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Topology Optimization in Structural and Continuum Mechanics

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Topology Optimization in Structural and Continuum Mechanics

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Editors

George I. N. Rozvany
Budapest University of Technology and Economics, Hungary

Tomasz Lewiński
Warsaw University of Technology, Poland

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PREFACE

Structural Topology Optimization (STO) is a relatively new, but rapidly expanding and extremely popular field of structural mechanics. Various theoretical aspects, as well as a great variety of numerical methods and applications are discussed extensively in international journals and at conferences. The high level of interest in this field is due to the substantial savings that can be achieved by topology optimization in industrial applications. Moreover, STO has interesting theoretical implications in mathematics, mechanics, multi-physics and computer science.

This is the third CISM Advanced Course on Structural Topology Optimization. The two previous ones were organized by the first author of this Preface, the current one – by both authors.

The aim of the present course is to cover new developments in this field since the previous CISM meeting on STO in 1997. The topics reviewed by various lecturers of this course are summarized briefly below.

In his first lecture, George I. N. Rozvany reviews the basic features and limitations of Michell's (1904) truss theory, and its extension to a broader class of support conditions.

In the second lecture, George Rozvany and Erika Pinter give an overview of generalizations of truss topology optimization, via the Prager-Rozvany (1977) optimal layout theory, to multiple load conditions, probabilistic design and optimization with pre-existing members, also briefly reviewing optimal grillage theory and cognitive processes in deriving exact optimal topologies.

George Rozvany's third lecture discusses fundamental properties of exact optimal structural topologies, including (non)uniqueness, symmetry, skew-symmetry, domain augmentation and reduction, and the effect of non-zero support cost.

In a joint lecture with Tomasz Sokół, the verification of various numerical methods by exact analytical benchmarks is explained, and conversely, the confirmation of exact analytical solutions by Sokół's numerical method is discussed. The latter can currently handle ground structures with several billion potential members. In his final lecture, George Rozvany gives a concise historical overview of structural topol-

ogy optimization, and critically reviews various numerical methods in this field.

The lecture by Tomasz Lewiński and Tomasz Sokół is focused on one aspect of the lectures by George Rozvany, namely on the Michell continua. This theory is constructed for volume minimization of trusses which finally reduces to a locking material problem.

The Michell problem belongs to the class of optimization of statically determinate structures whose behavior is governed only by the equilibrium conditions and constraints bounding the stress level. More complex problems arise if one optimizes the shape of elastic bodies, even those being homogeneous and isotropic. In general, the layout problems in linear elasticity are ill-posed, which is the central question of the lecture by François Jouve. This author discusses the above problem and clears up the remedies: either to extend the design space and to relax the problem, or to reduce the design space by introducing new regularity constraints. The relaxation by homogenization method is outlined in Sec. 2 of this lecture, along with numerical techniques. The method is efficient due to fundamental results concerning optimal bounds on the energy. Although this exact and explicit result is restricted to the compliance minimization for a single load condition, it has served as the basis for various researchers to develop other homogenization-based methods, such as the one by Grégoire Allaire, Eric Bonnetier, Gilles Francfort and François Jouve in 1997. In his lecture François Jouve discusses also the methods of partial relaxation of selected problems for which the exact relaxations are not at our disposal, or they assume a non-explicit form. The last chapter of the lecture concerns the level set method proposed in the early 2000's, which gives very promising results, even in an industrial context, with complex state equations, objective functions and constraints. This author shows how this method can be combined with shape derivatives and by the topology derivatives of selected functionals.

The lecture by Grzegorz Dzierżanowski and Tomasz Lewiński delivers a complete derivation of the crucial result mentioned: the optimal bounds on the energy. The derivation is based on the translation method for the case of two isotropic constituents and then reduced to the case if one constituent is a void.

Structural topology optimization comprises also the design of material characteristics without linking them with the density of mass.

This optimization field is called the Free Material Design (FMD). The classical FMD problem is aimed at finding the optimum values of all components of the Hooke tensor from the criterion of compliance minimization, under the isoperimetric condition of boundedness of the integral of the trace of the Hooke tensor. The lecture by Sławomir Czarnecki and Tomasz Lewiński shows that the FMD problem can be reduced to a locking material problem, even in the multi-load case.

The six lectures by Niels Olhoff, Jianbin Du and Bin Niu concern the optimization of structures subjected to dynamic loads. These authors explain how to design a structure such that the structural eigenfrequencies of vibration are as far away as possible from a prescribed external excitation frequency - or band of excitation frequencies - to avoid resonance phenomena with high vibration and noise levels. This objective may be achieved by

- *maximizing the fundamental eigenfrequency of the structure,*
- *maximizing the distance (gap) between two consecutive eigenfrequencies,*
- *maximizing the dynamic stiffness of the structure subject to forced vibration,*
- *minimizing the sound power flow radiated from the structural surface into an acoustic medium.*

A special lecture by Niels Olhoff and Bin Niu discusses how maximization of the gap between two consecutive eigenfrequencies generates significant design periodicity, and the final (sixth) lecture presents the application of a novel topology based method of simultaneous optimization of fiber angles, stacking sequence, and selection of materials, for vibrating laminate composite plates with minimum sound radiation.

In the first three of his five lectures, Kurt Maute discusses applications of the density method to diffusive and convective transport processes, as well as to multi-physics problems. The complexity of selecting appropriate objectives and constraints are emphasized in the chapter on diffusive transport optimization problems. The extension of the porosity model to fluid problems is presented for flow topology optimization problems, characterized by the Darcy-Stokes and Navier-Stokes equations at steady state conditions. The fundamental differences in solving multi-physics problems that are either coupled via constitutive laws or via surface interactions are discussed and illustrated with applications to piezo-electric coupling and fluid-structure *interaction problems. The fourth lecture introduces an alternative to topology optimization approaches that employ density or Ersatz material approaches to represent the material layout in the mechanical model. The integration of the eXtended Finite Element Method (XFEM) into a level-set topology optimization method is discussed and illustrated with applications to flow topology optimization. The last lecture by Kurt Maute is devoted to topology optimization methods that account for uncertainty in material parameters, geometry, and operating conditions. Here, the aim is to arrive at reliable and robust designs. This lecture introduces basic techniques in reliability based design optimization (RBDO) and robust design optimization (RDO), and discusses their application to topology optimization.*

G.I.N. Rozvany and T. Lewiński

CONTENTS

Structural Topology Optimization (STO) – Exact Analytical Solutions: Part I

George I. N. Rozvany[1]

[1] Department of Structural Mechanics, Budapest University of Technology and Economics, Budapest, Hungary

1. Introduction: Some Previous Meetings, Review Lectures and Review Articles on Structural Topology Optimization (STO)

As mentioned in the Preface, on the subject of STO the author has organized two previous CISM Advanced Courses in Udine (Rozvany 1992 and 1997), and with Niels Olhoff a NATO ARW in Budapest (Rozvany and Olhoff 2000).

In recent years the author gave principal, keynote or review lectures on STO at important meetings, such as 6^{th} European Solid Mechanics Conference (Budapest, 2006), World Congress of Computational Mechanics (Venice, 2008), US National Congress on Computational Mechanics (Columbus, Ohio, 2009), Chinese Solid Mechanics Congress (Dalian, 2009), Hungarian Academy of Science meeting (Budapest 2009), 8^{th} ASMO/ISSMO Conference (London, 2010), Computational Structures Technology Conference (Valencia, Spain, 2010), Computational Methods in Mechanics Conference (Warsaw, 2011), and Engineering Computational Technology Conference (Dubrovnik, 2012).

The author (and co-authors) have also written some much-cited review articles on STO (Rozvany, Bendsoe and Kirsch 1995, Rozvany 2001 and 2009). A most comprehensive and authoritative book on STO is by Bendsoe and Sigmund (2003).

2. Preliminaries

Since the above lectures dealt mostly with numerical topology optimization, the first three lectures of the author will discuss *exact analytical solutions*. These are extremely important as benchmarks, since numerical methods of topology optimization have many sources of error.

A basic feature of the author's lectures is that they employ *very simple examples*, so that fundamental principles of mechanics are not obscured by computational or mathematical complexities. Moreover, following the advice of the late Professor William Prager (Brown University), various general features of optimal topologies will be explained and proved at an 'engineering level', using principles of mechanics in preference to higher mathematical concepts.

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3. Subfields of STO

As explained previously (see e.g. the review article, Rozvany, Bendsoe and Kirsch 1995), structural topology optimization can be divided into two subfields.

Layout optimization (LO) deals with grid-like (low volume fraction) structures (trusses, grillages, shell-grids, and dense systems of intersecting shells, see Fig. 1), optimizing simultaneously the topology (connectivity of the members), geometry (location of the joints) and sizing (cross-sectional dimensions).

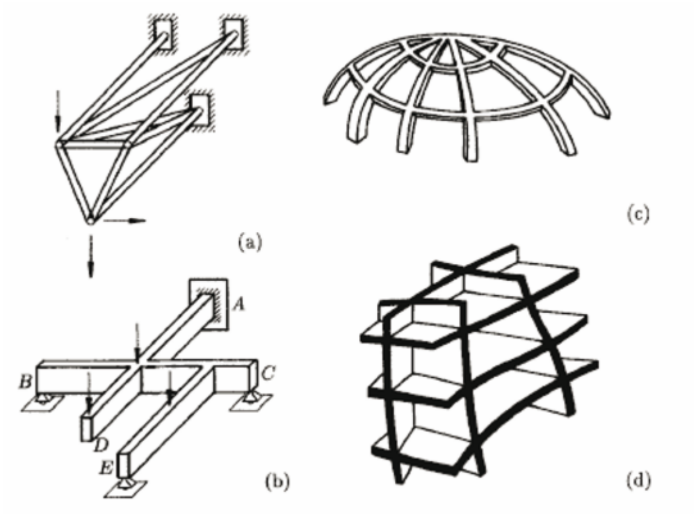

Figure 1. Some examples of grid-like structures: (a) truss, (b) grillage, (c) shell-grid, (d) intersecting shells (honeycomb).

Generalized shape optimization (GSO) finds the best topology of the internal boundaries, and the shape of both internal and external boundaries of perforated or composite continua.

3.1. Layout Optimization (LO)

In layout optimization (LO), we start off with a ‘ground structure’, which contains all potential members (‘ground elements’). During the optimization procedure, we remove non-optimal ground elements and determine the optimal sizes of the optimal ones.

In discretized LO, the number of ground elements is finite. In exact LO, we have an infinite number of ground elements. A ground structure with an infinite number of elements is sometimes called 'structural universe' (e.g. Rozvany and al. 1995). Alternatively, a structural universe may be replaced by a 'design domain', at all points of which members may run in any direction.

3.2. Generalized Shape Optimization (GSO)

The simplest form of structural topology optimization concerns so-called *ISE topologies* (with Isotropic Solid or Empty elements, see e.g. Rozvany 2001).

In discretized GSO, the structural domain is divided into a finite number of ground elements. For ISE topologies, the thickness or density of each ground element may only take on a zero value ('white element') or a pre-assigned other value ('black element'). In a normalized formulation, the pre-assigned value is unity, and then we have a 0-1 problem.

