sigma}_{\mathbf{X}}^{\mathrm{NG}}$ in the original space is:

$$\mathbf{\Sigma}_{\mathbf{X}}^{\mathrm{NG}} = \mathbf{\Gamma}_G[\mathbf{I}_n + \mathbf{S}]\mathbf{\Gamma}_G^T \tag{82}$$

This equation represents the NGSL solution for the covariance matrix, which requires the knowledge of the series coefficients $C_{i_1,i_2,\ldots,i_n}$. As shown later, they can be simply approximated as solution of a system of linear equations.

Approximation of the modified A-type Gram-Charlier series coefficients. The modified Hermite moments $C_{i_1,i_2,\ldots,i_n}$ are defined as the averages of products of one-dimensional Hermite polynomials. They can be evaluated by using the Itï¿œ rule or, alternatively, they can be approximated by taking advantage by the theory of Markov process. The probability density function $p_{\mathbf{Y}}(\mathbf{y})$ of the vector process $\mathbf{Y}(t)$ is the solution of the Fokker-Planck-Kolmogorov (FPK) equation (Risken, 1989):

$$\frac{\partial}{\partial t} p_{\mathbf{Y}}(\mathbf{y}) = \mathcal{L} p_{\mathbf{Y}}(\mathbf{y}) \tag{83}$$

where

$$\mathcal{L}(\,\cdot\,) = \frac{\partial}{\partial y_i}\left[r_i(\mathbf{y})(\,\cdot\,)\right] - \frac{1}{2} Q_{\ell s} U_{i\ell} U_{js} \frac{\partial^2}{\partial y_i \partial y_j}(\,\cdot\,) \tag{84}$$

is the FPK operator (where repeated indices mean summation convention).

It can be easily shown that the differential equation governing the time evolution of the average of a function $\varphi(\mathbf{Y})$ can be written as follows

$$\dot{E}\left[\varphi(\mathbf{y})\right] = E\left\{\mathcal{L}^+\left[\varphi(\mathbf{y})\right]\right\} \tag{85}$$

where

$$\mathcal{L}^+(\,\cdot\,) = r_i(\mathbf{y}) \frac{\partial}{\partial y_i}(\,\cdot\,) + \frac{1}{2} Q_{\ell s} U_{i\ell} U_{js} \frac{\partial^2}{\partial y_i \partial y_j}(\,\cdot\,) \tag{86}$$

is the adjoint Fokker-Planck-Kolmogorov (AFPK) differential operator.

An approximation of the modified A-type Gram-Charlier coefficients can be obtained by choosing $\varphi(\mathbf{Y}) = H_{j_1}(Y_1) H_{j_2}(Y_2) \ldots H_{j_n}(Y_n)$, for $j_1 + j_2 + \ldots + j_n = 2, 4, \ldots, M$, with $M \geq 4$. Then, the following differential equations of $C_{j_1,j_2,\ldots,j_n} = E[H_{j_1}(Y_1) H_{j_2}(Y_2) \ldots H_{j_n}(Y_n)]$ are obtained:

$$\dot{C}_{j_1,j_2,\ldots,j_n} = E\left\{\mathcal{L}^+\left[H_{j_1}(Y_1)H_{j_2}(Y_2)\ldots H_{j_n}(Y_n)\right]\right\} \tag{87}$$

In the stationary state the average quantities are not time dependent and consequently $\dot{C}_{j_1,j_2,\ldots,j_n} = 0$. Therefore, the last equation can be simplified as follows:

$$\sum_{\substack{j=2,4,\ldots \\ i_1+i_2+\ldots+i_n=j}}^{M} D^{j_1,j_2,\ldots,j_n}_{i_1,i_2,\ldots,i_n} C_{i_1,i_2,\ldots,i_n} + b^{j_1,j_2,\ldots,j_n} = 0 \tag{88}$$

where

$$D^{j_1,j_2,\ldots,j_n}_{i_1,i_2,\ldots,i_n} = \frac{1}{i_1!i_2!\ldots i_n!}\times$$

$$\times\left\{\hat{E}_G\left[r_i(\mathbf{Y})H_{i_1}(Y_1)H_{i_2}(Y_2)\ldots H_{i_n}(Y_n)\frac{\partial}{\partial Y_i}\left(H_{j_1}(Y_1)H_{j_2}(Y_2)\ldots H_{j_n}(Y_n)\right)\right]\right.$$

