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Collective Dynamics from Bacteria to Crowds

An Excursion Through Modeling, Analysis and Simulation

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Adrian Muntean
Eindhoven University of Technology, The Netherlands

Federico Toschi
Eindhoven University of Technology, The Netherlands

ISSN 0254-1971
ISBN 978-3-7091-1784-2 ISBN 978-3-7091-1785-9 (eBook)
DOI 10.1007/978-3-7091-1785-9
Springer Wien Heidelberg New York Dordrecht London

All contributions have been typeset by the authors
Printed in Italy

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

PREFACE

The motion of individual active agents (like e.g. bacteria or ant colonies, birds flocks, etc.) give rise to fascinating large scale collective behaviours. How does this large-scale emerge from the small-scale dynamics? How does the large-scale conditions influence the dynamics of the individuals? There are many fundamental scientific questions with important practical implications that have attracted the attention of various scientific communities, ranging from logistics, theoretical biology, ecology, statistical physics and mathematics. On the one hand, one would like to understand the formation of large-scale patterns in large colonies, this may be relevant to fisheries and fishing strategies optimization. On the other hand, in crowded pedestrian flows, the behaviour of individuals display significant differences from that of undisturbed free walking. Furthermore, when panic situations occur, small microscopic (i.e. individual-level) interactions can amplify leading to macroscopic patterns (e.g. shock-like waves) that can cause jamming during evacuation but also losses of human lives.

Multiscale models in social applications combine mean-field and kinetic equations with either microscopic or macroscopic level descriptions. These are approaches of strongly increasing importance with high potential for quantitative research. Typically, individual-based models need to be accurately coarse-grained to translate the relevant microstructure information to a mesoscopic (Boltzmann-level) or to a macroscopic (continuum) level. Relevant questions include: What is the natural scaling for the averaging? How much micro-level information needs to be retained in order to capture the specific individual-level interaction responsible for the formation and propagation of the macroscopically-observed patterns (for instance, lane formation in pedestrian counterflow). What are the main microscopic interactions responsible for the macroscopic transport mechanism displacing pedestrian flows?

Within this book an attempt is made to cover a limited number of these questions with an eye on multidisciplinary approach to the topics:

J.-A. Carrillo, Y.-P. Choi and M. Hauray focus on the derivation of mean-field models for swarming proving, by means of con-

verging Wasserstein distances of empirical measures, the discrete-to-continuum passage from a first-order system of interacting particles to a continuity-like equation with nonlocal kernel. Their technique is applicable to a large class of first-order interaction models.

B. Maury treats hard congestions in models for crowd motion. The hard-core part of the interactions naturally leads to non-smooth evolution systems. The handling of the contacts translates here into suitable (quasi-)variational inequalities. Rigorous numerics show that such contacts can be quantitatively evaluated.

Pedestrians moving in the dark are modeled by A. Muntean, E. Cirillo, O. Krehel, and M. Böhm in terms of Becker-Döring interaction rules for two possible kinds of scenarios: (i) a continuum PDE model in term of measures and (ii) a lattice automaton. They show that adhering to large groups is not necessarily the right thing to do if one wishes to find invisible exits.

S. Pigolotti, R. Benzi, M. Jensen, P. Perlekar, F. Toschi discuss a model for stochastic competitions of biological species in space focusing on how the macroscopic equations for individual species density can be derived within the formalism of master equations.

F. Tesser and Ch. Doering review the non-equilibrium statistical mechanics models of reaction and interaction kinetics. Among others, they show that traditional mean-field or "mass-action" reaction kinetics theories are useful but that there are also limits to their validity.

A. Tosin reviews multiscale crowd dynamics scenarios posed in terms of conservation laws for (discrete and absolutely continuous) mass measures from a threefold perspective: modeling, solvability, and approximation.