In exact GSO, the number of ground elements tends to infinity, and therefore they shrink to points. In other words, exact GSO consists of assigning optimally either 'material' or 'no material' to each point of the design domain.

GSO can be generalized to *multi-material optimization*, in which case for each ground element (or at each point of the design domain) we may chose one out of several given materials or no material.

Structural topology optimization may consist *of minimizing an objective function* (typically the total structural volume, weight or cost) subject to *constraints* on the structural response (e.g. stresses, displacements, buckling load, natural frequency, etc.). Additional constraints may put limits on the member sizes (side constraints), ensure manufacturability, or prescribe discrete cross-sectional dimensions. Alternatively, we may maximize or minimize some structural response or a weighted combination of several responses for given volume, weight or cost.

4. Michell's Truss Theory

The first paper on exact structural topology optimization was published more than a century ago by the ingenious Australian inventor A. G. M. ('George') Michell (1904). It dealt with the least-volume topology of trusses with a single load condition and a stress constraint.

4.1. Some Bio-data about Michell

A. G. M. Michell (1870-1959) went to school in Cambridge, UK, and lived later in Melbourne, Victoria, Australia. He was still alive, when the author moved from France to that state of Australia. Michell had a number of ingenious inventions of great importance to the industry, such as thrust-bearing for ships, crankless engine, lubrication systems, timber preservatives, hydraulic power transmissions, etc. However, these were not followed by financial success for Michell. He liked solitude, was not interested in team sports (never got married), and liked to walk hundreds of miles in the Australian bush.

4.2. The Significance of Michell's Paper on Trusses

Michell's (1904) milestone contribution to truss topology optimization was undoubtedly both revolutionary and ingenious. He introduced essential elements of what we now call layout optimization, continuum-type optimality criteria, adjoint strain field and ground structure (structural universe). He achieved all this over a century ago, when almost nothing was known about essential techniques of structural optimization. For this reason, Michell had to rely on new and imaginative ideas in deriving his optimality criteria.

5. Michell's Optimality Criteria

In current notation, Michell's problem can be stated as follows. Considering a structural domain D with given external forces, minimize the volume V of a truss subject to the stress constraint

$$-\sigma_C \le \sigma \le \sigma_T \tag{1}$$

where σ is the longitudinal stress in any truss member, σ_C is the permissible stress in compression and σ_T is the permissible stress in tension. Assuming that all truss members develop one of the permissible stresses (see end of Section 2.1 in Lecture 2), the total truss volume V can be expressed as

$$V = -\sum_{\mathrm{C}} L_i F_i / \sigma_C + \sum_{\mathrm{T}} L_i F_i / \sigma_T \tag{2}$$

where L_i and F_i are the length of and the force in the member i, and the summation over C and T refer to all the compression and tension members, respectively. A Michell truss may have an infinite number of members.

Note: In Michell's theory, neither upper nor lower limits are imposed on the cross-sectional areas, which vary continuously.

In current notation Michell's optimality criterion can be summarized as follows.

$$\bar{\varepsilon} = k \,\mathrm{sgn}\, F \quad (\text{for } F \ne 0), \quad |\bar{\varepsilon}| \le k \quad (\text{for } F = 0) \tag{3}$$

where $\bar{\varepsilon}$ is a (small) kinematically admissible strain and k is a positive constant

The member forces $\boldsymbol{F}$ must be statically admissible.

The term 'kinematically admissible' implies that the considered strain field (or the corresponding displacement field) satisfies kinematic continuity (compatibility) and boundary (support) conditions. 'Statically admissible' means that the considered forces/stresses satisfy equilibrium and static boundary conditions.

Michell (1904) used the symbol ε for the limit on reference strain (instead of k in the relation (3) above).

5.1. Limitations of Michell's Optimality Criteria

After re-examining Michell's proof, the author concluded (Rozvany 1996a) that Michell's original optimality criteria are valid, if one or both of the following two *sufficient* conditions are satisfied:

(a) *the structural domain is subject to prescribed external forces only, or equivalently, to 'statically determinate supports'* (supports in which the reactions can be uniquely determined from equilibrium, i. e. they do not depend on the choice of a stable topology, nor on the cross-sectional areas within a given topology), *or*

(b) *the permissible stresses in tension and compression are the same.*

These limitations will be illustrated with examples in Section 8.

5.2. Researchers' Response to Michell's (1904) Paper

Michell's pioneering effort was completely ignored for over half a century, after which it was re-discovered by Cox in 1958 (Cox 1958, 1965), who applied Michell's criteria to a few new, but simple load conditions. Following this, Owen (1965) also reviewed briefly Michell's work in his book. But a detailed investigation of this topic is due to Hemp, who (with important contributions by his co-workers, ASL Chan, HSY Chan and McConell) studied Michell structures most of his professional life. Valuable results by him and his research associates were summarized in his book (Hemp 1973).

6. Extension of Michell's Theory to Statically Indeterminate Support Conditions and Unequal Permissible Stresses

If neither of the conditions (a) or (b) under Section 5.1 are satisfied, then Michell's optimality criteria are not valid, and must be modified. Such more general optimality criteria were stated by Hemp (1973), and also follow readily from the Prager-Shield (1963) conditions, which was used for the optimal layout theory by Prager and Rozvany (1977a), discussed in Lecture 2, Section 2.

For different permissible stresses in tension and compression, and statically indeterminate support conditions, one must use the following optimality criteria.

$$\begin{aligned} &\bar{\varepsilon} = -1/\sigma_C \text{ (for } F < 0\text{)}, \ \bar{\varepsilon} = 1/\sigma_T \text{ (for } F > 0\text{)},\\ &-1/\sigma_C \le \bar{\varepsilon} \le 1/\sigma_T \ \text{ (for } F = 0\text{)}, \end{aligned} \tag{4}$$

in which $\bar{\varepsilon}$ represents a fictitious strain-field termed 'adjoint' strain field. We sometimes use the shorter notation

$$1/\sigma_C = k_C, \ 1/\sigma_T = k_T \tag{5}$$

For $\sigma_T = \sigma_C = \sigma_p$ we have

$$k_T = k_C = k = 1/\sigma_p \tag{6}$$

and then (4) reduces to (3).
Note: Relations (3) or (4) are *necessary and sufficient conditions* of optimality for the relevant class of problems if a feasible solution exists, the adjoint strains $\bar{\varepsilon}$ are kinematically admissible, and the member forces F statically admissible (i. e. satisfying equilibrium). This is because Michell's original problem, and its extension in (4) are convex problems.

Moreover, for equal permissible stresses in tension and compression Michell's problem is self-adjoint i. e. the 'real' elastic strains and the adjoint strains are linearly proportional.

It is to be remarked that during his post-doctoral work with Hemp in Oxford around 1970, the first author used the Prager-Shield (1967) conditions for all topology problems, which of course clearly imply the extended optimality conditions in (4).

The structural volume is given by the sum of products of cross sectional areas A_i and member length L_i, divided by the appropriate permissible stress (primal formulation), as in (2), or from the 'dual formula',

$$V = \mathbf{P}^T \, \overline{\mathbf{\Delta}} \tag{7}$$

where the vector $\mathbf{P}$ denotes the external forces and $\overline{\mathbf{\Delta}}$ the adjoint displacements (given by the adjoint stain field $\bar{\varepsilon}$) at points of application of these forces.

7. Review of the Author's Paper on the 'Shortcomings' of Michell's Theory (Rozvany 1996)

In the above paper, the author (Rozvany 1996)

(i) pointed out the limitations of Michell's (1904) proof,

(ii) stated the range of validity of Michell's original criteria for unequal permissible stresses,
(iii) gave a simple example to show that the corrected optimality criteria in (4) result in a lower volume than the original criteria by Michell (1904), see Figs 2b and 3b later, and
(iv) pointed out the examples in Michell's paper, which in his opinion have a non-optimal topology (see Section 10 herein).

In his paper (Rozvany 1996), the author presented three proofs of the extended optimality criteria in (4). These proofs were based on

(a) the Prager-Shield (1967) condition (see Lecture 2, Section 2.2),
(b) the Karush-Kuhn-Tucker (KKT) condition,
(c) the principle of virtual work.

In his book, Hemp (1973) stated the more general optimality conditions (see (4) above), but did not present any examples for which these give an optimal topology that is different from those given by Michell's more restricted criteria. He did not point out either, which ones of Michell's examples represent a non-optimal solution for unequal permissible stresses, nor the part of Michell's proof that renders his criteria of more restricted validity. Possibly, Hemp carefully avoided all statements, which could tarnish Michell's well-deserved historical image. His gentlemanly attitude was also obvious when he reviewed the author's 1996 paper, and suggested tactful formulations (e.g. 'shortcomings' instead of 'errors').

8. Examples Illustrating the Limited Range of Validity of Michell's Optimality Criteria.

8.1. Statically Determinate Support Conditions, Equal Permissible Stresses

In Fig. 2a, we show an example, in which the support conditions (a pin and a roller) are statically determinate, i. e. equilibrium uniquely determines the reactions in them, irrespective of any (statically admissible) topology of the truss. This means that for the load ***P*** we always have the reactions $|\mathbf{Q}| = |\mathbf{P}|$ and $|\mathbf{R}| = \sqrt{2}\ |\mathbf{P}|$.

From the viewpoint of truss topology optimization, this is completely equivalent to having the given external loads ***Q***, ***R*** and ***P***, as in Michell's original formulation. Taking first the special case $\sigma_T = \sigma_C = \sigma_p$ implying $k_T = k_C = k$, in Fig. 2a we have $\bar{\varepsilon}_\theta = -\bar{\varepsilon}_r = k$, and then the quarter circular fan with equal adjoint strains in both radial and circumferential directions clearly satisfies kinematic admissibility with

$$\bar{u} = \bar{v} = 0 \ \text{ at point B, } \ \bar{u} = 0 \ \text{ at point A} \tag{8}$$

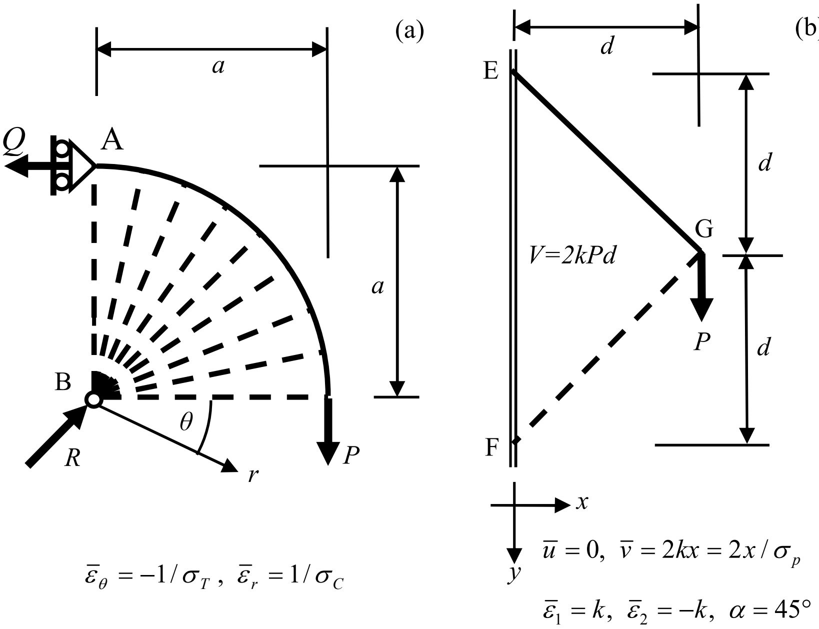

Figure 2. Examples illustrating the limited range of validity of Michell's optimality criteria.

where $\bar{u}$ and $\bar{v}$ are the adjoint displacements in directions x and y.

