Mathematics Education in the Digital Era

Sergei Abramovich

Computational Experiment Approach to Advanced Secondary Mathematics Curriculum

Springer

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MATHEMATICS EDUCATION IN THE DIGITAL ERA
Volume 3

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ISSN 2211-8136 ISSN 2211-8144 (Electronic)
ISBN 978-94-017-8621-8 ISBN 978-94-017-8622-5 (eBook)
DOI 10.1007/978-94-017-8622-5
Springer Dordrecht Heidelberg New York London

Library of Congress Control Number: 2014931654

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This book stems from the author's interest in and experience of using computing technology in K-12 mathematics teacher education. In particular, two decades of teaching prospective secondary mathematics teachers in technology-rich environments have led the author to believe that one of the most productive ways to teach mathematics in the digital era is through experimentation with mathematical concepts that takes advantage of computers' capability to plot sophisticated graphs, construct dynamic geometric shapes, generate interactive arrays of numbers, and perform symbolic computations. Whereas the notion of experiment in mathematics has several meanings (Baker 2008; Van Bendegem 1998), assigning the adjective *computational* to the word *experiment* implies that the meaning of the latter is narrowed down to the use of electronic computers as the means of mathematical experimentation.

The book utilizes a number of commonly available computer applications that allow for lucid presentation of advanced, though grade-appropriate, mathematical ideas. One application is the Graphing Calculator 4.0 produced by Pacific Tech (Avitzur 2001) that facilitates experimentation in algebra through the software's capability of constructing graphs from any two-variable equation, inequality, or a combination of those. Another application is an electronic spreadsheet used to support numerical experimentation, in particular, when carrying out probability simulations and modeling elementary number theory concepts. The book also takes advantage of Maple (Char et al. 1991) and Wolfram Alpha developed by Wolfram Research—software tools that allow for different types of experimentation with mathematical concepts, including the construction of graphs of functions and relations and carrying out complicated symbolic computations. Also, the book uses The Geometer's Sketchpad (GSP) created by Nicholas Jackiw in the late 1980s. Yet this dynamic geometry program is used more as a technical tool rather than as an experimental device. This lesser focus on experimentation with GSP is due to the book's stronger focus on algebra in comparison to geometry. Nevertheless, the idea of geometrization of algebraic concepts is one of the major mathematical ideas used in the book.

Throughout the book, a number of the modern day secondary mathematics education documents developed throughout the world are reviewed as appropriate.

These include Common Core State Standards (2010) for mathematical practice and the Conference Board of the Mathematical Sciences[1] (2001, 2012) recommendations for the preparation of mathematics teachers—the United States (the context in which the author prepares teachers); National mathematics curriculum (National Curriculum Board 2008)—Australia; Ontario mathematics curriculum (Ontario Ministry of Education 2005) and British Columbia mathematics curriculum (Western and Northern Canadian Protocol 2008)—Canada; Programs of mathematics study (Department for Education 2013a, b)—England; A secondary school teaching guide for the study of mathematics (Takahashi et al. 2006)—Japan; and Secondary mathematics syllabi (Ministry of Education, Singapore 2006)—Singapore.

The book consists of eight chapters. The first chapter provides theoretical underpinning of computational experiment approach to advanced secondary mathematics curriculum. In the focus are mathematics education research publications that started appearing in the second half of the twentieth century with the advent of computers as tools for the teaching of mathematics. The role of mathematics education reform in bringing computers first to the undergraduate level and gradually extending their use to include experimentation at the primary level is highlighted. Several theoretical frameworks leading to the development of the notion of technology-enabled mathematics pedagogy referred to as TEMP throughout the book are discussed. It is suggested that TEMP can become a major pillar of modern signature pedagogy of mathematics as it can focus on the unity of computational experiment and formal mathematical demonstration. The relationship between technology-enabled experiment and solution-enabled experiment is introduced as a structure that makes computational experiment a meaning making process. It is shown how visual imagery can support deductive reasoning leading to an error-free computational experiment.