The multiscale nature of interacting particle systems gives rise to many interesting and challenging mathematical problems. In this book, the reader will find not only a wide spectrum of multiscale analysis results (like convergence proofs), but also practically important information such as derivations of mean-field equations, methods to handle hard contacts numerically, to model group behavior, to quantitative estimate microscopic/macroscopic segregation of competing species, to quantitative understand the limits of validity of mass-action kinetics for simple reactions.

Adrian Muntean and Federico Toschi

CONTENTS

The derivation of swarming models: Mean-field limit and Wasserstein distances

José Antonio Carrillo[†], Young-Pil Choi[†], and Maxime Hauray[‡]
[†] Department of Mathematics, Imperial College London, SW7 2AZ London, United Kingdom
[‡] Centre de Mathématiques et Informatique, Université d'Aix-Marseille, Technopôle Château-Gombert, Marseille, France

Abstract These notes are devoted to a summary on the mean-field limit of large ensembles of interacting particles with applications in swarming models. We first make a summary of the kinetic models derived as continuum versions of second order models for swarming. We focus on the question of passing from the discrete to the continuum model in the Dobrushin framework. We show how to use related techniques from fluid mechanics equations applied to first order models for swarming, also called the aggregation equation. We give qualitative bounds on the approximation of initial data by particles to obtain the mean-field limit for radial singular (at the origin) potentials up to the Newtonian singularity. We also show the propagation of chaos for more restricted set of singular potentials.

1 Introduction

In the last years, we have seen the development of a great deal of different models in the biology, applied mathematics, and physics literature to describe the collective behavior of individuals. Here, individuals may mean animals (insects, fish, birds,...), bacteria, and even robots. Most of these models involve the nonlocal character of the interaction as a basic modelling pillar, see for instance Camazine, Deneubourg, Franks, Sneyd, Theraulaz, and Bonabeau (2003); Couzin, Krause, Franks, and Levin (2005); Li, Lukeman, and Edelstein-Keshet (2008); Vicsek, Czirok, Ben-Jacob, Cohen, and Shochet (1995). In fact, one of largest source of collective behavior models comes from control engineering. There, the aim is to produce a suitable control of the movement of small squads of robots in order to perform unmanned vehicle operations, for instance Perea, Gómez, and Elosequi (2009). These ideas even have been proposed to model crowd motion, including

A. Muntean, F. Toschi (Eds.), *Collective Dynamics from Bacteria to Crowds*, CISM International Centre for Mechanical Sciences, DOI 10.1007/978-3-7091-1785-9_1, © CISM, Udine 2014

more "intelligent" particles deciding their movement based on optimization of certain quantities: time to exit from a room or a stadium, for instance Burger, Markowich, and Pietschmann (2011).

Either in social or in biological sciences, these models encounter many interesting features such as the spontaneous formation of different pattern behaviors. When we talk about patterns, we do not mean static patterns like in the study of crystals but rather dynamic patterns leading to the collective motion of the individual ensemble. For instance, two of the main collective motion patterns studied in different models are the flock and the milling behavior, see D'Orsogna, Chuang, Bertozzi, and Chayes (2006); Carrillo, D'Orsogna, and Panferov (2009); Cañizo, Carrillo, and Rosado (2010); Carrillo, Klar, Martin, and Tiwari (2010); Carrillo, Panferov, and Martin (2013). In the flock pattern, individuals achieve a consensus on the direction or orientation towards some objective, producing as a consequence a particular spatial shape showing their preferred comfort structure. This kind of swiftly moving flocks have been reported in many species although the most spectacular or bucolic ones are the bird flocks, starlings for instance. In the mill pattern, individuals arrange into a kind of vortex like motion around some point. This particular moving pattern has been observed in fish schools. Hundreds of movies can be easily accessed through internet search showing them.

There are many reasons one can argue, why such a large number of individuals react to external stimuli producing these macroscopic patterns without seemingly the presence of a leader in the swarm. Hydrodynamic enhancement, predators avoidance, social interactions, spawning survival rate, and many others have been proposed to explain this behavior in different species, see Parrish, and Edelstein-Keshet (1999).

