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Ülo Lepik
Helle Hein

Haar Wavelets

With Applications

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Ülo Lepik · Helle Hein

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Ülo Lepik
Institute of Mathematics
University of Tartu
Tartu
Estonia

Helle Hein
Institute of Computer Science
University of Tartu
Tartu
Estonia

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Preface

In many disciplines, problems appear which can be formulated with the aid of differential or integral equations. In simpler cases, such equations can be solved analytically, but for more complicated cases, numerical procedures are needed. At present we have, for this purpose, several methods and programs. In recent times, the wavelet-based methods have gained great popularity, where different wavelet families such as Daubechies Coiflet, Symlet, etc., wavelets are applied. A shortcoming of these wavelets is that they do not have an analytic expression. For this reason, differentiation and integration of these wavelets are very complicated and doubts about the expediency of these wavelets in Calculus arise.

From wavelets which have an analytic expression mathematically the simplest are the Haar wavelets, which consist of pairs of piecewise constant functions. Such functions were introduced by Alfred Haar in 1910 and they have been used for solving problems of Calculus only from 1997.

When compared with other methods of solution, the Haar wavelet approach has some preferences, as mathematical simplicity, possibility to implement standard algorithms, and high accuracy for a small number of grid points. The solutions based on the Haar wavelets are usually simpler and faster than in the case of other methods. For these reasons, the Haar wavelets have obtained a great popularity and the number of papers about Haar Wavelets is rapidly increasing. According to the Science Direct in 12.04.13, there were 3,295 publications about Haar wavelets, among these 1,266 items are on differential and integral equations. To the reader it is difficult to find his way among the great number of these publications; therefore, a text-book about the applications of the Haar wavelets in Calculus is extremely necessary. Unfortunately such a book has been missing up to now.

The aim of the present book is to fulfill this gap, even if partially. At present, time different variants of the Haar method exist. It is not reasonable to handle and analyze all of them in detail; it would make the book less understandable and could confuse the reader. Therefore, we have decided to choose a method of solution, which is sufficiently universal and is applicable to solve all the problems by a unit approach. Other treatments will be referred and discussed in the section related papers, which is added to each chapter.

The book is put together on the basis of 19 papers, which we have published in prereviewed international journals. A unit method of solution is applied for solution of a wide range of problems (different types of differential and integral equations, fractional integral equations, optimal control theory, buckling and vibrations of elastic beams). To demonstrate efficiency and accuracy of the proposed method, a number of examples is solved. Mostly test problems, for which the exact solution or solution obtained by other methods is known, are considered.

The book is meant for researchers in applied mathematics, physics, engineering, and related disciplines, also for teachers of higher schools, graduate and post-graduate students. To make the book accessible for a wider circle of readers, some mathematical finesses are left out.

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Ülo Lepik
Helle Hein

Contents

Chapter 1
Preliminaries

1.1 Why we Need the Wavelets?

Consider a function of time $f = f(t)$. The Fourier transform of this function is

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt. \tag{1.1}$$

Here the transform variable ω represents angular frequency. If the signal $f(t)$ is put together of harmonic components then the frequency diagram $F = F(\omega)$ consists only of sharp peaks. This approach has been successfully applied for solving many problems (especially for signal processing), but it also has an essential disadvantage. The Fourier method analyses the signal over the whole domain, but does not characterize the motion in time.

Let us illustrate this with an example. Consider two motions

$$(i)\ f(t) = \sin 3t + 0.8 \sin 10t \ \mathit{for}\ t \in [0, 10], \tag{1.2}$$

and

$$(ii) \quad f(t) = \begin{cases} \sin 3t & \mathit{for}\ t \in [0, 5), \\ \sin 10t & \mathit{for}\ t \in (5, 10]. \end{cases} \tag{1.3}$$

These motions and their Fourier diagrams are plotted in Fig. 1.1. Although the motions are quite different, their Fourier diagrams are very similar. The only information we get from the Fourier diagram is that in both motions components with the frequencies $\omega_1 = 3$ and $\omega_2 = 10$ dominate.

The other example is from music. If we analyse a piece of music by the Fourier method we can find out which frequencies (notes) occured, but we do not get any information about the melody. To sum up: the Fourier transform localizes the time series in frequency, but the results are completely delocalized in time.