$$\left.+\,\hat{E}_G\left[H_{i_1}(Y_1)H_{i_2}(Y_2)\ldots H_{i_n}(Y_n)\frac{\partial^2}{\partial Y_i\partial Y_j}\left(H_{j_1}(Y_1)H_{j_2}(Y_2)\ldots H_{j_n}(Y_n)\right)\right]\right\} \tag{89}$$

and

$$b^{j_1,j_2,\ldots,j_n} = \hat{E}_G\left[r_i(\mathbf{Y})\frac{\partial}{\partial Y_i}\left(H_{j_1}(Y_1)H_{j_2}(Y_2)\ldots H_{j_n}(Y_n)\right)\right]$$

$$+\frac{1}{2}Q_{\ell s}U_{i\ell}U_{js}\hat{E}_G\left[\frac{\partial^2}{\partial Y_i\partial Y_j}\left(H_{j_1}(Y_1)H_{j_2}(Y_2)\ldots H_{j_n}(Y_n)\right)\right] \tag{90}$$

In the last equations the repeated indices mean summation convention, with $i,j = 1,2,\ldots,n$ and $\ell,s = 1,2,\ldots,m$. Note that eqs.(88) constitute a system of linear algebraic equations with unknowns $C_{j_1,j_2,\ldots,j_n}$, whose solution is an approximation of these coefficients.

The NGSL applied to the power-law nonlinear oscillator. Let us consider the following non-linear oscillator subjected to a Gaussian white noise input with intensity q:

$$\ddot{U}(t) + \beta\dot{U}(t) + g_\gamma\left|U(t)\right|^\gamma sgn[U(t)] = \sqrt{2\beta}W(t) \tag{91}$$

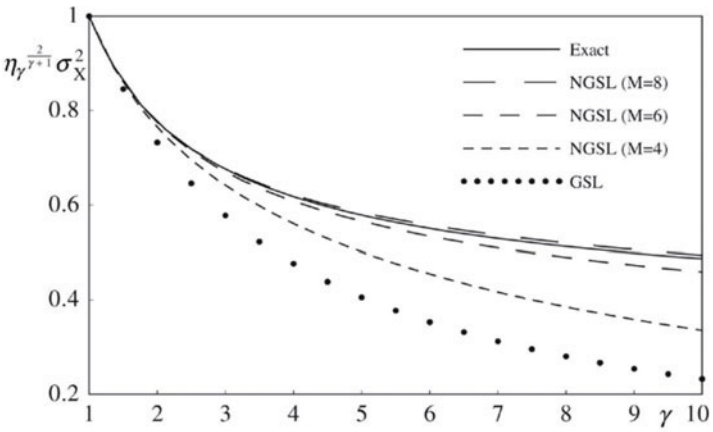

Figure 3. Displacement variance versus γ for the hardening power-law oscillator.

where g_γ is a positive constant and $\gamma > 0$ is a parameter related to the non-linear restoring force. If $0 < \gamma < 1$ or $\gamma > 1$, the oscillator exhibits softening or hardening behavior, respectively. Roberts and Spanos (1991) considered such an oscillator by the Gaussian stochastic linearization method. The exact variance of the displacement is:

$$\sigma_{U,ex}^2 = \left(\frac{\gamma+1}{\eta_\gamma}\right)^{\frac{2}{\gamma+1}} \Gamma\left(\frac{3}{\gamma+1}\right) \Gamma^{-1}\left(\frac{1}{\gamma+1}\right) \tag{92}$$

where $\eta_\gamma = g_\gamma/q$ and $\Gamma(\,\cdot\,)$ is the Gamma function. The GSL method leads to the following approximation:

$$\sigma_{U,G}^2 = \frac{1}{2}\left[\frac{\sqrt{\pi}}{\eta_\gamma \Gamma(1+(\gamma/2))}\right]^{\frac{2}{\gamma+1}} \tag{93}$$

This value has been assumed as a first approximation for the NGSL method.