Using the primal formulation in (2), it is known (e.g. Rozvany et al 1997) that for any circular fan (see Fig. 3a), the sum of the product of forces and member lengths for both (i) the circular chord and (ii) the radial members is the same, and equals

$$\sum_i F_i L_i = Fr\theta \tag{9}$$

where F is the constant force in the circular chord, r is the radius of the fan and θ is the central angle of the fan. In our case $F = P$, $r = a$ and $\theta = \pi / 2$, so the expression in (9) reduces to $\pi PL/2$. Then by (2) we get a truss volume of

$$V = k\pi Pa = \pi Pa / \sigma_p \tag{10}$$

Using the dual formulation in (7), it can be shown (e.g. Rozvany et al 1997) that the relative circumferential displacement at the end of the circular bar of a fan is (Fig. 3a)

$$\overline{\Delta} = r\theta(\bar{\varepsilon}_\theta - \bar{\varepsilon}_r) \tag{11}$$

Substituting $\varepsilon_\theta = k,\ \varepsilon_r = -k,\ r = a$ and $\theta = \pi/2$ into (11), we find that $P\overline{\Delta}$ gives the same truss volume as the primal formulation.
Note: In the author's lectures, thick continuous lines denote tension members, and thick broken lines compression members.

8.2. Statically Determinate Support Conditions, Unequal Permissible Stresses

Considering again the problem in Fig. 2a, assign the values $\sigma_C = \sigma_T/3$ to the permissible stresses. With the adjoint strains given by (4), we have

$$\bar{\varepsilon}_\theta = 1/\sigma_T,\ \bar{\varepsilon}_r = 1/\sigma_C = 3/\sigma_T \tag{12}$$

The above adjoint strains are kinematically admissible (see Fig. 2a). Then from *primal formulation* by (2) with (9)

$$V = \pi Pa(1/\sigma_T + 1/\sigma_C)/2 = \pi Pa(1+3)/2\sigma_{\mathrm{T}} = 2\pi Pa/\sigma_T \tag{13}$$

Using *dual formulation*, we have by (4), and Figs 2a and 3a

$$\bar{\varepsilon}_\theta = 1/\sigma_T,\ \bar{\varepsilon}_r = -3/\sigma_T,\ r = a \ \text{ and } \ \theta = \pi/2 \tag{14}$$

and then (7) and (11) confirm the volume value in (13).
Note: For the statically determinate support conditions of this example, Michell's optimality criteria in (3) would also give the correct truss topology. However, for calculating the volume from the dual formula in (7), one would have to take $k = 1$ in (3), and then multiply the result by $(1/\sigma_T + 1/\sigma_C)/2$. Similar formulae are used in the examples of Michell (1904). However, this 'dual' method is rather cumbersome, and is only valid if the sum of products of bar forces and bar lengths is equal for compression and tension bars:

$$-\sum_C L_i F_i = \sum_T L_i F_i \tag{15}$$

which was actually pointed out to the author by Mariano Vázquez Espi. Clearly, a great advantage of the extended optimal criteria in (4) is, that the corresponding dual formula in (7) is valid even if the restriction (15) is not satisfied.

8.3. Statically Indeterminate Support Conditions, Equal Permissible Stresses

Considering the problem in Fig. 2b, the external load P can be transmitted by a truss to any arbitrary points of the rigid line support EF. This support condition is statically indeterminate, because the location, direction and magnitude of the reaction forces depend on the layout of the truss. A similar problem was considered by Prager and Rozvany (1977a, Fig. 1 in that paper).

The adjoint displacement field in Fig. 1b gives principal adjoint strains of

$$\bar{\varepsilon}_1 = -\bar{\varepsilon}_2 = k \tag{16}$$

at $\pm 45°$ to the axis x. This strain field is kinematically admissible, because it satisfies

$$\bar{u} = \bar{v} = 0 \tag{17}$$

along the rigid line support EF. It also satisfies the inequality in (3), because strains in non-principal directions cannot have greater absolute value than the principal strains in (16).

For the result in Fig. 2b, the primal volume-formula in (2) with $\sigma_T = \sigma_C = \sigma_p$ gives

$$V = 2\sqrt{2}d\,(P/\sqrt{2})/\sigma_p = 2Pd/\sigma_p \tag{18}$$

For the dual formula in (7) the adjoint displacement at the point G in Fig. 2b is (by the framed relations in Fig. 2b)

$$\left|\bar{\Delta}\right| = \bar{v}_G = 2d/\sigma_p, \tag{19}$$

and then (7) confirms the result in (18).

8.4. Statically Indeterminate Support Conditions, Unequal Permissible Stresses

We consider the problem in Fig. 2b, but with unequal permissible stresses in tension and compression $\sigma_C = \sigma_T/3$.

For that problem, Michell's original optimality criteria would give the layout in Fig. 2b, but with the above permissible stresses the volume would then become

$$V = \sqrt{2}\,d(1/\sigma_T + 3/\sigma_T)P/\sqrt{2} = 4d\,P/\sigma_T \tag{20}$$

On the other hand, the solution based on the extended optimality criteria in (4), shown in Fig. 3b, gives the following volume by primal formulation

$$V = 2d(\sqrt{3}P/2)/\sigma_T + (2d/\sqrt{3})(P/2)(3/\sigma_T) = 2\sqrt{3}/\sigma_T \approx 3.4641d\,P/\sigma_T \tag{21}$$

which is about 13.4 per cent smaller than the volume of the solution based on Michell's criteria. The dual formula in (7), with $\left|\bar{\boldsymbol{\Delta}}\right| = \bar{v}_G = 2\sqrt{3}d/\sigma_T$ confirms the result in (21).

This shows the significance of using the modified criteria for statically indeterminate support conditions and unequal permissible stresses. The above (counter) example was originally presented by Rozvany (1996).

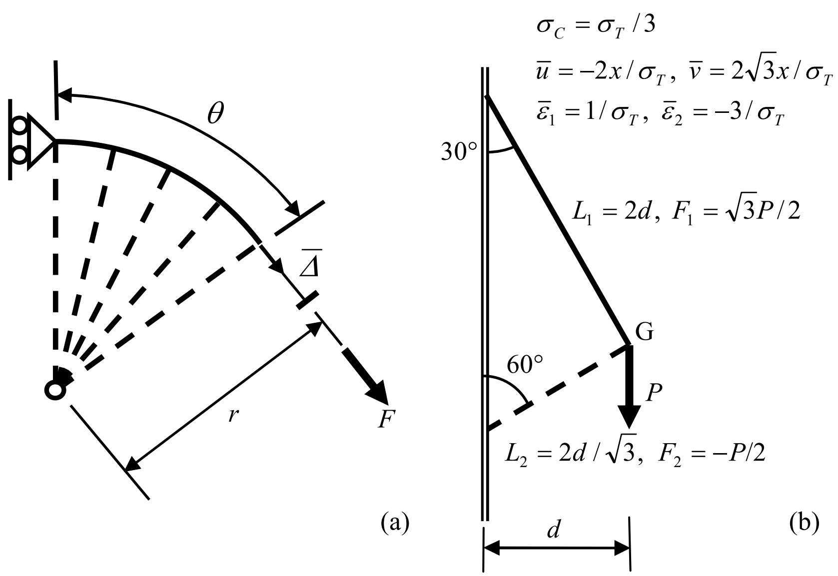

Figure 3. (a) Adjoint displacement for a circular fan, (b) The correct solution of the problem in Fig. 2b for unequal permissible stresses

9. The Optimal Orientation of Truss Members at a Point of a Line Support

Considering a point A of a line support (Fig. 4a), by (4) the adjoint strains along two truss members are $\overline{\varepsilon}_1 = 1/\sigma_T$ and $\overline{\varepsilon}_2 = -1/\sigma_C$, which are also principal strains due to the inequality in (4).

In a Mohr-circle representing the adjoint strains at point A, we have (Fig. 4b)

$$r = (1/\sigma_T + 1/\sigma_C)/2,\ \ a = (1/\sigma_C - 1/\sigma_T)/2 \tag{22}$$

$$\cos(2\kappa) = a/r \tag{23}$$

Then elementary transformations imply

$$\kappa = \frac{1}{2}\arccos\frac{\sigma_T - \sigma_C}{\sigma_T + \sigma_C} \tag{24}$$

For $\sigma_T = \sigma_C = \sigma_p$ (24) gives $\kappa = 45°$, and for $\sigma_C = \sigma_T / 3$ we get $\kappa = 30°$, providing an independent confirmation of the results in Figs 2b and 3b.

The optimal orientation of members along line supports was derived previously (Rozvany 1996) from Hemp's (1973) equations, but the above derivation is much simpler and easier to understand for non-specialists.

10. Critical Re-examination of the First Example by Michell

For a point load and 'forces equivalent to a force…and a couple…over a small circle' Michell (1904) presents the solution in Fig. 4c (here redrawn, after Rozvany 1996). For the optimal volume Michell gives (in our notation)

$$V = Fa \ln\frac{a}{r_0}\left(\frac{1}{\sigma_T} + \frac{1}{\sigma_C}\right) \tag{25}$$

In our Fig. 4c, we show a circular support, but Michell refers to distributed forces along a circle.

It is unfortunate that Michell is not with us to clarify his results, but this author has the following difficulties in accepting the solution in Fig. 4c and (25).

(a) The relation (25) clearly implies that the above solution is meant to be also for different permissible stresses in tension and compression. For that case, Michell's optimality criteria in (3) are *restricted to given external forces* (or equivalently, statically determinate support conditions). For the problem in Fig. 4c, the optimal distribution of the forces along the circle with the radius r_0 depends on the ratio a / r_0, so *these forces are not given*. This means that we are choosing the forces along the circle optimally, which is *equivalent to optimizing for a circular line support*. In that case however, we have to use the extended optimality criteria in (4).

(b) The solution in Fig. 4c fails to satisfy the condition (24) for optimal member directions along line supports, so it cannot be optimal

(c) The problem in Figs 2b and 3b is the special (limiting) case of the problem in Fig. 4c, with $r_0 \to \infty$. For this special case it was shown that the extended optimality criteria in (4) give a lower volume than Michell's original criteria.

(d) For the problem in Fig. 4c, Dewhurst and Srithongchai (2005) obtained a lower volume than Michell's solution, when they used the optimal member angles of 30° and 60° given by the extended optimality criteria.