One of the major differences between TEMP and a mathematics pedagogy (MP) that does not incorporate technology pertains to the interplay between mathematical content under study and the scope of student population to which this content can be made available. Whereas many problems discussed in the book under the umbrella of TEMP are fairly complex, using technology as a support system makes it possible to develop mathematical insight, facilitate conjecturing, and illuminate plausible problem-solving approaches to those problems. To a certain extent, the use of TEMP may be comparable to the use of computers in the modern day investigation of dynamical systems allowing one to carry out numeric/symbolic computations and graphical constructions not possible otherwise, yet being critical for understanding the behavior of those systems. In comparison with MP which, in particular, lacks empirical support for conjectures, using TEMP has great potential to engage a much broader student population in significant mathematical explorations. TEMP provides teachers with tools and ideas conducive to engaging students in the project-based, exploratory learning of mathematics by dividing a project in several stages—empirical, speculative, formal, and reflective. Even if TEMP helps

[1] The Conference Board of the Mathematical Sciences is an umbrella organization consisting of 16 professional societies in the United States.

a student to reach the level of conjecturing without being able to proceed to the next level, its use appears to be justified.

The second chapter is devoted to the use of computational experiment as support system for solving one-variable equations and inequalities. The importance of revealing the meaning of problem-solving techniques frequently considered as “tricks” by students and their teachers alike is emphasized. It is demonstrated how technology can be used as an agency for mathematical activities associated with formal demonstration of theoretical concepts that underpin commonly used algebraic techniques. The idea of parameterization of one-variable equations is explained through the lens of problem posing in the digital era, an approach conducive to the development of “mathematical reasoning and competence in solving increasingly sophisticated problems” (Department for Education 2013b, p. 1). Also, the chapter shows how through the integration of technology and historical perspectives the secondary mathematics content can be connected to its historical roots.

The third chapter is devoted to the study of quadratic equations and functions with parameters. Here, two methods of exploration made possible by computational experiment are discussed. One method deals with the possibility of transition from the traditional (x, y)-plane typically used to construct the graphs of functions to the (variable, parameter)-plane commonly referred to as the phase plane. Using the diagrams (loci) constructed in the phase plane, one can discern the most important information about a quadratic equation with a parameter, namely, the influence of the parameter on the solutions (roots) of the equation. Another method, in the case of equations with two parameters, deals with the qualitative study of solutions in the plane of parameters. It demonstrates how one can make a transition from representations in the plane of variables to representations in the plane of parameters when investigating the properties of quadratic functions and associated equations depending on parameters. In particular, qualitative methods for deciding the location of roots of quadratic equations with parameters about a point as well as about an interval are discussed. These methods make it possible to determine the location of the roots without finding their exact values. The need for such methods proved to be very useful in the context of the “S” and “E” components of STEM (science, technology, engineering, mathematics) where qualitative techniques are commonly used in exploring the corresponding mathematical models.

The fourth chapter is devoted to the systematic study of algebraic equations with parameters (including simultaneous equations) using the computationally supported locus approach when explorations take place in the (variable, parameter)-plane. Here, the computational experiment approach is applied to “make use of structure” (Common Core State Standards 2010) of a complicated mathematical situation and to develop its deep understanding by using locus as a thinking device. In doing so, one can come across various extensions of the situation to include new concepts, representations, and lines of reasoning that connect different grade appropriate mathematical ideas. The notion of collateral learning in the spirit of Dewey (1938) is highlighted. In the case of two-variable simultaneous equations with parameters it is shown how the parameter can be given a proper geometric interpretation enabling the computational experiment to be carried out in a two-dimensional context, that is, in the plane of variables.