One of the main question in describing this behavior by mathematical models is how to include the interaction between individuals. In any case, there is a consensus that the modelling starts from particle-like models as in statistical physics. These particle models are also called Individual-Based Models (IBMs) in the community. They are usually formed by a set of differential equations of Newton type (called 2nd order models) or by kinematic equations where the inertia terms are neglected (called first order models). Essentially, by assuming that the inertia term is negligible, we assume that individuals can adjust to the velocity field instantaneously, an approximation valid when their speed is not too large. In any case, these first order models were proposed in the literature derived in a phenomenological manner; see Mogilner, Edelstein-Keshet, Bent, and Spiros (2003); Mogilner and Edelstein-Keshet (1999); Parrish, and Edelstein-Keshet (1999); Topaz and Bertozzi (2004); Topaz, Bertozzi, and Lewis (2006); Eftimie, de Vries, and

Lewis (2007). The literature on first and second order models for swarming has increased exponentially fast in the last few years. Many of these models find also their origin in social sciences, where consensus or opinion formation was also described in similar grounds. Another typical ingredient in these models is some kind of noise leading to systems of SDEs. In this work, we will not discuss how to incorporate noise in these models, we refer to Bolley, Cañizo, and Carrillo (2011) and the references therein.

Most of these models are based on discrete approaches incorporating certain effects that we like to call the "first principles" of swarming. These first principles are based on modelling the "sociological behavior" of animals with very simple rules such as the social tendency to produce grouping (attraction/aggregation), the inherent minimal space they need to move without problems and feel comfortably inside the group (repulsion/collisional avoidance) and the mimetic adaptation or synchronization to a group (orientation/alignment). Even if these minimal models contain very basic rules, the patterns observed in their simulation and their complex asymptotic behavior are already very challenging from the mathematical viewpoint. The 3-zone models including attraction, repulsion, and alignment effects are classical in fish modelling; see Aoki (1982); Huth and Wissel (1992) for instance. Based on them, one can incorporate may other effects to render more realistic the outputs of the simulations and the models, see Barbaro, Taylor, Trethewey, Youseff, and Birnir (2009) for fish schools or Hemelrijk and Hildenbrandt (2008) for birds flocks. We also refer to the reader to the recent review Carrillo, Fornasier, Toscani, and Vecil (2010) about the kinetic modelling of swarming.

To the eyes of a kinetic theorist or a statistical physicist, studying such systems of ODEs when the number of individuals becomes large is doomed to fail. Dynamical system approaches are quite useful but they typically have huge problems to describe large systems of particles. A classical approach to attack the problem is to pass to a continuous description of the system. This means to go from particle descriptions to kinetic descriptions where the unknown is the particle density distribution in position-velocity (phase) space for 2nd order models or in position space for 1st order models.

Going from particle to continuum descriptions is one of the most classical problems in kinetic theory. It is at the basis of the derivation of the mother and father kinetic equations, namely: the Vlasov and the Boltzmann equations. A rigorous derivation of the Boltzmann equation from the Newtonian dynamics has only been given for short times (of the order of the average time of first collision), see Lanford (1974) Gallagher, St-Raymond, and Texier (2012). In that case, interactions between the particles are modelled by short-range potentials leading to collision kernels. The question of the derivation of the Boltzmann equation from particles with jump processes was also raised and solved by Kac (1956), and further results are given in the recent important work by Mischler and Mouhot (2013). The derivation of the Vlasov equation is well understood only for regular or not too singular potentials; see Braun and Hepp (1977); Neunzert (1984); Dobrushin (1979); Hauray and Jabin (2012). In fact, a full derivation of the Vlasov-Poisson system in 3D is also lacking. The problem of passing to the limit from particle to continuum models like the Vlasov equation is called the mean-field limit. This name just comes from the fact that the resulting equation is a kind of averaged version of the interaction between the large number of individuals. Moreover, the resulting equation gives the typical behavior of one isolated individual among all the others since they are assumed to be completely indistinguishable.