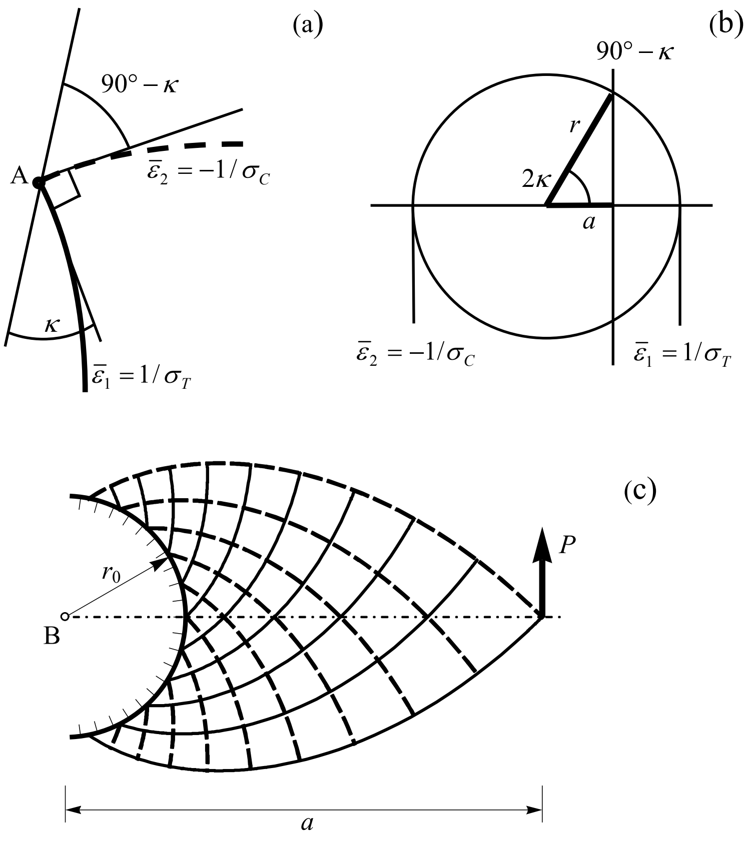

Figure 4. (a and b) Optimal directions of truss members at line supports, (c) Michell's first example

Note: Michell's solution is completely correct for equal permissible stresses $\sigma_T = \sigma_C = \sigma_p$. In his first example (Fig. 4c herein), he starts off with having a concentrated couple and force at point B in Fig. 4c, which is equivalent to $r_0 \to 0$. For that case, (25) gives an infinite volume. A minimization problem giving an infinite objective function value does not make much sense, because this would imply that any other feasible solution is equally optimal.

Michell's second, third and fourth examples are correct, because they are for given external forces (equivalent to statically determinate support conditions). His fifth example has the same problem as the one discussed above for the first example.

11. Concluding Remarks

(i) Exact analytical benchmarks are extremely important in structural topology optimization for checking on the validity, efficiency and convergence of *numerical* methods, which have many sources of error.
(ii) Michell's (1904) landmark paper is of pivotal importance to this field. However, it is beneficial to any theory to determine its range of validity, and to derive its extension to a much broader class of problems. This has been attempted in this lecture. Further generalizations of Michell's theory will be presented in the next lecture.

References are listed after the last lecture of the author.

Structural Topology Optimization (STO) – Exact Analytical Solutions: Part II

George I. N. Rozvany[1] and Erika Pinter[1]

[1] Department of Structural Mechanics, Budapest University of Technology and Economics, Budapest, Hungary

1. Introduction

In this lecture we review the origin and basic features of the optimal layout theory (Prager and Rozvany 1977a), including optimal regions and difficulties with so-called O-regions. Then we discuss various extensions of this theory, including multiple load conditions and multiple design constraints, probabilistic design and pre-existing members.

2. Optimal Layout Theory (Prager and Rozvany 1977a)

2.1. Preliminaries

It has been shown (e.g. Drucker, Greenberg and Prager 1951), that for elastic-ideal plastic structures a lower bound on the collapse load is given by any statically admissible stress field, which nowhere violates the yield condition. 'Statically admissible' means that a stress field satisfies static boundary conditions and equilibrium. A design based on the above 'lower bound theorem' is termed 'plastic lower bound design'. It need not take kinematic (compatibility) conditions into consideration.

In the problem in Fig. 5, we have a beam with equal positive and negative yield moments, which are constant over the entire beam length. The moment diagram shown in thick line is statically admissible, but kinematically inadmissible for a linearly elastic beam, violating kinematic boundary conditions. Yet in 'plastic lower bound design' it can be used, because the bending moments nowhere exceed the yield moments.

Sved (1954) has shown that optimal (least-weight) elastic trusses under a stress constraint and a single load condition are always 'statically determinate' (or a convex combination of statically determinate solutions). This term means that (i) there are no 'redundant' members in the structure (a 'redundant member' can be removed without making the structure unstable), (ii) its internal forces can be computed on the basis of

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static equations only, ignoring kinematic (compatibility) requirements, and therefore (iii) the internal forces do not depend on member sizes.

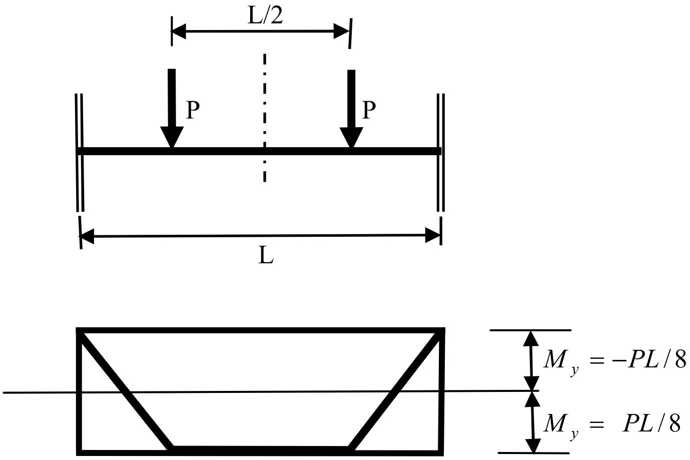


Figure 5. Example of using the lower bound theorem of plastic design.

Barta (1957) extended Sved's theorem on statical determinacy to trusses with local buckling and McKeown (1974, 1997) to any combination of stress and displacement constraints under a single load. (Pedersen (1969, 1970) obtained more rigorous proofs of Barta's (1957) theorem, and extended the statical determinacy property to trusses with variable support conditions (Pedersen 1992, 1993). The equivalent of Sved's theorem was also proved rigorously by Achtziger (1997), in a comprehensive review on optimization of discrete structures.

We may add that according to Wasiutynski and Brandt (1963), the first proof of statical determinacy of single-load trusses, expressed as the 'theorem on the non-existence of statically indeterminate lattices of uniform strength' (meaning fully stressed trusses), is due to Lévy (1873).

Since the optimal solution for certain classes of redundant structures (e.g. trusses, grillages, rigid frames) with one load condition is statically determinate, we can enlarge our feasible set for optimal elastic design (with static and kinematic admissibility) to include all statically admissible solutions (also those violating kinematic admissibility).This is because by the above theorems the final solution will be statically determinate, and that automatically satisfies elastic compatibility. In other words, the optimal solution for the enlarged feasible set will always be contained in the original, smaller feasible set. This means that optimal elastic design of certain structures with one load condition reduces to the optimal plastic design of these structures. Moreover, a stress-based design of a statically determinate truss can only be optimal, if all members develop the permissible stress.

2.2. Basic Features of the Optimal Layout Theory

This theory was already introduced in effect in the author's first book (Rozvany 1976) for flexural systems (e.g. grillages, shells), and formulated in a more concise form later (Prager and Rozvany 1977a). It is based on the Prager-Shield (1967) optimality condition for plastic design. However, the new element in the layout theory is, that it also gives optimality conditions for 'vanishing' members (of zero cross-sectional area) in terms of 'adjoint' strains along these members. In other words, optimal layout theory starts off with a ground structure or structural universe of all potential members, and selects the optimal members (of non-zero cross-sectional area) out of those.

Using either one of the above theories, we need to find

(i) a kinematically admissible 'adjoint' displacement/strain field (satisfying kinematic continuity and boundary conditions),
(ii) a statically admissible stress field (satisfying equilibrium and statical boundary conditions),
(iii) such that certain ('static-kinematic') optimality criteria are also fulfilled.

In these optimality criteria, the adjoint strains are given by the subgradient of the 'specific cost function' with respect to stresses or stress resultants (e.g. member forces F or beam moments M).

The subgradient of a function is the usual gradient, but at discontinuities of the gradient any convex combination of the adjacent gradients can be taken.
For sign-independent, stress-based design of trusses and grillages of given depth, for example, the 'specific cost functions' (representing in this text cross sectional areas A) are

$$A = k|F| \quad \text{and} \quad A = k|M| \tag{26}$$

where k is a constant, F is a member force, and M is a bending moment.

Then, e.g. for trusses, the optimality conditions reduce to those of Michell (1904)

$$\bar{\varepsilon} = k \operatorname{sgn} F \quad (\text{for } F \neq 0), \qquad |\bar{\varepsilon}| \leq k \quad (\text{for } F = 0) \tag{27}$$

where $\bar{\varepsilon}$ is the adjoint strain.

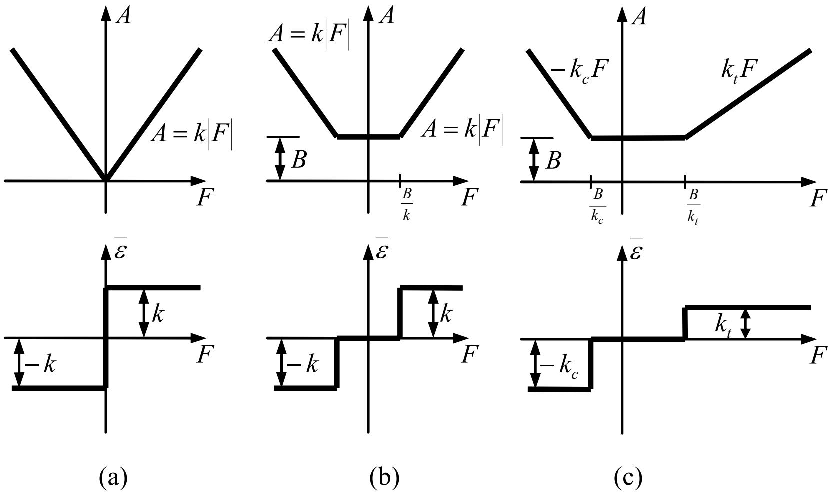


Figure 6. Examples of specific cost functions and the corresponding adjoint strains.

For grillages, we have

$$\bar{\kappa} = k \operatorname{sgn} M \quad (\text{for } M \neq 0), \quad |\bar{\kappa}| \leq k \quad (\text{for } M = 0) \tag{28}$$

where $\bar{\kappa} = -\bar{u}''$ is the adjoint beam curvature, and $\bar{u}$ is the adjoint beam deflection.

The specific cost function and adjoint strain values for Michell trusses are shown in Fig. 6a. These relations are extended to trusses with a prescribed minimum cross section, and, respectively, equal and unequal permissible stresses in tension and compression, in Figs. 6b and c.