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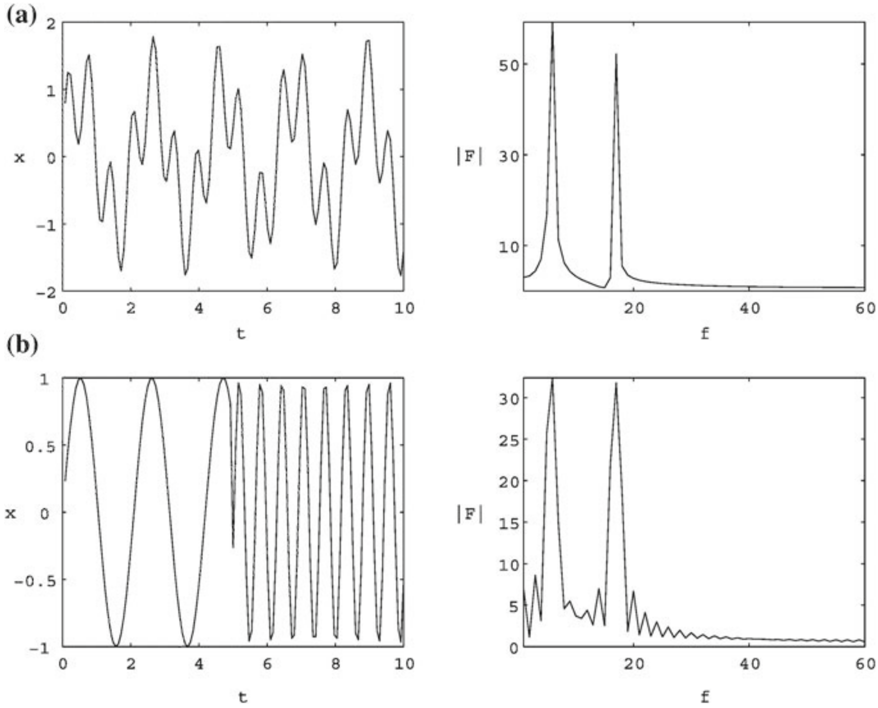

Fig. 1.1 Time history and Fourier diagram. **a** $x = \sin 3t + 0.8 \sin 10t$ for $t \in [0, 10]$ and **b** $x = \sin 3t$ for $t \in [0, 5]$; $x = \sin 10t$ for $t \in [5, 10]$ [7]

Evidently we need a method which analyses our signal over the whole domain and is also able to characterize the motion in time. This can be done with the aid of wavelets.

1.2 Wavelet Families

For generating a wavelet system two basic functions—the scaling function $\varphi = \varphi(t)$ and the mother wavelet $\psi = \psi(t)$ are required. These functions must satisfy the admissibility conditions

$$\int_{-\infty}^{\infty} \frac{\hat{\varphi}(\omega)}{\omega} d\omega < \infty, \quad \int_{-\infty}^{\infty} \frac{\hat{\psi}(\omega)}{\omega} d\omega < \infty. \tag{1.4}$$

Here $\hat{\varphi}(\omega)$ and $\hat{\psi}(\omega)$ denote the Fourier transforms of $\varphi(t)$ and $\psi(t)$, respectively. These conditions imply that if $\hat{\varphi}(\omega)$ and $\hat{\psi}(\omega)$ are smooth then $\hat{\varphi}(0) = \hat{\psi}(0) = 0$.

For getting a wavelet series, which describes the motion both in space and time, we must introduce two parameters. This can be done by assuming

$$\varphi_b(t) = \varphi(t-b),\, \psi_{a,b}(t) = a^{-1/2}\psi(\frac{t-b}{a}). \tag{1.5}$$

The parameter a means rescaling and b-shifting along the t-axis. In some cases it is more convenient to make use of the binary system and take

$$\varphi_k(t) = \varphi(t-k),\;\; \psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k). \tag{1.6}$$

Here j and k are integers; j is called the *dilation parameter*, k—the *translation parameter*.