In Fig. 3 and Fig. 4 the displacement variance and the relative error are plotted versus the non-linear parameter γ, in the case of hardening behavior of the oscillator ($\gamma > 1$). In these figures the NGSL results, for different closure order ($M = 4, 6, 8$), are compared with the GSL results, revealing that the NGSL method leads to better results, also for low closure order ($M = 4$). Accurate results are obtained for higher order closures, with errors

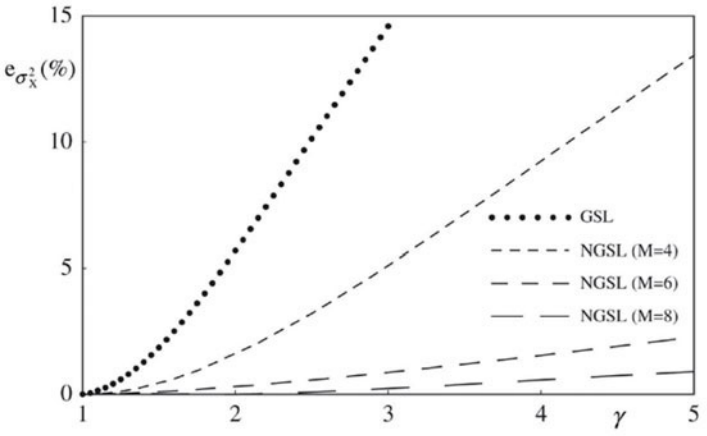

Figure 4. Relative error of displacement variance versus γ for hardening the power-law oscillator.

lesser then 1% for $1 < \gamma < 5$ and $M = 8$. From these figures, it appears that the NGSL method seems to give results very close to the exact ones as M increases. This is due to the fact that the series approximation used by the NGSL method converges to the exact probability density function of the response, given as:

$$p_{U,ex}(u) = \frac{1}{2}\left(\frac{\eta_\gamma}{\gamma+1}\right)^{\frac{1}{\gamma+1}} \Gamma^{-1}\left(\frac{\gamma+2}{\gamma+1}\right) \exp\left(-\eta_\gamma \frac{|u|^{\gamma+1}}{\gamma+1}\right) \tag{94}$$

In Fig. 5 and Fig. 6 the displacement variance and the relative error are plotted, in the case of softening behavior of the oscillator ($0 < \gamma < 1$). From these figures the lack of convergence of the series expansion used by the NGSL is evident. However, in the range $0.2 < \gamma < 1$, the NGSL method gives better results than the GSL, with errors lesser than 5% for all the closure orders considered ($M = 4, 6, 8$). In particular, for $M = 4$, the error is lesser than 5% in the entire range $0 < \gamma < 1$.

The NGSL applied to the double-well Duffing oscillator. Let us consider the following Duffing oscillator:

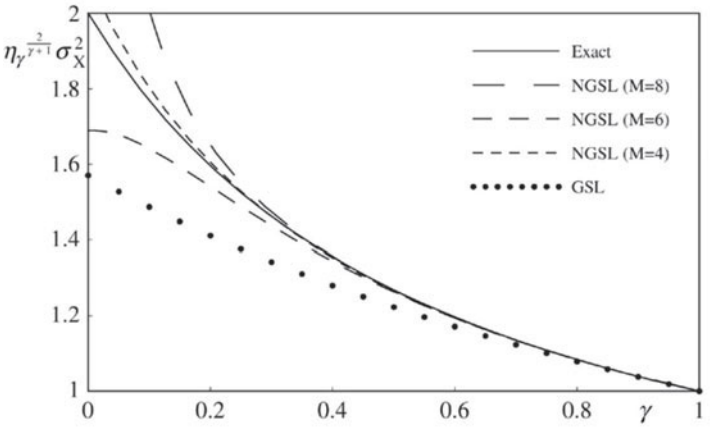

Figure 5. Displacement variance versus γ for the softening power-law oscillator.

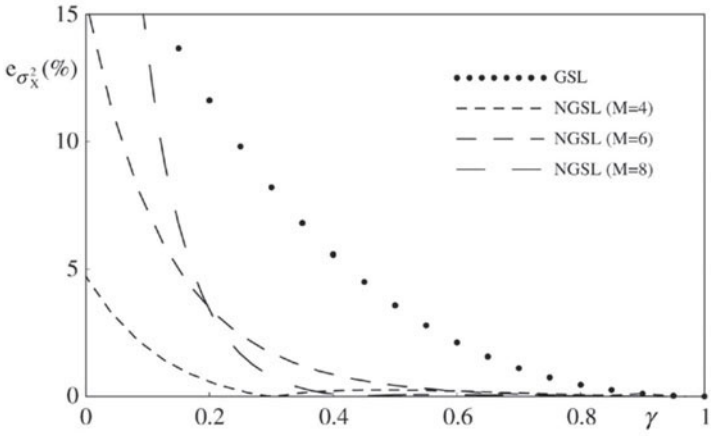

Figure 6. Relative error of displacement variance versus γ for the softening power-law oscillator.