Finally, there are other famous mean-field limit equations, such as the Euler and the Navier-Stokes equations for incompressible fluids, see Marchioro and Pulvirenti (1994); Majda and Bertozzi (2002). It has been extensively used for numerical purposes that both equations in the 2D incompressible case can be derived from particle approximations, called vortex point approximations. The convergence in the viscous case has been rigorously proved for very general initial data; see Osada (1985); Founier, Hauray, and Mischler (2012). In the non-viscous case Schochet (1996) proves that particle approximations converge towards solutions of the Euler equation, but they may not converge to the good solution because of the lack of uniqueness in the Euler equation, see De Lellis and L. Székelyhidi (2009). However, in the case where the initial particles are equally spaced on a grid to approximate a smooth solution of the Euler equation, the convergence was shown in Goodman, Hou, and Lowengrub (1990). These vortex methods have been proven to be convergent and estimates of the error committed have been obtained in recent works using optimal transport techniques (Hauray (2009)) but not for the real Euler equation in 2D.

The aim of this work is to show in detail a particular example of the mean field limit in the case of first order models not covered in the previous literature. Nevertheless, we will first discuss some of these issues for 2nd order models summarizing results in Cañizo, Carrillo, and Rosado (2011); Bolley, Cañizo, and Carrillo (2011). We will also discuss that the spatial shape of the main patterns, flock and mills, are given by stationary solutions of the 1st order models. This gives another reason from a more conceptual mathematical viewpoint of reducing to 1st order models. Section 3 will be devoted to obtain the mean field limit to the so-called aggregation equation for singular potentials recovering some of the models studied in Bertozzi, Carrillo, and Laurent (2009); Bertozzi, Laurent, and Rosado (2010). Here, the idea is to assume that we have solutions of the model in better functional spaces due to the singularity of the potential, but we have to pay in terms of conditions on the initial distribution of particles (how they are distributed) in such a way that the particle solution converges to the continuum solution of the aggregation equation as $N \to \infty$. We will make use of similar arguments to Hauray (2009) to show the mean-field limit for first order swarming models with singular potentials up to the Newtonian singularity. In Section 4, we study a local existence of a unique L^p-solution for the aggregation equation. This complements the well-posedness theory in Bertozzi, Laurent, and Rosado (2010). Finally, Section 5 is devoted to show the propagation of chaos property for the aggregation equation. This property is very important from the physical relevance of the kinetic and aggregation models, since it states that one can derive the mean-field equations under quite generic randomly generated initial location of the particles. We are only able to show it for a more restricted set of singular potentials with respect to the mean-field limit.

2 The Dobrushin approach

2.1 Some Individual Based Models

As we described in the introduction, the modelling in swarming starts by introducing some particle models, IBMs in the jargon of this community, incorporating some of the basic effects: repulsion, attraction, and alignment. Let us discuss briefly some of these models, starting with the ones that have recently attracted more attention due to their simplicity while having a rich mathematical structure and pattern formation. One of these models was introduced by the UCLA group in D'Orsogna, Chuang, Bertozzi, and Chayes (2006) and it consists of Newton-like equations where all the effect of repulsion and attraction is encoded via a pairwise potential $W : \mathbb{R}^d \to \mathbb{R}$. A popular choice for the interaction potential W is the Morse potential given by