Next consider a function $f = f(t)$ for which

$$\int_{-\infty}^{\infty} \mid f(t) \mid^2 dt < \infty. \tag{1.7}$$

This function can be expanded into the wavelet series as follows

$$f(t) = \sum_b c_b \varphi_b(t) + \sum_a \sum_b d_{a,b}\psi_{a,b}(t), \tag{1.8}$$

or in the case of binary system

$$f(t) = \sum_k c_k \varphi_k(t) + \sum_k \sum_j d_{j,k}\psi_{j,k}(t). \tag{1.9}$$

Here the symbols c_b, $d_{a,b}$ (or respectively c_k, $d_{j,k}$) denote the wavelet coefficients. For the case that $\psi_{j,k}(t)$ are orthonormal and $\varphi_k(t)$ is orthogonal to $\psi_{j,k}(t)$ these coefficients can be found according to the formulae

$$\begin{aligned} c_k &= \int f(t)\varphi_k(t)dt, \\ d_{j,k} &= \int f(t)\psi_{j,k}(t)dt, \end{aligned} \tag{1.10}$$

where the integrations are carried out over the whole time domain.

Since we are quite free to choose the scaling function $\varphi(t)$ and mother wavelet $\psi(t)$ different wavelet families can then be generated.

The wavelet transform was introduced in the works of J. Morlet in the early 1980s. From the outset, the wavelet method was considered as a mathematical curiosity but due to extensive research in the 1990s was turned into a well grounded and powerful mathematical tool with many practical applications. Paving the way for this is the paper by Ingrid Daubechies [3].

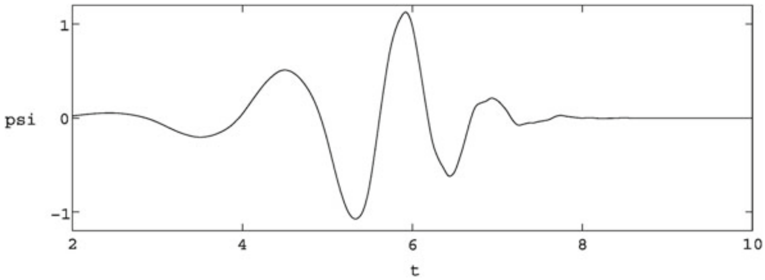

Fig. 1.2 Daubechies mother wavelet of order $J = 6$

The wavelets introduced by Daubechies are quite frequently used for solving different problems. The Daubechies mother wavelet is plotted in Fig. 1.2. These wavelets are differentiable and have a minimum size support.

A shortcoming of the Daubechies wavelets is that they do not have an explicit expression and therefore analytical differentiation or integration is not possible. This complicates the solution of differential equations, where the following type integrals

$$\int_a^b G(t, \psi_{i,k}, \frac{d\psi_{i,k}}{dt}, \frac{d^2\psi_{i,k}}{dt^2}, \ldots)dt \tag{1.11}$$

must be computed (G is generally a nonlinear function). For calculating such integrals the conception of *connection coefficients* is introduced. Calculation of these coefficients is very complicated and must be carried out separately for different types of integrals see e.g. [1, 6]. Besides, it can only be done for some simpler types of nonlinearities (mainly for quadratic nonlinearity). This remark holds also for other types of wavelets (as Symlet, Coiflet, etc. wavelets).

The wavelet method was first applied to solving differential and integral equations in the 1990s. Due to the complexity of the wavelet solutions some pessimistic estimates exist. So Strang and Ngyen in 1996 wrote in their text-book [9] "... the competition with other methods is severe. We do not necessarily predict that wavelets will win" (p. 394). Jameson [4] writes "... nonlinearities etc., when treated in a wavelet subspace, are often unnecessarily complicated ... There appears to be no compelling reason to work with Galerkin-style coefficients in a wavelet method" (p. 1982).

Obviously attempts to simplify solutions based on the wavelet approach are wanted.

Nowadays we possess several wavelet families for which analytic expressions for scaling function and mother wavelet are defined. Since such wavelets have been applied for solving differential and integral equations in several papers, we will describe some of them here.

In the case of the *Morlet wavelets* (also known as Gabor wavelets) the mother wavelet is taken in the form

$$\psi(t) = e^{i\omega_0 t} e^{-0.5t^2},\ i = \sqrt{-1},\ \omega_0 \geq 5, \tag{1.12}$$

the scaling function $\varphi(t)$ is not introduced.