2.3. A Simple Illustrative Example Using the Prager-Shield Condition

Consider a clamped beam of constant depth and variable width with a central point load (Fig. 7a). Then by (28) the adjoint curvatures (i.e. second derivatives of the adjoint beam deflections) for positive and negative moments, respectively, are $\bar{\kappa} = k$ and $\bar{\kappa} = -k$.

For any system of downward forces on the beam, the sign of the moment diagram (M) may only change at two places. The simplest of such loads, a single point load is considered in Fig. 7. For two negative and one positive segments of the moment diagram one gets the adjoint curvatures in Fig. 7b. For our simple load, this gives the moment diagram in Fig. 7c, but the zero moment points would be the same for most downward loads.

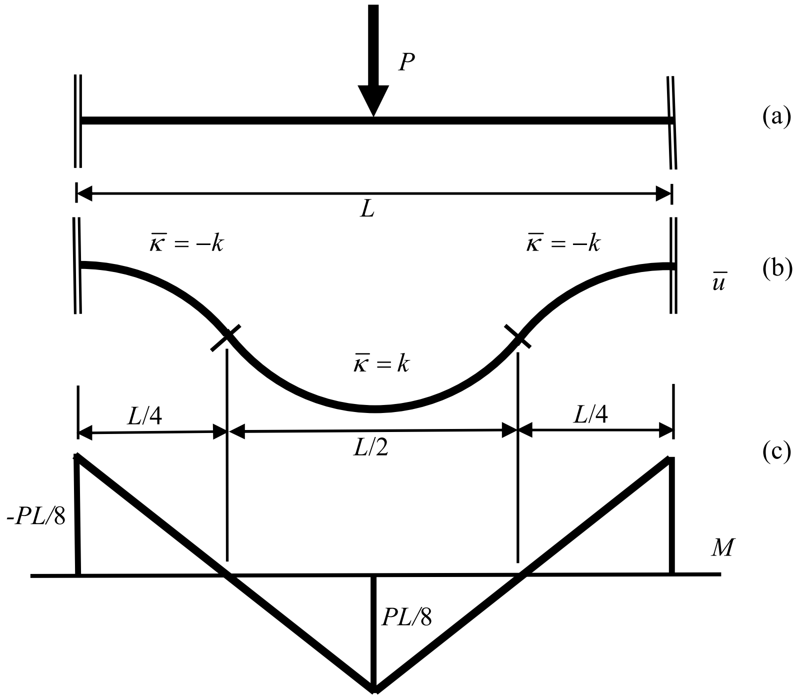


Figure 7. Example of applying the Prager-Shield optimality condition (problem of Heyman, 1959).

The general feature of earlier applications of the Prager-Shield condition was that no members or components disappeared from the structure. This condition was often applied to plates or shells with several stress components (see the author's first book, Rozvany 1976). In layout optimization, however, most of the original members vanish from the ground structure, but optimality criteria (usually inequalities) must be satisfied along vanishing members also.

2.4. Optimal 'Regions' of the Adjoint Strain Field for Michell Structures

It follows from (27) that in Michell trusses, the members are in the direction of the adjoint principal strains of a constant magnitude (k), and the adjoint strains may nowhere exceed this value. The optimal topology usually consists of several 'regions'. Denoting the principal adjoint strains by $(\bar{\varepsilon}_1, \bar{\varepsilon}_2)$, at any point of a 2D truss, *for equal permissible stresses in tension and compression* we may have a

T-region with a tensile and a compression member at right angles, $\bar{\varepsilon}_1 = -\bar{\varepsilon}_2 = k$,
S-region with members having forces of the same sign in any direction, $\bar{\varepsilon}_1 = \bar{\varepsilon}_2,\ \left|\varepsilon_i\right| = k\ \ (i = 1{,}2)$,
R-regions with only one member at any point, $\left|\bar{\varepsilon}_1\right| = k,\ \ \left|\bar{\varepsilon}_2\right| \le k$, or
O-region with no members $\left|\bar{\varepsilon}_1\right| \le k,\ \ \left|\bar{\varepsilon}_2\right| \le k$, with $k = 1/\sigma_p$, where σ_p is the permissible stress in both tension and compression.

The symbols used for optimal regions are shown in Fig. 8a.
For unequal permissible stresses in tension and compression, we have

for **T**-regions $\bar{\varepsilon}_1 = k_T,\ \bar{\varepsilon}_2 = -k_C$,
for **R**-regions $\bar{\varepsilon}_1 = k_T$, or $\bar{\varepsilon}_1 = -k_C$, *and* $k_T \ge \bar{\varepsilon}_2 \ge -k_C$ with $1/\sigma_C = k_C$, $1/\sigma_T = k_T$.

Depending on the sign of the forces, **S** and **R** regions may be further subdivided into $\mathrm{S}^+, \mathrm{S}^-, \mathrm{R}^+$ or R^- regions.
Note: Most of the Michell literature deals with T-regions, because of the link with slip-lines in plasticity (Hencky-Prandtl nets), which was developed earlier.

Some new types of regions for Michell structures will be introduced in Section 5.

Optimal regions were also derived for 2D perforated and composite continua by Rozvany, Olhoff, Bendsoe, Ong and Szeto (1987), Ong, Rozvany and Szeto (1988) and Szeto (1989), for a review see Rozvany, Bendsoe and Kirsch 1995 or a book by Rozvany (1989). The optimal regions for 3D perforated continua were presented in an outstanding paper by Olhoff, Rønholt and Scheel (1998).

2.5. A Simple Example of Applying the Prager-Rozvany (1977a) Layout Theory to a Michell-Type Problem

In Fig. 8b we show some optimal topologies and in Fig. 8c the corresponding optimal adjoint displacement fields. There are two line supports at right angles. We use a normalized formulation with $k = 1$.

Elementary calculations show that for the adjoint displacements $\bar{u}$ and $\bar{v}$ in the upper region we have the principal adjoint strains

$$\bar{\varepsilon}_1 = -\bar{\varepsilon}_2 = 1 \tag{29}$$

at $\pm 45°$ to the horizontal. This corresponds to a T-region, denoted by an arrow cross in Fig. 8c. In the lower region we have

$$\bar{\varepsilon}_1 = 0,\ \ \bar{\varepsilon}_2 = -1 \tag{30}$$

which signifies an R^- region denoted by a double arrow. Both *satisfy the optimality conditions* in (27).

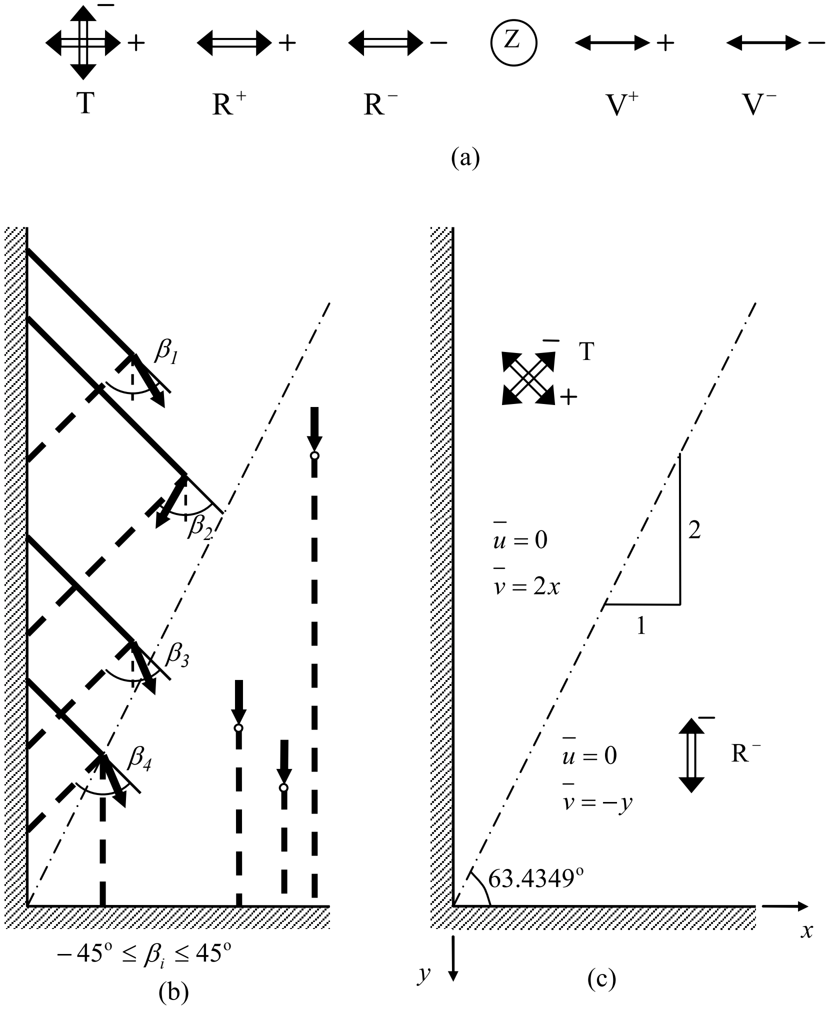


Figure 8. (a) Symbols used for optimal regions. (b,c) Example of application of optimal layout theory to Michell trusses.

It can be readily seen that along the line supports with $x = 0$ and $y = 0$ we have $\overline{u} = \overline{v} = 0$, satisfying the *kinematic boundary conditions*. Moreover, along the region boundary with $y = -2x$ both adjoint displacement fields in Fig. 8c give

$$\bar{u} = 0,\ \bar{v} = -2x \tag{31}$$

satisfying *kinematic continuity*.

It can be seen from Fig. 8b that in the upper region we have two-bar trusses with one tension and one compression bar, in the lower region we have single bar trusses in compression. If the load is acting on the region boundary, we may have any convex combination of a two bar truss and a single bar truss.

3. Optimal Topologies for Least/Weight Grillages

The theory of the optimal topology of grillages (beam systems) actually preceded the development of the optimal layout theory (Prager and Rozvany 1977a). The grillage theory (e.g. Rozvany 1972a and b) was developed by the author's research group, but it has some elements of Morley's (1966) theory for optimal reinforcement in concrete slabs. Reviews of the grillage theory may be found in several texts (e.g. Rozvany and Hill 1976, Prager and Rozvany 1977b, and Rozvany, Bendsoe and Kirsch 1995).

Exact optimal grillage topologies should actually be used more often as benchmarks, because they have the following advantages.

(i) The optimal grillage theory has advanced much further than the truss theory, because exact optimal grillage topologies are available for almost all possible support and load conditions, and these analytical solutions can even be generated by computer (see e.g. Rozvany, Bendsoe and Kirsch 1995, Fig. 38). A complex optimal grillage topology is shown on the cover of the author's first book (Rozvany 1976), see Fig. 9 herein.

(ii) Michell trusses ignore buckling, which plays a much more important role for trusses than for grillages.

(iii) Grillages may have three types of boundary conditions (clamped and simply supported boundaries, and free edges, whilst trusses may only have two (support or no support).

(iv) A large number of extensions of the grillage theory exist (see Rozvany 1992b, pp 125-127)

(v) Whilst most Michell trusses are mechanisms, most optimal grillages are stable structures.