Inequalities are the major means of investigation in mathematics, both pure and applied. The fifth chapter is devoted to the systematic study of inequalities, including systems of inequalities, with parameters. In this context, it is demonstrated how computational experiment approach facilitates solving traditionally difficult problems not typically considered in the secondary school mathematics curriculum. The idea of extending an original exploration of an inequality with a parameter to allow for a deep inquiry into closely related ideas is discussed. One of the aspects associated with the dependence of solution to an inequality on a parameter is a possibility of using locus of an inequality with a parameter as a tool for posing one-variable inequalities with no parameter. In the digital era, such pedagogical perspective makes it possible to present problem solving and problem posing as two sides of the same coin. Also, the chapter focuses on the so-called technology-enabled/technology-immune tasks in the sense that whereas technology may be used in support of problem solving, its direct application is not sufficient for achieving the end result. It is shown that such tasks can be developed in the context of inequalities with parameters. A point is made that the computational experiment approach may be inconclusive, thereby requiring an analytic clarification of the experiment to make sense of the structure of a situation. A two-dimensional sign-chart method that can be used for solving inequalities with parameters is presented. Finally, the applied character of problem-solving techniques developed for solving inequalities with parameters is illustrated through their application to quadratic equations when the location of roots about a given point can be determined without solving an equation.

Trigonometry is known as a subject matter of great importance for the study of engineering disciplines. The sixth chapter shows how concepts in trigonometry can be approached from a computational perspective. Here, a single trigonometric equation with parameters is used as a springboard into several geometric ideas, thereby, demonstrating a closed connection of the two contexts. Technology such as Wolfram Alpha with its own unique algorithm of solving trigonometric equations and inequalities is presented as an agent of rather sophisticated mathematical activities stemming from the need to justify the equivalence of different forms of solution expressed through inverse circular functions. Whereas in the presence of technology (including just a calculator) such equivalence can be easily established numerically, the appropriate use of technology should motivate learners of mathematics to appreciate rigor and to enable the development of formal reasoning skills. Having experience with proving the equivalence of two solutions obtained through different methods can be construed as support system for research-like experience that prospective secondary mathematics teachers need for the successful teaching of the subject matter. This further provides experience with STEM-related techniques, something that can contribute to the efforts of introducing secondary mathematics teacher candidates and their future students alike to the ideas that develop the foundation of engineering profession.

The seventh chapter deals with geometric probabilities. It shows how the computational experiment approach can work in calculating probabilities of events associated with the behavior of solutions of algebraic equations with parameters. In particular, both theoretical and experimental probabilities are computed in the space of parameters and compared within a spreadsheet. Here, most of the explorations extend the ideas considered in the previous chapters and developed in the context of making a transition from the plane of variables to the plane of parameters. The material of this chapter is aimed at providing teacher candidates with experience important for applying mathematics to science and engineering the models of which typically depend on parameters. Whereas the construction of spreadsheet-based computational environments for computing geometric probabilities experimentally does not require any mathematical or technological sophistication, the skills in using such environments are important for understanding how to do explorations of mathematical models in engineering and science.

The last chapter illustrates how the computational experiment approach can make concepts of number theory more accessible to prospective teachers and their students alike. It provides a number of illustrations of using modern technology tools in exploring classic topics in elementary theory of numbers through a computational experiment. Here, one can learn how technology can be used to develop theoretical knowledge on the basis of a simple experiment so that, in turn, the knowledge so developed can inform and facilitate similar yet more complicated experiments. The chapter highlights the duality of computational experiment and formal demonstration in the sense that whereas one needs theory to validate experimental results, once can benefit from computing when discovering and correcting unexpected flaws that theory may sometimes comprise. The chapter demonstrates how TEMP that encourages collateral learning can be brought to bear by emphasizing geometrization of algebraic concepts and the appropriate use of digital tools. The deficiency of reasoning by induction in the context of basic summation formulas that can result in overgeneralization is discussed. The topic of Pythagorean triples is explored in depth using jointly a spreadsheet and Wolfram Alpha. Within this topic, it is demonstrated how computational experiment approach can motivate mathematical insight and encourage natural curiosity of the learners of mathematics.