The *harmonic wavelet transform* proposed by Newland [8] in 1993 is defined as

$$\varphi(t) = \frac{e^{i2\pi t} - 1}{i2\pi t},\ \psi(t) = \frac{1}{i2\pi t}(e^{i4\pi t} - e^{i2\pi t}). \tag{1.13}$$

These wavelets are remarkable in the sense that they have a very simple Fourier transform for the scaling function [1].

$$\hat{\varphi}(\omega) = \begin{cases} \frac{1}{2\pi} \text{ for } 0 \leq \omega < 2\pi\ , \\ 0 \text{ elsewhere.} \end{cases} \tag{1.14}$$

The *Shannon wavelets* are defined by [2]

$$\begin{aligned} \varphi(t) &= \text{sinc}\pi t = \frac{\sin \pi t}{\pi t}, \\ \psi(t) &= \text{sinc}\frac{t}{2} \cos \frac{3\pi t}{2}. \end{aligned} \tag{1.15}$$

The benefit of Shannon wavelet is that the functions $\varphi(t)$ and $\psi(t)$ also have a simple Fourier transform

$$\hat{\varphi}(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}} \text{ for } \mid \omega \mid \leq \pi\ , \\ 0 \text{ otherwise,} \end{cases} \tag{1.16}$$

and

$$\hat{\psi}(\omega) = \begin{cases} \frac{1}{\sqrt{2\pi}} \exp(-i\omega/2) \text{ for } \pi \leq \mid \omega \mid \leq 2\pi\ , \\ 0 \text{ otherwise.} \end{cases} \tag{1.17}$$

In many papers the Legendre wavelets have been applied. For these wavelets different variants exist. We follow here the method proposed for linear Legendre multiwavelets in [5].

For constructing these wavelets two scaling functions are introduced and they are described as

$$\varphi^0(t) = 1,\ \varphi^1(t) = \sqrt{3}(2t - 1),\ 0 \leq t < 1. \tag{1.18}$$

The corresponding mother wavelets are defined as

$$\psi^0(t) = \begin{cases} -\sqrt{3}(4t - 1) \text{ for } 0 \leq t \leq 1/2\ , \\ \sqrt{3}(4t - 3) \text{ for } 1/2 \leq t < 1, \end{cases} \tag{1.19}$$

and

$$\psi^1(t) = \begin{cases} 6t - 1 \text{ for } 0 \leq t \leq 1/2\ , \\ 6t - 5 \text{ for } 1/2 \leq t < 1. \end{cases} \tag{1.20}$$

With dilation and translation, we obtain the Legendre wavelets

$$\psi_{k,n}^{j}(t) = \begin{cases} 2^{k/2}\psi^{j}(2^{k}t - n) \text{ for } 2^{-k}n \le t \le 2^{-k}(n+1), \\ 0 \; \textit{otherwise}. \end{cases} \tag{1.21}$$

Here $n = 0, 1, \ldots, 2^k - 1$, k is an non-negative integer, $j = 0, 1$. A function $f(t)$ defined over $[0, 1)$ may be expanded as

$$f(t) = c_0\varphi^0(t) + c_1\varphi^1(t) + \sum_{k=0}^{M}\sum_{j=0}^{1}\sum_{n=0}^{2^k-1} c_{k,n}^{j}\psi_{k,n}^{j}(t). \tag{1.22}$$

Due to orthonormality of the Legendre wavelets, the wavelet coefficients can be calculated from (1.10). In some papers the Laguerre, Chebyshev or B-spline wavelets are used.

Haar wavelets are based on the functions which were introduced by Hungarian mathematician Alfred Haar in 1910. The Haar wavelets are made up of piecewise constant functions and are mathematically the simplest among all the wavelet families. A good feature of these wavelets is the possibility to integrate them analytically arbitrary times. They can be interpreted as a first order Daubechies wavelet. The Haar wavelets have been applied for solving several problems of mathematical calculus. Some of these results will be discussed and analysed in the following chapters of this book.

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Chapter 2
Haar Wavelets

2.1 Haar Wavelets and their Integrals

This section is based on paper [4]. Let us consider the interval $x \in [A, B]$, where A and B are given constants. We define the quantity $M = 2^J$, where J is the maximal level of resolution. The interval $[A, B]$ is divided into $2M$ subintervals of equal length; the length of each subinterval is $\Delta x = (B - A)/(2M)$. Next two parameters are introduced: $j = 0, 1, \ldots, J$ and $k = 0, 1, \ldots, m - 1$ (here the notation $m = 2^j$ is introduced). The wavelet number i is identified as $i = m + k + 1$.