Elementary calculations show that for the adjoint displacements $\bar{u}$ and $\bar{v}$ in the upper region we have the principal adjoint strains

$$\bar{\varepsilon}_1 = -\bar{\varepsilon}_2 = 1 \tag{29}$$

at $\pm 45°$ to the horizontal. This corresponds to a T-region, denoted by an arrow cross in Fig. 8c. In the lower region we have

$$\bar{\varepsilon}_1 = 0,\ \ \bar{\varepsilon}_2 = -1 \tag{30}$$

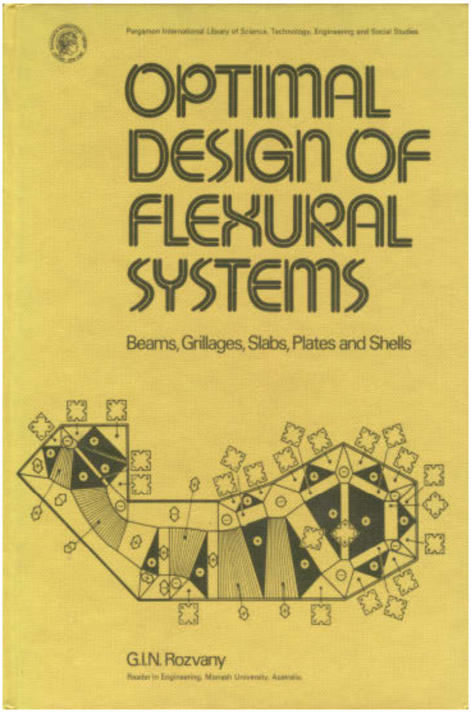


Figure 9. Cover of the author's first book, showing a complex optimal grillage topology.

In spite of these advantages, only Sigmund (Sigmund et. al 1983) has compared numerical and analytical solutions for grillages (and got a very good agreement).

We are not going to review here the grillage theory in detail, because it was discussed at considerable length at another CISM Advanced Course (see Rozvany 1992b, pp. 107-127).

The optimal layout theory was also used for least-weight shell-grids (starting with Rozvany and Prager 1979) including self-weight, for a review see the author's book (Rozvany 1989, pp. 338-341).

4. Cognitive Processes in Deriving Exact Analytical Structural Topologies

The optimal topology for a completely new problem cannot be derived by a deductive process (using logical reasoning). The rough arrangement of optimal regions has to be first 'dreamt up' (Melchers 2005), i.e. assumed by 'intelligent' guessing, or invention, and then checked if such a solution satisfies all the optimality criteria. During this second stage, the exact shape of the region boundaries is also assessed. The first stage requires considerable insight, the second stage a lot of high level mathematical work.

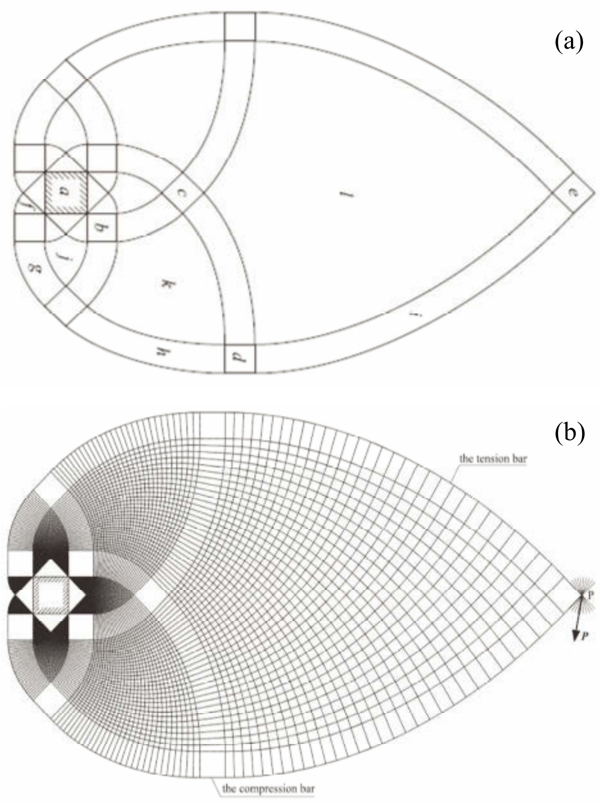


Figure 10. Optimal truss topology for a square support. (a) solution guessed by Rozvany (1991), (b) exact solution by Lewinski (Lewinski and Rozvany 2008).

As an example, for a square support the author guessed the optimal topology fairly accurately in a text by the author (Rozvany 1995, presented 1993 here Fig. 10a, exact reproduction), and this was confirmed by Lewinski fifteen years later (Lewinski and Rozvany 2008) by an extremely long derivation (Fig. 10b).

More recently, however, it has been possible to compute highly accurate numerical solutions e.g. for Michell trusses, using over a billion potential members in the ground structure (Gilbert and Tyas 2003, Sokol 2011a and b, Pichugin, Tyas and Gilbert 2012, see also Lecture 4 in this course). These numerical solutions give a very good general idea about the adjoint displacement field for the exact optimal topology, except for O-regions (regions without members), which are discussed in the next section.

5. Difficulties in Deriving the Adjoint Displacement/Strain Field for O-Regions – Recent Developments

O-Regions were used by the author in some contributions to papers by Lewinski, Zhou and Rozvany (1994b), Rozvany, Gollub and Zhou (1997) and Sokol and Rozvany (2012).
It is explained in the last of these papers that O-Regions may contain the usual R-Regions and T-Regions, but also the following new types of regions **Z**-Regions, in which both principal strains are zero (rigid region with $\bar{\varepsilon}_1 = \bar{\varepsilon}_2 = 0$), and **V**-regions (with $|\bar{\varepsilon}_1| \le k,\ \bar{\varepsilon}_2 = 0$).

Examples of O-regions consisting of T, R, and Z-Regions are given in Fig. 11b, c and d (after Sokol and Rozvany 2012). An example of an O-Region consisting of T and V-Regions is given in Fig. 12. The above diagrams show only one quarter of the domain, axes of symmetry are indicated in dash-dot lines. They are generalizations of Michell's solution shown in Fig. 11a.

The state of adjoint strains in these T and Z-Regions in Fig 12a, respectively, is represented by the Mohr-circles in Figs 12b and c. It can be seen from Fig. 12b that the strain along the boundary of the T- and V-regions is

$$\bar{\varepsilon}_{\mathrm{B}} = k\cos(2\alpha) \tag{32}$$

Moreover, one can infer from Fig. 12c that for the V-region, we have

$$k\cos(2\alpha) = (\bar{\varepsilon}_1/2)(1+\cos(2\alpha)) \Rightarrow \bar{\varepsilon}_1 = \bar{\varepsilon}_y = k(1-\tan^2\alpha) \tag{33}$$

It can be seen that for $\alpha = 0$ and $\alpha = 45°$ (33) gives the correct $\bar{\varepsilon}_1$ values for the limiting cases in Figs 11a and b. It is important to note that *in O-regions the adjoint strain/displacement fields may be non-unique.*

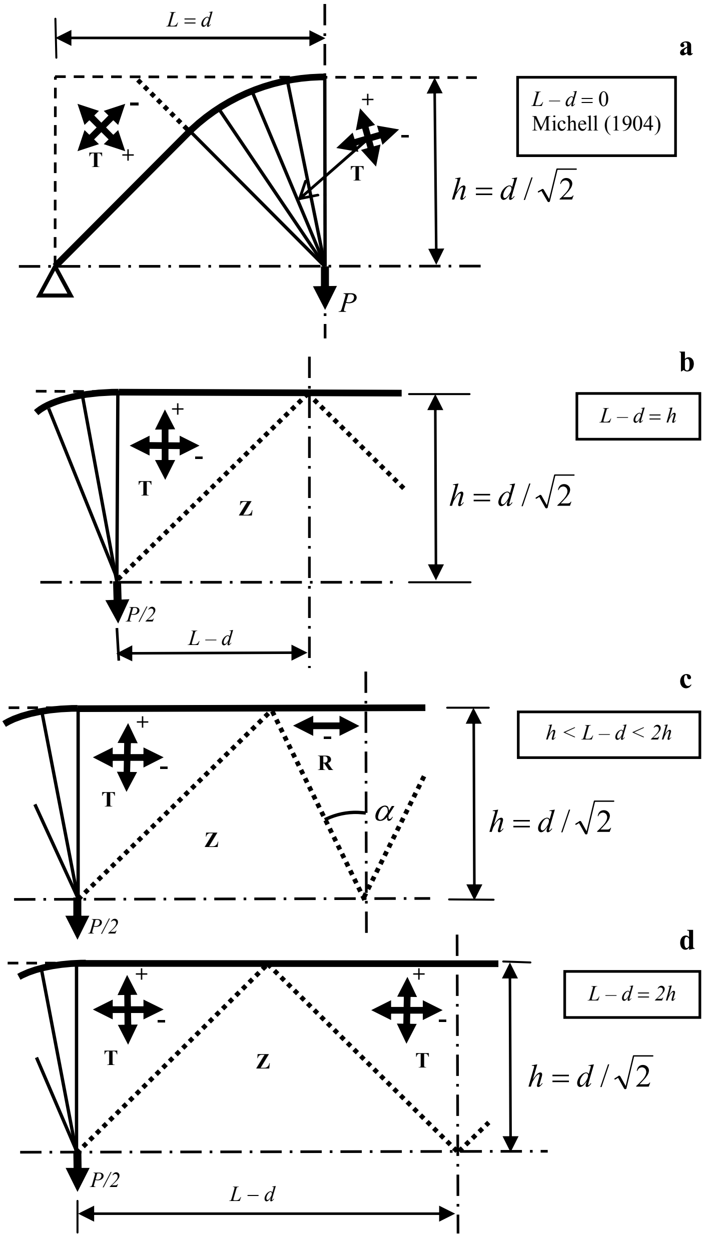


Figure 11. Some O-regions consisting of T, R and Z-regions.

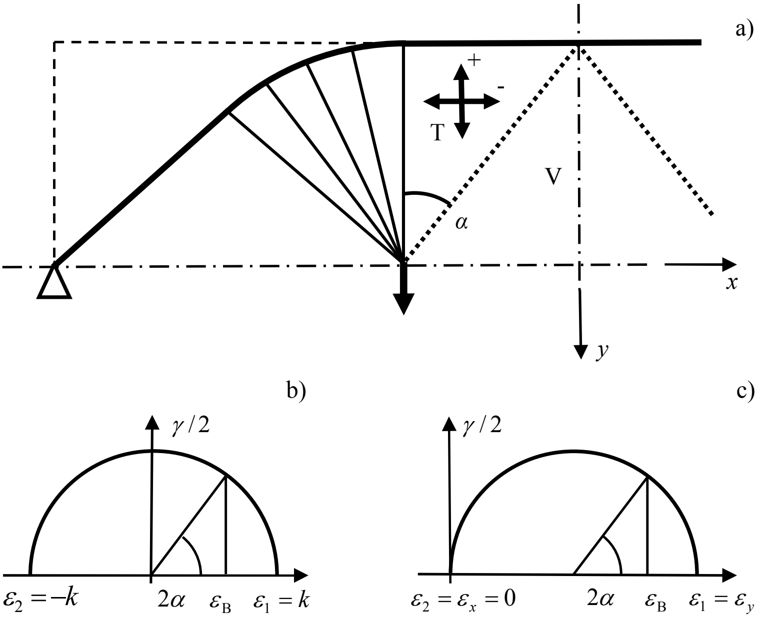


Figure 12. O-Region consisting of T and V-Regions.