$$W(x) = -C_A e^{-|x|/\ell_A} + C_R e^{-|x|/\ell_R}, \tag{2.1}$$

where C_A, C_R and ℓ_A, ℓ_R are the strengths and the typical lengths of attraction and repulsion, respectively. They are chosen for having biologically reasonable potentials with $C = C_R/C_A > 1$ and $\ell_R/\ell_A < 1$, see Carrillo, Panferov, and Martin (2013) for other nice choices of the interaction potentials and a deeper discussion on the issue of biologically relevant interaction potentials. Apart from this, the other effect included is the tendency of the particles to travel asymptotically at a fixed speed as in Levine, Rappel, and Cohen (2000). Consequently, a term producing a balance between self-propulsion and friction is introduced imposing an asymptotic speed to the particles (if other effects are ignored), but it does not influence the orientation vector. The resulting ODE system reads as:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, & (i = 1, \dots, N), \\ \dfrac{dv_i}{dt} = (\alpha - \beta\,|v_i|^2)v_i - \dfrac{1}{N}\displaystyle\sum_{j\neq i} \nabla W(|x_i - x_j|), & (i = 1, \dots, N). \end{cases}$$

where α, β are nonnegative parameters, determining the asymptotic speed of particles given by $\sqrt{\alpha/\beta}$. Here, the potential has been scaled depending on the mass of each particle as in Carrillo, D'Orsogna, and Panferov (2009) and in such a way that the effect of the potential per particle diminishes while the energy is of constant order as the number of particles N diverges. This scaling is the so-called mean-field scaling, see the introduction of Bodnar and Velazquez (2012) for a nice discussion of the different scalings in first order models.

Another popular IBM including only the alignment effect is the so-called Cucker and Smale (2007) model. Each individual in the swarm changes its velocity vector based on the other individuals by adjusting/averaging their relative velocity with all the others. This averaging is weighted in such a way that closer individuals have more influence than further ones. For a system with N individuals the Cucker-Smale model reads as

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\ \dfrac{dv_i}{dt} = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} w_{ij}\,(v_j - v_i)\,, \end{cases}$$

with the *communication rate* $w(x)$ given by:

$$w_{ij} = w(x_i - x_j) = \frac{1}{(1 + |x_i - x_j|^2)^\gamma}\,,$$

for some $\gamma \geq 0$.

Associated to the above models, one can formally write the expected Vlasov-like kinetic equations as $N \to \infty$, see for instance Carrillo, D'Orsogna, and Panferov (2009), leading to

$$\partial_t f + v \cdot \nabla_x f - (\nabla W * \rho) \cdot \nabla_v f + \mathrm{div}_v((\alpha - \beta|v|^2)vf) = 0, \tag{2.2}$$

where ρ represents the macroscopic *density* of f:

$$\rho(t,x) := \int_{\mathbb{R}^d} f(t,x,v)\,dv \quad \text{for } t \geq 0, \quad x \in \mathbb{R}^d.$$

The Cucker-Smale particle model leads to the following kinetic equation:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \nabla_v \cdot [\xi[f]\, f] \,, \tag{2.3}$$

where $\xi[f](x,v,t) = (H * f)\,(x,v,t)$, with $H(x,v) = w(x)v$ and $*$ standing for the convolution in both position and velocity (x and v). We refer to Cucker and Smale (2007); Ha and Tadmor (2008); Ha and Liu (2009); Carrillo, Fornasier, Rosado, and Toscani (2010) for further discussion about this model and qualitative properties.

Moreover, quite general models incorporating the three effects previously discussed with additional ingredients, such as vision cones or topological interactions, have been considered in Carrillo, Fornasier, Toscani, and Vecil (2010); Li, Lukeman, and Edelstein-Keshet (2008); Agueh, Illner, and Richardson (2011); Albi and Pareschi (2013); Haskovec (2013). In particular Li, Lukeman, and Edelstein-Keshet (2008) consider that the N individuals follow the system:

$$\begin{cases} \dfrac{dx_i}{dt} = v_i, \\[2ex] \dfrac{dv_i}{dt} = F_i^A + F_i^I, \end{cases} \tag{2.4}$$

where F_i^A is the self-propulsion generated by the ith-individual, while F_i^I is due to interaction with the others. The interaction with other individuals can be generally modeled as:

$$F_i^I = F_i^{I,x} + F_i^{I,v} = \sum_{j=1}^{N} g_\pm(|x_i - x_j|)\frac{x_j - x_i}{|x_i - x_j|} + \sum_{j=1}^{N} h_\pm(|v_i - v_j|)\frac{v_j - v_i}{|v_i - v_j|}.$$