The i-th Haar wavelet is defined as

$$h_i(x) = \begin{cases} 1 & \text{for } x \in [\xi_1(i), \xi_2(i)), \\ -1 & \text{for } x \in [\xi_2(i), \xi_3(i)), \\ 0 & \text{elsewhere,} \end{cases} \tag{2.1}$$

where

$$\begin{aligned} \xi_1(i) &= A + 2k\mu\Delta x\,, & \xi_2(i) &= A + (2k+1)\mu\Delta x\,, \\ \xi_3(i) &= A + 2(k+1)\mu\Delta x\,, & \mu &= M/m\,. \end{aligned} \tag{2.2}$$

These equations are valid if $i > 2$. The case $i = 1$ corresponds to the scaling function: $h_1(x) = 1$ for $x \in [A, B]$ and $h_1(x) = 0$ elsewhere. For $i = 2$ we have $\xi_1(2) = A$, $\xi_2(2) = 0.5(2A + B)$, $\xi_3(2) = B$. The parameters j and k have concrete meaning. The support (the width of the i-th wavelet) is

$$\xi_3(i) - \xi_1(i) = 2\mu\Delta x = (B - A)m^{-1} = (B - A)2^{-j} \tag{2.3}$$

It follows from here that if we increase j then the support decreases (the wavelet becomes more narrow). By this reason it is called the *dilatation parameter*. The other parameter k localises the position of the wavelet in the x-axis; if k changes from 0 to $m - 1$ the initial point of the ith wavelet $\xi_1(i)$ moves from $x = A$ to $x = [A + (m - 1)B]/m$. The integer k is called the *translation parameter*.

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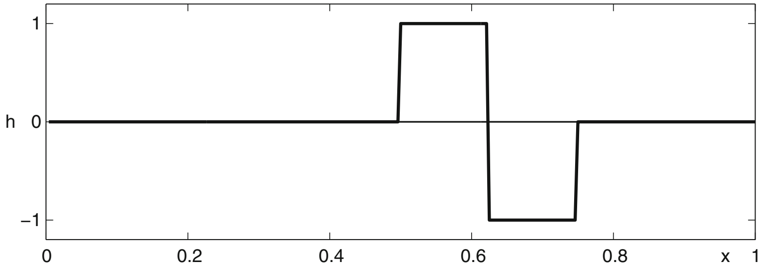

Fig. 2.1 Haar wavelet for $J = 2$; $i = 7$

Let us take an example. If $j = 2$, $J = 2$, $k = 2$, $A = 0$, $B = 1$ we have $m = M = 2^2 = 4$, $\mu = 1$, $\Delta x = 0.125$ and the wavelet number is $i = 7$. According to (2.2) $\xi_1(7) = 0.5, \xi_2(7) = 0.625, \xi_3(7) = 0.75$. This wavelet is plotted in Fig. 2.1.

Eight first wavelets h_1-h_8 are shown in Fig. 2.2.

If the maximal level of resolution J is prescribed then then it follows from (2.1) that

$$\int_A^B h_i(x)h_l(x)dx = \begin{cases} (B-A)2^{-j} & \text{for } l = i, \\ 0 & \text{for } l \neq i. \end{cases} \tag{2.4}$$

So we see that the Haar wavelets are orthogonal to each other.

In the following we need the integrals of the Haar functions

$$p_{\nu,i}(x) = \underbrace{\int_A^x \int_A^x \ldots \int_A^x}_{\nu-\text{times}} h_i(t)dt^\nu = \frac{1}{(\nu-1)!}\int_A^x (x-t)^{\nu-1}h_i(t)dt \tag{2.5}$$
$$\nu = 1, 2, \ldots, n, \quad i = 1, 2, \ldots, 2M.$$

Taking account of (2.1) these integrals can be calculated analytically; by doing it we obtain

$$p_{\alpha,i}(x) = \begin{cases} 0 & \text{for } x < \xi_1(i), \\ \frac{1}{\alpha!}[x-\xi_1(i)]^\alpha & \text{for } x \in [\xi_1(i), \xi_2(i)], \\ \frac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2[x-\xi_2(i)]^\alpha\right\} & \text{for } x \in [\xi_2(i), \xi_3(i)], \\ \frac{1}{\alpha!}\left\{[x-\xi_1(i)]^\alpha - 2[x-\xi_2(i)]^\alpha + [x-\xi_3(i)]^\alpha]\right\} & \text{for } x > \xi_3(i). \end{cases} \tag{2.6}$$