Acknowledgements

While working on this book, I have been greatly supported by the Editors of the Series, Dragana Martinovic of University of Windsor and Viktor Freiman of University of Moncton. Their collegial encouragement and professional advice during the discussion of my possible contribution to the series are highly appreciated. I owe a special debt of gratitude to two unanimous reviewers whose comprehensive reviews and insightful comments significantly improved the quality of the book. Many thanks are extended to Natalie Rieborn, Senior Editorial Assistant (Mathematics Education) of Springer for invaluable assistance during the preparation of the book for publication. Finally, I am grateful to my son Leonard Abramovich for much needed help with graphics.

Contents

Chapter 1
Theoretical Foundations of Computational Experiment Approach to Secondary Mathematics

1.1 Introduction

This chapter provides theoretical underpinning of computational experiment approach to pre-college mathematics curriculum. It reviews mathematics education research publications and (available in English) educational reform documents from Australia, Canada, England, Japan, Singapore and the United States related to the use of computers as tools for experimenting with mathematical ideas. The chapter links pioneering ideas by Euler about experimentation with mathematical ideas to the use of the word experiment in the modern context of pre-college mathematics curricula. It emphasizes the role of mathematics education reform in bringing computers first to the undergraduate level and gradually extending their use to include experimentation at the primary level. Several theoretical frameworks including signature pedagogy, Type I/Type II technology applications, parallel structures of teaching and learning, agent-consumer-amplifier framework, and collateral learning in the digital era are highlighted leading to the development of the notion of technology-enabled mathematics pedagogy (TEMP). One of the major characteristics of TEMP is its focus on the idea with ancient roots—the unity of computational experiment and formal mathematical demonstration. The relationship between technology-enabled experiment and solution-enabled experiment is introduced as a structure that makes computational experiment a meaning making process. It will be demonstrated how visual imagery can support deductive reasoning leading to an error-free computational experiment.

1.2 Experiment in Mathematics Education

In this book, the word experiment is connected to the use of electronic computers in the context of advanced secondary mathematics curriculum and, in particular, mathematics teacher education. These modern tools when used in mathematics instruction create and enhance conditions for one's inquiry into mathematical structures, which may include interactive graphs, dynamic geometric shapes, and electronically generated and controlled arrays of numbers. That is, the modern experiment in

S. Abramovich, *Computational Experiment Approach to Advanced Secondary Mathematics Curriculum*, Mathematics Education in the Digital Era 3,
DOI 10.1007/978-94-017-8622-5_1,

Fig. 1.1 One out of 12 Latin squares of order 3

1	2	3
2	3	1
3	1	2

mathematics can be a computational one. In the context of mathematical education in general, a computational experiment approach to mathematics makes use of such computer-enabled experiments designed by a teacher and carried out by students, or jointly by teacher and students. Hereafter, the words teacher and student are understood broadly: the former is one who teaches and the latter is one who is taught. Because through such interaction, both parties can learn, the word learner will be applied to any individual engaged in the learning of mathematics.

Whereas the notion of experiment in the context of education has multiple meanings, learning as the goal of experiment is what all the meanings have in common. In a seminal book on experiment in education, McCall (1923) recognized the power of experiment as a milieu where "teachers join their pupils [i.e., students] in becoming question askers" (p. 3). Similarly, about a century later, Hiebert et al. (2003) emphasized the importance for teachers to treat lessons as experiments towards the end "of making some aspects of teachers' routine, natural activity more systematic and intensive" (p. 207). In other words, by treating lessons as experiments, teachers, "by focusing attention on, and making more explicit, the process of forming and testing hypotheses" (ibid, p. 207), are expected to learn both about and from teaching. Mathematics is especially conducive to the development of an environment in which reflective inquiry—a problem-solving method that blurs the distinction between knowing and doing by integrating knowledge with experience (Dewey 1933)—is the major learning strategy.