Here, g_+ and h_+ (g_- and h_-) are chosen when the influence comes from the front (behind), i.e., if $(x_j - x_i) \cdot v_i > 0$ (< 0); choosing $g_+ \neq g_-$ and $h_+ \neq h_-$ means that the forces from particles in front and those from particles behind are different. The sign of the functions $g_\pm(r)$ encodes the short-range repulsion and long-range attraction for particles in front of (+) and behind (-) the ith-particle. Similarly, $h_+ > 0$ (< 0) implies that the velocity-dependent force makes the velocity of particle i get closer to (away from) that of particle j.

Some of these models, for instance Agueh, Illner, and Richardson (2011); Albi and Pareschi (2013); Haskovec (2013), include sharp boundaries for the vision cone or for the interaction with the nearest neighbors. As we shall see later, these are typical situations in which the mean-field limit for general measures will not work. By sharp boundaries we mean that the functions involved in the kernels such as $w(x)$, $g_\pm$, or $h_\pm$ are given by characteristic functions on sets depending on the location/velocity of the agent.

2.2 First-order models: Aggregation Equation

In this work, the objective is to show how to obtain the continuum limits of these particle models in a simpler situation than the ones in the previous section. However, at the same time we will allow for more singular kernels. We will showcase these tools in the case of the so-called aggregation equation. Let us assume that we have just particles interacting through the pairwise potential $W(x)$. Assuming that the variations of the velocity and speed are much smaller than spatial variations, see Mogilner and Edelstein-Keshet (1999), then one can neglect the inertia term in Newton's equation to deduce that

$$\frac{dX_i}{dt} = -\sum_{j\neq i} \nabla W(X_i - X_j) \text{ in the } N \to \infty \text{ limit} \Rrightarrow \begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0 \\ u = -\nabla W * \rho \end{cases} . \tag{2.5}$$

Another reason to study this first order equation is that the stationary states of the first order model determine the spatial shape of the flock solutions to the second order models, see Carrillo, Panferov, and Martin (2013).

Let us note that some of the difficulties to overcome are already in this model. Next subsection is devoted to review the classical Dobrushin strategy for the mean-field limit when all functions involved in the model are smooth enough. This strategy applies to the aggregation equation for $\mathcal{C}^2(\mathbb{R}^d)$ smooth potential with at most quadratic growth at infinity by following the same argument as in Theorem 2.4 below. This argument was detailed in a nice summer school notes in Golse (2003). The goal of this chapter is to show how to deal with more singular potentials. The main message is that in order to obtain the mean-field limit, whose precise statement is given later on, you need to impose certain conditions on the approximation of the initial data avoiding the possible singularities (collisions) in finite time of the particles. We will elaborate on this at the begining of next section. In order to deal with these questions, it is quite convenient to work with transport distances between probability measures that we quickly review next.

2.3 Basic tools in transport distances

In this subsection, we present several definitions of Wasserstein distances and their properties.

Definition 2.1. (Wasserstein p-distance) Let $\rho_1, \ \rho_2$ be two Borel probability measures on $\mathbb{R}^d$. Then the Euclidean Wasserstein distance of order $1 \le p < \infty$ between ρ_1 and ρ_2 is defined as

$$d_p(\rho_1, \rho_2) := \inf_\gamma \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \, d\gamma(x,y) \right)^{1/p},$$

and, for $p = \infty$ (this is the limiting case, as $p \to \infty$),

$$d_\infty(\rho_1, \rho_2) := \inf_\gamma \left(\sup_{(x,y) \in \mathrm{supp}(\gamma)} |x-y| \right),$$

where the infimum runs over all transference plans, i.e., all probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ρ_1 and ρ_2 respectively,

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(x) d\gamma(x,y) = \int_{\mathbb{R}^d} \phi(x) \rho_1(x) dx,$$

and

$$\int_{\mathbb{R}^d \times \mathbb{R}^d} \phi(y) d\gamma(x,y) = \int_{\mathbb{R}^d} \phi(y) \rho_2(y) dy,$$

for all $\phi \in \mathcal{C}_b(\mathbb{R}^d)$.