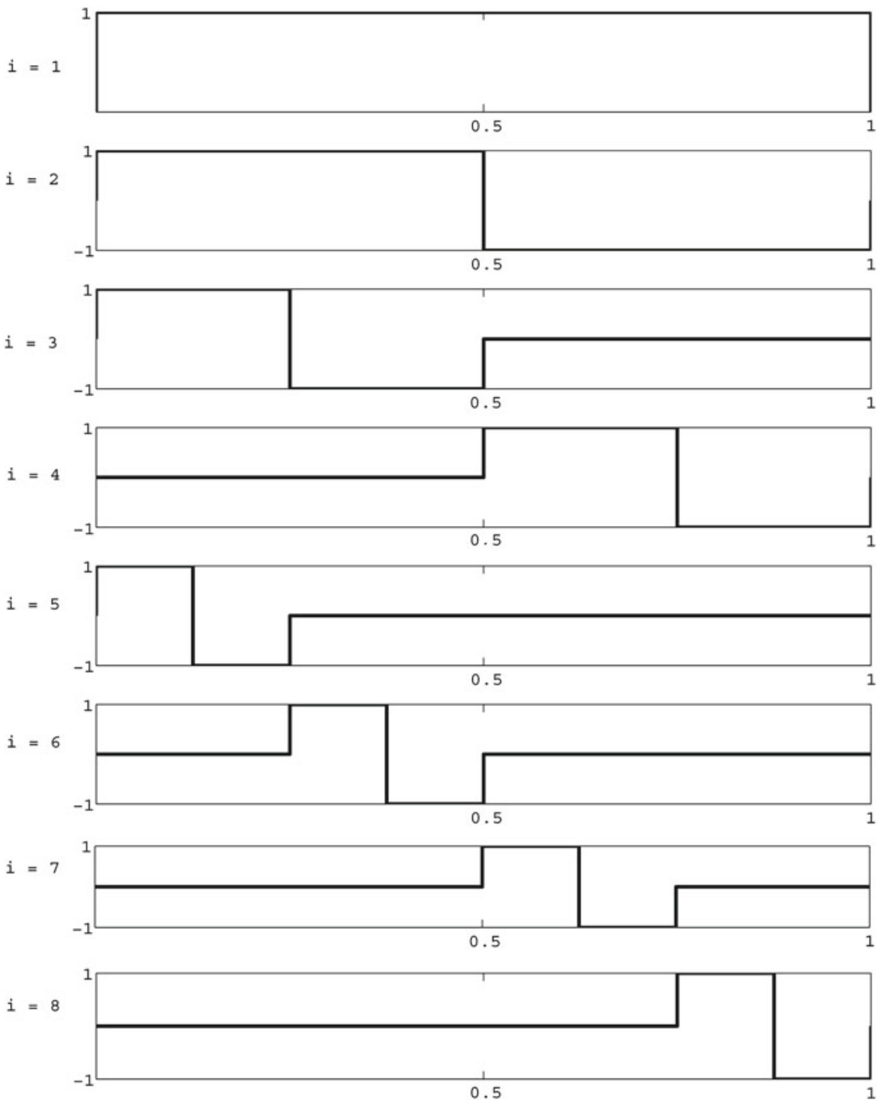

Fig. 2.2 Eight first Haar wavelets

These formulas hold for $i > 1$. In the case $i = 1$ we have $\xi_1 = A, \xi_2 = \xi_3 = B$ and

$$p_{\alpha,1}(x) = \frac{1}{\alpha!}(x - A)^\alpha. \tag{2.7}$$

2.2 Haar Matrices

If we want to use the Haar wavelets for the numerical solutions we must put them into a discrete form. There are different ways to do it; in this paper the collocation method is applied.

Let us denote the grid points by

$$\tilde{x}_l = A + l\Delta x, \quad l = 0, 1, \ldots, 2M \tag{2.8}$$

For the collocation points we take

$$x_l = 0.5(\tilde{x}_{l-1} + \tilde{x}_l), \quad l = 1, \ldots, 2M \tag{2.9}$$

and replace $x \to x_l$ in Eqs. (2.1), (2.6) and (2.7). It is convenient to put these results into the matrix form. For this we introduce the Haar matrices $H, P_1, P_2, \ldots, P_\nu$ which are $2M \times 2M$ matrices. The elements of these matrices are $H(i, l) = h_i(x_l)$,$P_\nu(i, l) = p_{\nu i}(x_l)$, $\nu = 1, 2, \ldots$.