There is an interesting connection between the notions of experiment in mathematics and experiment in education. This connection can be revealed through the concept of Latin square. The latter is a square matrix each row and column of which contains any element one and only one time (Fig. 1.1). Latin squares have been commonly utilized in the design of educational experiments (Fisher 1935; Campbell and Stanley 1963) in different disciplines where it is required to construct a matrix under specific conditions on the location of its entries. For example, in the study by Gall et al. (1978) involving 12 teachers from 12 classrooms (experimental units), three recitation treatments (Probing and Redirection, No Probing and Redirection, Filler Activity) and one control treatment (Art Activity) were arranged in three 4×4 Latin squares (with each treatment, randomly assigned to the experimental units, appearing only once in a row and once in a column), provided that each teacher taught all four treatments. Another major application of Latin squares is in agriculture (Lakić 2001). Here, a field can be divided into sections and different seeds sown or treatments applied are recorded in the form of a Latin square with the goal to diminish the influence of other factors. In sum, Latin squares are great tools within which data can be conveniently stored, meaningfully observed, and appropriately analyzed.

In mathematics Latin squares were used by Euler, the great Swiss mathematician of the eighteenth century considered the father of all modern mathematics. In particular, Euler's name is associated with the so-called *Officers Problem* of placing 36 officers of six ranks and six regiments in a Latin square so that no officer of the same rank or of the same regiment would be in the same row or in the same column (MacNeish 1922). In his insightful approaches to mathematics, especially to number theory, Euler emphasized the importance of observations and so-called quasi-experiments or thought processes (experiments) that stem from observations: "the properties of the numbers known today have been mostly discovered by observation, and discovered long before their truth has been confirmed by rigid demonstration" (Pólya 1954, p. 3; according to Lakatos (1976), Pólya, who made the translation, "mistakenly attributes the quotation to Euler" (p. 9) instead of crediting it to the Editor of Euler's work). So, quite unexpectedly, a mathematical tool used by a pioneer of experimentation with mathematics, nowadays is utilized for the rigorous description of educational experiments.

The ideas about making numeric quasi-experiments as part of pre-college mathematics curricula with an emphasis on discovery learning, mathematical investigations, and drawing conclusions informed by inductive reasoning have begun gaining popularity around the world in the second part of the twentieth century. This is evidenced by a number of publications on and standards for mathematics teaching and learning (Cambridge Conference on School Mathematics 1963; Fletcher 1964; National Council of Teachers of Mathematics 1970, 1989; Peterson 1973; Wheeler 1967). According to Mason (2001), in England, this approach to mathematics can be traced back to the writings of Wallis[1] (1685) who used the word investigation to refer to 'my method of investigation' which, however, when supported by (empirical) induction alone can lead to erroneous conjectures (see Chap. 8 for examples). Therefore, it has been cautioned, "we should take great care not to accept as true such properties of the numbers which we have discovered by observation and… should use such a discovery as an opportunity to investigate more exactly the properties discovered and to prove or disprove them; in both cases we may learn something useful" (Pólya 1954, p. 3). In that, the importance of theory that augments mathematical experimentation by appropriate demonstration and formal justification was equally emphasized. Therefore, as an experiment provides basis for insight, one can conclude that observation is at the core of any experiment. By the same token, experiment leads to the development of theory, which, in turn, can inform experiment as its conditions grow in complexity. All this is true for a modern day computational experiment.

1.3 Computational Experiment and its Validation

In education, any experiment can be associated with two types of validity: internal and external (Campbell and Stanley 1963). Internal validity of experiment is characterized by the basic set of skills and abilities without which any experiment

[1] John Wallis (1616–1703)—an English mathematician whose work, in particular, provided foundation for the development of integral calculus.