We also remind the definition of the push-forward of a measure by a mapping in order to give the relation between Wasserstein distances and optimal transportation.

Definition 2.2. Let ρ_1 be a Borel measure on $\mathbb{R}^d$ and $\mathcal{T} : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping. Then the push-forward of ρ_1 by $\mathcal{T}$ is the measure ρ_2 defined by

$$\rho_2(B) = \rho_1(\mathcal{T}^{-1}(B)) \quad \text{for} \quad B \subset \mathbb{R}^d,$$

and denoted as $\rho_2 = \mathcal{T} \# \rho_1$.

The set of probability measures with bounded moments of order p, denoted by $\mathcal{P}_p(\mathbb{R}^d)$, $1 \le p < \infty$, is a complete metric space endowed with the p-Wassertein distance d_p, see Villani (2003). We refer to Givens and Shortt (1984); McCann (2006) for more details in the case of the d_∞ distance.

Remark 2.3. The definition of $\rho_2 = \mathcal{T} \# \rho_1$ is equivalent to

$$\int_{\mathbb{R}^d} \phi(x) \, d\rho_2(x) = \int_{\mathbb{R}^d} \phi(\mathcal{T}(x)) \, d\rho_1(x),$$

for all $\phi \in \mathcal{C}_b(\mathbb{R}^d)$. Given a probability measure with bounded p-th moment ρ_0, consider two measurable mappings $X_1, X_2 : \mathbb{R}^d \to \mathbb{R}^d$, then the following inequality holds.

$$d_p^p(X_1\#\rho_0, X_2\#\rho_0) \le \int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p d\gamma(x,y) = \int_{\mathbb{R}^d} |X_1(x)-X_2(x)|^p d\rho_0(x).$$

Here, we used as transference plan $\gamma = (X_1 \times X_2)\#\rho_0$ in Definition 2.1.

2.4 A quick review of the classical Dobrushin result

Under smoothness assumptions on the ingredient functions of the swarming models, one can use adaptations of the classical result of Dobrushin (1979) to obtain what is called the mean-field limit equation for general particle approximations of any initial measure. These arguments are classical in kinetic theory and were also introduced in Braun and Hepp (1977); Neunzert (1984), making use of the bounded Lipschitz distance, and reviewed in Spohn (1991); Villani (2002), see also Sznitman (1991); Méléard (1996) for the case with noise. The bounded Lipschitz distance or dual $W^{1,\infty}$-norm is equivalent to the Wasserstein distance d_1 for compactly supported measures. This strategy works as soon as the velocity field defining the characteristics of the model is a bounded and globally Lipschitz function whose dependence on the measure itself is Lipschitz continuous in the d_1 sense. These ideas were improved to allow for locally Lipschitz velocity fields for compactly supported initial measures in Cañizo, Carrillo, and Rosado (2011) and for suitable decay conditions at infinity and with noise in Bolley, Cañizo, and Carrillo (2011). With these techniques one can include quite general kinetic models for swarming in this well-posedness theory.

Let us introduce some notation for this section: $\mathcal{A} = \mathcal{P}_c(\mathbb{R}^d\times\mathbb{R}^d)$ denotes the subset of $\mathcal{P}(\mathbb{R}^d\times\mathbb{R}^d)$ consisting of measures of compact support in $\mathbb{R}^d\times\mathbb{R}^d$. On the other hand, we consider the set of functions $\mathcal{B} := \mathrm{Lip}_{loc}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$, which in particular are locally Lipschitz with respect to (x, v). B_R will denote the ball centered at 0 of radius R in $\mathbb{R}\times\mathbb{R}$.