For illustration consider the case $A = 0, B = 1, J = 1$. Now $2M = 4$ and the grid points are $\tilde{x}_0 = 0, \tilde{x}_1 = 0.25, \tilde{x}_2 = 0.5, \tilde{x}_3 = 0.75, \tilde{x}_4 = 1$. By calculating the coordinates of the collocation points from (2.9) we find $x_1 = 0.125, x_2 = 0.375, x_3 = 0.625, x_4 = 0.875$. The Haar matrices H, P_1, P_2 are

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}, \quad P_1 = \frac{1}{8}\begin{pmatrix} 1 & 3 & 5 & 5 \\ 1 & 3 & 3 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad P_2 = \frac{1}{128}\begin{pmatrix} 1 & 9 & 25 & 49 \\ 1 & 9 & 23 & 31 \\ 1 & 7 & 8 & 8 \\ 0 & 0 & 1 & 7 \end{pmatrix} \tag{2.10}$$

2.3 Expanding Functions into the Haar Wavelet Series

Consider a square integrable function $f = f(x)$ for $x \in [A, B]$. This function can be expanded into the Haar wavelet series

$$f(x) = \sum_{i=1}^{2M} a_i h_i(x). \tag{2.11}$$

The symbol a_i denotes the Haar wavelet coefficients. The discrete form of (2.11) is (x_l are the collocation points):

$$\hat{f}(x_l) = \sum_{i=1}^{2M} a_i h_i(x_l). \tag{2.12}$$

The matrix form of (2.12) is

$$f = aH. \tag{2.13}$$

Here H is the Haar matrix; a and f are defined as $a = (a_i), f = (f_l)$; both are $2M$ dimensional row vectors. Solving the matrix equation (2.13) with regard to the coefficient vector a we find (H^{-1} denotes the inverse of H)

$$a = fH^{-1}. \tag{2.14}$$

Replacing a into (2.11) we obtain the wavelet approximation of the function $f(x)$ for the level of resolution J. The question arises as to what the degree of exactness of the approximation is (2.11). There are different possibilities to estimate the error function Δ of the wavelet approximations. Here we define the error function as

$$\Delta = \int_A^B \left[f(x) - \hat{f}(x)\right]^2 dx, \tag{2.15}$$

where $\hat{f}(x)$ denotes the approximation of $f(x)$. The discrete form of (2.15) is

$$\Delta_J = \Delta x \sum_{l=1}^{2M} \left[f(x_l) - \hat{f}(x_l)\right]^2. \tag{2.16}$$

The Haar wavelets belong to the group of piecewise constant functions. It is known that if the function is sufficiently smooth, then the convergence rate for the piecewise constant function is $O(M^{-2})$; this result can be transferred also to the Haar wavelet approach. So it could be expected that by doubling the number of collocation points the error roughly decreases four times. Consider two examples.

Example 1: Let $f(x) = \sqrt{x}$ and $x \in (0, 1)$. The Haar matrix is put together as shown in Sect. 2.2. The wavelet coefficients were calculated according to (2.14) and for $J = 3$ they are plotted in Fig. 2.3. Wavelet approximation for some values of J are presented in Fig. 2.4.

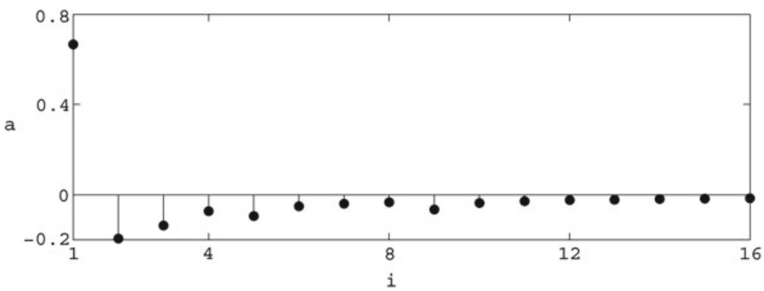

Fig. 2.3 Wavelet coefficients for the equation $f = \sqrt{x},\ x \in (0.1),\ J = 3$