A simple example of non-uniqueness of the adjoint strain field is shown in Fig. 13. In this problem we have two vertical line supports at a distance of $3L$ from each other, and a horizontal point load at a distance L from the right hand support. To the permissible stresses the same value is assigned in tension and compression. The optimal layout obviously consists of a single horizontal bar between the load and the nearer support. The principal adjoint strain in the vertical direction is everywhere zero: $\bar{\varepsilon}_y = 0$. In Fig. 13a we have an **R**-region on the right hand side and an **O**-region on the left hand side, where the inequality in (27) admits a horizontal strain of $-k/2$. An alternative, discontinuous adjoint strain field is given in Fig. 13b, in which we have a **Z**-region and an **R**-region on the left, both admissible by (27).

Another new type of region within an O-region is a ***scaled T-region*** termed **T'** region, having the property $\bar{\varepsilon}_1 = -\bar{\varepsilon}_2 = \lambda k$ with $\lambda < 1$ for equal permissible stresses, and $\bar{\varepsilon}_1 = \lambda k_T$, $\bar{\varepsilon}_2 = -\lambda k_C$ for unequal permissible stresses. Fig. 14 shows an optimal topology with a **T'** region, for which we have

$$\lambda = (4\pi/3 + 2\sqrt{3})/(8\pi/3 + 2\sqrt{3}) \tag{34}$$

The solution in Fig. 14 has been verified numerically to a high degree of accuracy by

Sokol (see the paper by Rozvany and Sokol 2012).

Similarly, we have scaled **R**-regions termed **R'**-regions.

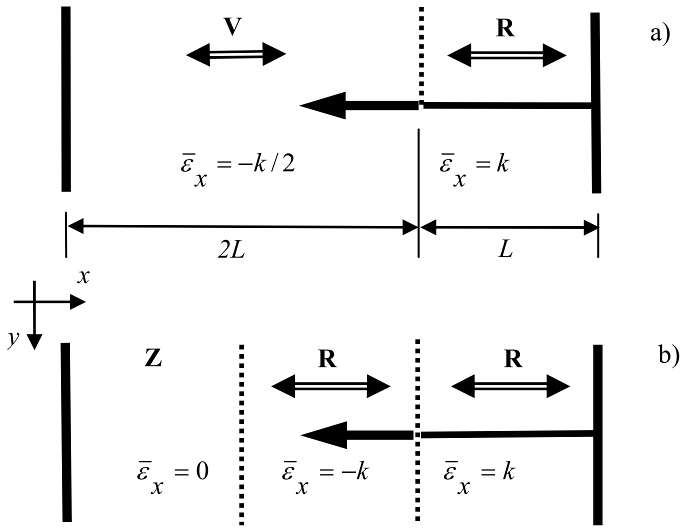


Figure 13. A trivially simple example of non-uniqueness of adjoint strain/displacement fields in O-Regions.

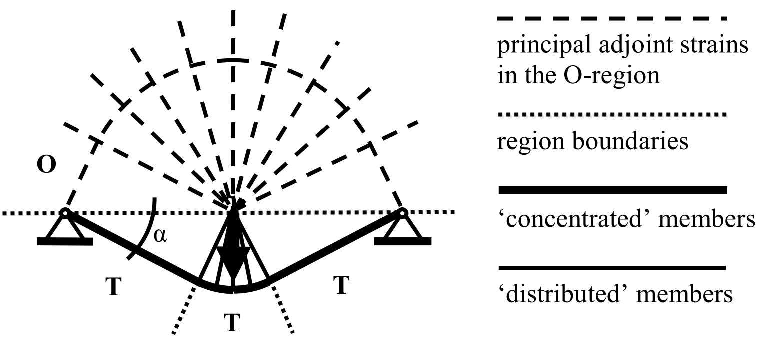


Figure 14. Optimal topology with three **T'**- regions (in the **O**-region)

6. Extension of the Layout Theory to Elastic Design with Multiple Load Conditions

The original optimal layout theory (Prager and Rozvany 1977a) has been extended to many additional classes of problems, including combined stress and displacement constraints (e.g. Rozvany, Birker and Gerdes 1994).

General optimality conditions for trusses with several load conditions and several displacement constraints were derived by the author (Rozvany 1992a). A compliance constraint puts a limit on the sum of the scalar product of external forces and the corresponding displacements for any one of the load conditions (sometimes called the 'worst scenario' constraint). This is a special case of displacement constraint, in which a weighted combination of the displacements at the external loads is constrained, and the weighting factors are the external forces.

For a compliance constraint the author's (Rozvany 1992a) optimality conditions reduce to the following (Rozvany et al. 1993).

$$\begin{aligned} &\varepsilon_{ik} = \frac{F_{ik}}{E_i A_i},\ \bar{\varepsilon}_{ik} = \frac{\nu_k F_{ik}}{E_i A_i},\ A_i = \sqrt{\sum_k (\nu_k F_{ik}^2 / E_i \rho_i)}, \\ &(E_i / \rho_i) \sum_k \nu_k \varepsilon_{ik}^2 = 1 \quad (\text{for } A_i > 0), \\ &(E_i / \rho_i) \sum_k \nu_k \varepsilon_{ik}^2 \le 1 \quad (\text{for } A_i = 0), \end{aligned} \tag{35}$$

where ε_{ik} and $\bar{\varepsilon}_{ik}$ are *kinematically admissible* real (elastic) and adjoint strains in the member i under the load k, ν_k Lagrange multipliers, $F_{ik} = \overline{F}_{ik}$ the real and adjoint member forces, whilst $A_i = E_i = \rho_i$ denote the cross-sectional area, Young's modulus and density, respectively, for the member i.

For the particular case of a vertical line support and of two alternative loading cases, consisting of concentrated forces at $\pm\beta$, the above optimality criteria lead (Rozvany et al. 1993) to an optimal topology of two bars shown in Fig. 15, with a very neat closed form result for the bar angles (top of Fig. 15).

The optimal volume of the truss is given by (Rozvany et al. 1993)

$$V_{opt} = \frac{L^2 P^2}{CE \cos^2 \alpha} \left(\frac{\cos^2 \beta}{\cos^2 \alpha} + \frac{\sin^2 \beta}{\sin^2 \alpha} \right) \tag{36}$$

where L is the horizontal distance of the load from the vertical support, P is the magnitude of the loads, E is Young's modulus and C is the given compliance value.

In addition to global proof via layout theory, in this rather elaborate paper (Rozvany, Zhou and Birker 1993), optimality of the solutions was also checked by the co-authors numerically by

(a) using a dense grid of potential truss members,
(b) optimizing a perforated plate by means of SIMP (Bendsoe 1989, Zhou and Rozvany 1991, Rozvany, Zhou and Birker 1992)
(c) deriving for the given topology (two-bar truss) the optimal angle from the Kuhn-Tucker condition.

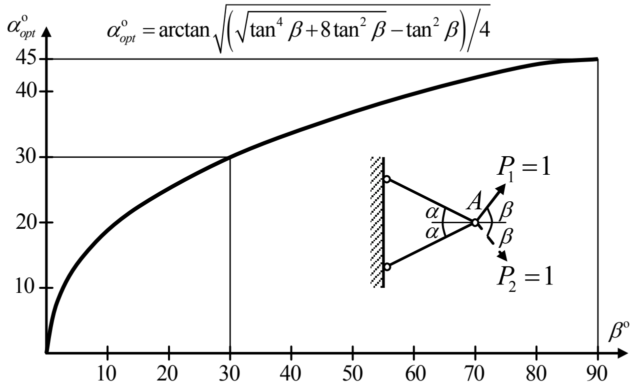


Figure 15. Optimal topology for two alternative loads.

All solutions derived by three different authors have shown a complete agreement. For this reason, “reliability” of the above analytical solutions is fairly high.

7. Extension of the Layout Theory to Probabilistic Design

The aim of reviewing the problem in the last section was to extend the same optimal topology to probabilistic loads. This topic was considered by Rozvany and Maute (2011).

The general form of the considered problem class is as follows.

$$\min V = \sum_i A_i L_i \tag{37}$$

subject to

$$\Pr\left[C \le K\right] \ge R \tag{38}$$

$$C = \sum_i \frac{F_i^2 L_i}{A_i E_i} \tag{39}$$

where V = truss volume, A_i = cross-sectional area of member i, L_i = length of member i, Pr = probability, C = total compliance, K = limiting value of total compliance, R = limiting value of probability, F_i = force in member i and E_i = Young's modulus of member i.

The particular example considered is shown in Fig. 16a. The topology of a truss is to be optimized within the design domain ABDG with supports along AB, subject to a non-random vertical load of V= 300 and a random horizontal load H, with a mean of zero and a normal distribution having a standard deviation of $\sigma = 100$. The truss volume is to be minimized for the conditions in (38)-(39). It will be shown that the solution for the considered problem is a *symmetric* two bar truss (as shown in Fig. 16.b). For simplicity, we assign a unit value to Young's modulus.

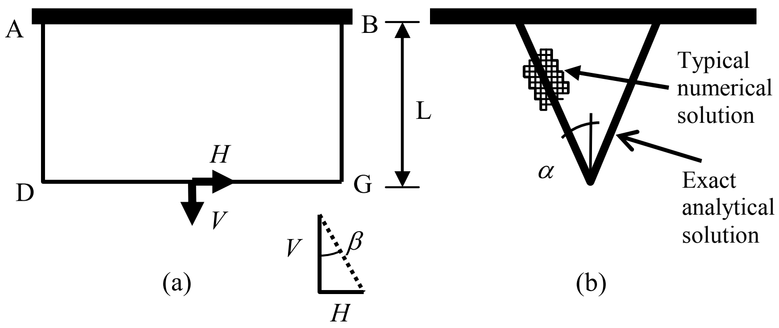


Figure 16. Elementary benchmark example.

Since this problem is symmetric in the only random variable (H), condition (38) is fulfilled if we consider the range of values for the random variable:

$$-H_0 \leq H \leq H_0 \tag{40}$$

where the value H_0 can be calculated from the inverse normal distribution cumulative probability function (also called "quantile" or "probit") function Φ^{-1} and R in (38):

$$H_0 / \sigma = \Phi^{-1}\left(\frac{1+R}{2}\right) \tag{41}$$

In (41), (1+R)/2 is used, since the failure may occur at both ends of the interval of (40). This implies

$$R = \Phi(\mu + H_0/\sigma) - \Phi(\mu - H_0/\sigma) = \\ \Phi(\mu + H_0/\sigma) - (1 - \Phi(\mu + H_0/\sigma)) = 2\Phi(\mu + H_0/\sigma) - 1 \quad (42)$$

from which with $\mu = 0$ follows (41).

For example, if we require a probability of 0.9999 (failure probability of 10^{-4}), then we have $\Phi^{-1}((1+0.9999)/2) = 3.89$ and hence

$$H_0 = 3.89\sigma = 389 \quad (43)$$

in (41).