Let us consider general operators from measures to vector fields, $\mathcal{H}[\cdot] : \mathcal{A} \to \mathcal{B}$, satisfying the following hypotheses: for any $R_0 > 0$ and $f, g \in \mathcal{A}$ such that $\operatorname{supp} f \cup \operatorname{supp} g \subseteq B_{R_0}$, there exists some ball $B_R \subset \mathbb{R}^d\times\mathbb{R}^d$ and a constant $C = C(R, R_0) > 0$, such that

$$\|\mathcal{H}[f] - \mathcal{H}[g]\|_{L^\infty(B_R)} \le C\, d_1(f, g), \tag{2.6}$$

$$\mathrm{Lip}_R(\mathcal{H}[f]) \le C, \qquad \|\mathcal{H}[f]\|_{L^\infty(B_R)} \le C. \tag{2.7}$$

Here, $\mathrm{Lip}_R(\cdot)$ denotes the Lipschitz constant of a function in B_R.

Given $f \in \mathcal{C}([0,T],\mathcal{P}_c(B_{R_0}))$, and for any initial condition $(X^0,V^0) \in \mathbb{R}^d \times \mathbb{R}^d$, the following system of ordinary differential equations has a unique locally defined solution

$$\frac{d}{dt}X = V, \qquad X(0) = X^0 \tag{2.8a}$$

$$\frac{d}{dt}V = \mathcal{H}[f(t)](X,V), \qquad V(0) = V^0. \tag{2.8b}$$

We will additionally require that the solutions to that system are "global". More precisely, we assume that for any $R_0, T > 0$, there exists $R > 0$ such that $(X(t),V(t)) \in B_R$ for all $t \in [0,T]$ and all $(X^0,V^0) \in B_{R_0}$. Of course, this is a requirement that has to be checked for every particular model. We prefer to give a general condition which reduces the problem of existence and stability to the simpler one of existence of the ODEs. Under the above conditions, the existence and uniqueness of associated transport equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_v \cdot [\mathcal{H}[f]f] = 0. \tag{2.9}$$

was obtained in Cañizo, Carrillo, and Rosado (2011) to which we refer for full details. In Cañizo, Carrillo, and Rosado (2011), the interactions $\mathcal{H}[f] = (\alpha - \beta|v|^2)v - \nabla W * \rho$ and $\mathcal{H}[f] = H * f$ corresponding to (2.2) and (2.3), respectively, and

$$\mathcal{H}[f] = F_A(x,v) + G(x) * \rho + H(x,v) * f,$$

with F_A, G and H given functions satisfying suitable hypotheses, such that the kinetic equation (2.9) corresponds to the model (2.4) are investigated.

Theorem 2.4. *Given an operator $\mathcal{H}[\cdot] : \mathcal{A} \to \mathcal{B}$ satisfying Hypotheses* (2.6) *and* (2.7) *for which the characteristics* (2.8a)-(2.8b) *are globally well-defined, and f_0 a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with compact support. There exists a solution f on $[0,+\infty)$ to equation* (2.9) *with initial condition f_0. In addition,*

$$f \in \mathcal{C}([0,+\infty);\mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)) \tag{2.10}$$

and there is some increasing function $R = R(T)$ such that for all $T > 0$,

$$\operatorname{supp} f_t \subseteq B_{R(T)} \subseteq \mathbb{R}^d \times \mathbb{R}^d \quad \textit{for all } t \in [0,T]. \tag{2.11}$$

This solution is unique among the family of solutions satisfying (2.10) *and* (2.11). *Moreover, given any other initial data $g_0 \in \mathcal{P}_c(\mathbb{R}^d \times \mathbb{R}^d)$ and g its corresponding solution, there exists a strictly increasing function $r(t)$:*