It can be shown (Rozvany and Maute 2011) that for the considered problem, only $H = \pm H_0$ can be critical for (38) Therefore, once we know the value of H_0, we can calculate the optimal topology and the optimal volume from the relation in Fig, 16 and (36).

The above results were confirmed by Maute both analytically and by a first order reliability approach (FORM) combined with a material distribution method (SIMP, Bendsoe 1989, Zhou and Rozvany 1991, Rozvany, Zhou and Birker 1992).

8. Extension of the Layout Theory to Pre-Existing Members

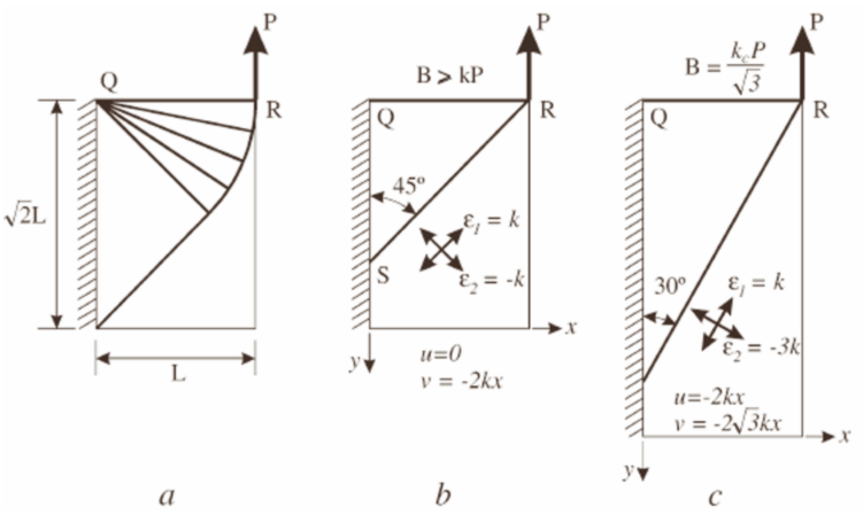


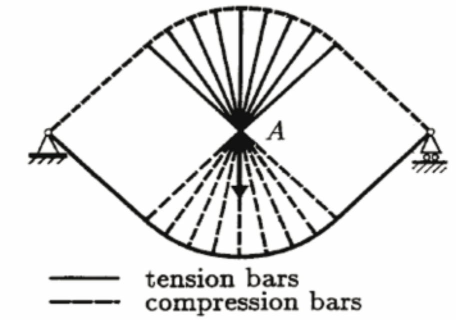
Figure 17. Optimal topology (a) without and (b and c) with pre-existing members.

This problem was discussed in a paper by Rozvany, Querin, Logo and Pomezanski (2006). If we have some already existing members of cross-sectional area ***B***, and want to add new members to satisfy some stress condition, then we have the following optimality criteria for equal permissible stresses (see Fig. 6b)

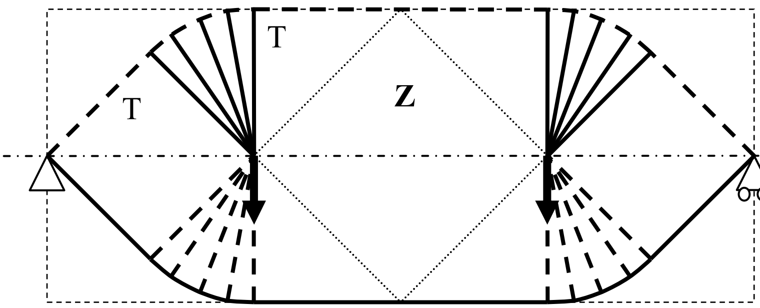
$$\begin{aligned}&(\text{for } |F| < B/k) \quad \bar{\varepsilon} = 0,\\ &(\text{for } |F| > B/k) \quad \bar{\varepsilon} = k \operatorname{sgn} F,\\ &(\text{for } |F| = B/k) \quad 0 \le |\bar{\varepsilon}| = k, \ \operatorname{sgn}\bar{\varepsilon} = \operatorname{sgn} F\end{aligned} \tag{44}$$

For unequal permissible stresses we have similar criteria based on Fig. 6c. Fig. 17a shows the optimal topology without pre-existing members, Figs. 17b and c are for pre-existing members along QR, with equal and unequal permissible stresses, respectively.

These analytical solutions were obtained by the author, with independent numerical confirmations by Querin, Logo and Pomezanski.



(a)



(b)

Figure 18. (a) Michell's (1904) first example, (b) its extension to two point loads (Rozvany 2011a, Sokol and Rozvany 2012).

9. Historic Aspects of Some Recent Developments

It is rather remarkable that the first example of Michell (1904) shown in Fig 18 (top) has not be extended from one point load to two point loads for over a century. This extension was first stated recently (Rozvany 2011a), and discussed in greater detail, also for other aspect ratios a year later (Sokol and Rozvany 2012).

It is even more surprising that Michell's (1904) truss theory has not been extended until this year to several load conditions (stress constraints, elastic design). This will be discussed in a forthcoming paper (Rozvany, Sokol and Pomezanski 2013).

10. Concluding Remarks

In this lecture, we have reviewed the optimal layout theory by Prager and Rozvany (1977a), and some of its extensions to various design constraints. Difficulties in obtaining solutions were also explained, and new types of optimal regions were introduced for overcoming them. Some historic aspects of current truss topology research were also noted.

References are listed after the last lecture of the author.

Some Fundamental Properties of Exact Optimal Structural Topologies

George I. N. Rozvany[1]

[1] Department of Structural Mechanics, Budapest University of Technology and Economics, Budapest, Hungary

1. Introduction

In Lecture 1 of this author we discussed Michell's (1904) theory of optimal truss design, examined its range of validity, and looked at extended optimality criteria for a broader class of boundary conditions.

In Lecture 2 the optimal layout theory (Prager and Rozvany 1977) as well as optimal regions in exact truss topologies were reviewed and several extensions of the layout theory presented.

Whilst these two lectures touched on some basic features of optimal structural topologies, fundamental properties of these will be examined in detail in this lecture.

2. Partial Relaxation of the Orthogonality Requirement for Michell Trusses

Hemp (1973) states about Michell trusses: 'If a pair of tension and compression members meet at a point, they must be orthogonal … no other member can be co-planar with them'. In the following, we mention cases in which the above orthogonality can be relaxed.

2.1. Non-Orthogonal Tension- and Compression-Members at Boundary Points of T-regions

As a simple example, we consider Michell's (1904) solution (see Fig. 18a in Lecture 2 or Fig. 26 in this lecture).

This solution consists of four T-regions, two of them have constant directions of principal adjoint strains and the other two are 'circular fans'. At the point A an infinite number of members meet, but only the outside ones satisfy the above orthogonality rule.

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2.2. Non-Orthogonal Tension and Compression Members Along the Boundary of an R^+ and an R^- Region

As an example we quote the results of Rozvany and Gollub (1990), who solved the Michell truss problem for any convex polygonal domain with line supports along the boundaries. The optimal truss topology for these support conditions was even calculated by a non-numeric computer program. One of these topologies is shown in Fig. 19, in which double arrows indicate principle directions of R-regions. At the point load, the compression and tension members are clearly not orthogonal.

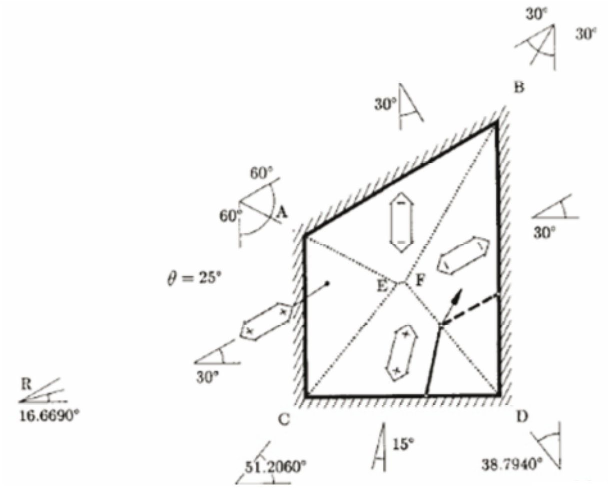

Figure 19. Non-orthogonal compression and tension members for loads at the boundary between an R^+ and an R^- region (after Rozvany and Gollub 1990).

Another example of non-orthogonality was derived by the author (Rozvany 1997) Considering a vertical point load and two pin supports, the optimal truss topologies for various horizontal distances of the point load are shown in Figs 20a, d, e and f. The kinematic boundary conditions for the adjoint strain fields are shown in Fig. 20b and the calculation of the principal adjoint strains $\bar{\varepsilon}_I$ and $\bar{\varepsilon}_{II}$ by means of a Mohr-circle in Fig. 20c. In Fig. 20 we use a normalized formulation with $k = 1$.

It can be seen that the optimal adjoint strain field in Figs 20a, b and c consist of an R^+ and an R^- region. Well known solutions in Figs 20d and e are special cases of the topology in Fig. 20a. The solution in Fig. 20f is a short 'Michell cantilever', (Hemp 1973, Lewinski, Zhou and Rozvany 1994a).

The topology in Fig. 20a has been extended to non-symmetric locations of the point load in a paper to be submitted (Sokol and Rozvany 2013). All new non-orthogonal optimal topologies have been confirmed numerically to a high degree of accuracy by Sokol's (2011a and b) truss optimization program.

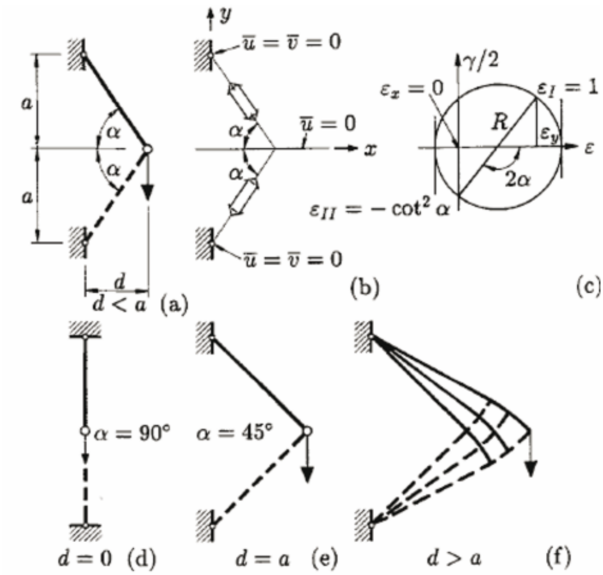

Figure 20. (a, d, e and f) Optimal truss topologies for a vertical point load and two pin supports. The adjoint strain field for the solution in sub-figure (a) is explained in sub-figures (b) and (c).

3. Domain Augmentation and Reduction

The *domain augmentation theorem* states that for a domain with line supports along the boundaries the optimal topology does not change if we modify the boundary such that

(a) active supporting points along the old boundary are contained in the new boundary, and
(b) no point of the new boundary is contained in the interior of the old domain.

A point of a line support is 'active' if members with non-zero cross sectional area connect to it.