CISM COURSES AND LECTURES NO. 495
INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

WAVES IN NONLINEAR PRE-STRESSED MATERIALS

EDITED BY

MICHEL DESTRADE
GIUSEPPE SACCOMANDI

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PREFACE

Pre-stressed bodies are ubiquitous in technical applications and in several fields of science. For instance in biomechanics, the mechanical behaviour in service of many soft biological tissues (such as arterial walls, veins, skin, tendons, etc.) can be explained by modelling them as pre-stressed viscoelastic materials. In civil engineering, bridge bearings or seismic shock absorbers under a building are clear examples of devices operating in conditions of pre-stress, and sometimes subject to dramatically large strains. Other examples can be found in the automotive industry, in seismology, in oil prospecting, in non-destructive ultrasonic evaluation, in high frequency signal processing for electronic devices, in fibers optics, etc.

The study of wave motion is a natural and revealing approach to the properties of a pre-stressed body. Indeed, acoustic waves may be used to evaluate the material parameters of a given elastic body or, if these are known, to evaluate the state of induced anisotropy or of residual stress (in fact, they may well be the only way to evaluate the mechanical characteristics of soft tissues in vivo). They may also be used to detect structural defects. Other major interests include the study of standing waves, with applications to stability and bifurcation analyses, and the study of nonlinear waves, with applications to shock formations and solitary waves generation. Hence, the understanding of the mathematics and of the mechanics of dynamical problems in pre-stressed elastic and viscoelastic materials is of paramount importance to many applications. Nevertheless, a recent comprehensive synthetic textbook is lacking in this field.

The aim of these lecture notes is to take a first step toward the eventual elaboration of such a reference volume, by providing a unique, state-of-the-art, multidisciplinary overview on the subject of linear, linearized, and nonlinear waves in pre-stressed materials. This is achieved through the interaction of several topics, ranging from the mathematical modelling of incremental material elastic response, to the analysis of the governing differential equations and related boundary-value problems, and to computational methods for the numerical solution to these problems, with particular reference to industrial, geophysical, and biomechanical applications. We have tried to achieve this goal by including:

- *A unified introduction to wave propagation (small-on-large and large-on-large);*
- *The basic and fundamental theoretical issues (mechanical modelling, exact solutions, asymptotic methods, numerical treatment);*
- *A perspective on classical (such as geophysics), current (such as the mechanics of rubber-like solids), and emergent (such as nonlinear solid biomechanics) applications.*

These Lectures Notes originate from a course held in September 2006 at the Centre International des Sciences Mécaniques (CISM) in Udine, Italy. As always, the CISM staff and the Rector, currently Professor G. Maier, have been very helpful and professional in coping with all the technical aspects of the course. The beautiful city of Udine and the magnificent Palazzo del Torso together provided a perfect location for the course. The Editors are eternally thankful to the Lecturers and to the Participants, without whom the Course would not have taken place and these Notes would not exist. Finally, they are glad to dedicate this volume in honour of the sixtieth birthday of Tommaso Ruggeri, a most sophisticated researcher in wave propagation.

Michel Destrade and Giuseppe Saccomandi

CONTENTS

Incremental Statics and Dynamics of Pre-Stressed Elastic Materials

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Abstract. In this chapter we provide a summary of the equations governing the incremental deformations superimposed on a finite deformation of an elastic solid. For the equilibrium equations the incremental theory is built on top of the underlying finite deformation theory, which includes discussion of constitutive laws for isotropic materials and for anisotropy associated with one or two preferred directions. Following the static theory the corresponding dynamic equations are summarized. The resulting equations for incremental motions superimposed on a static finite deformation are then used to examine some basic problems in the propagation of incremental plane waves in pre-stressed elastic solids in order to illustrate the influence of the pre-stress and the associated finite deformation on the wave propagation characteristics.

1 Introduction

This chapter provides the basic equations of nonlinear elasticity theory, both static and dynamic, that underpin the applications examined in the other chapters of this volume. More general and detailed treatments of this background material can be found in, for example, Ogden (1997) and Holzapfel (2001). In Section 2 the basic equations for nonlinear elastostatics are reviewed, including a discussion of the constitutive laws for isotropic elastic solids, both incompressible and unconstrained, and of corresponding constitutive laws for elastic solids with one or two preferred directions in their reference configuration. The concept of *strong ellipticity* is introduced in this section. Section 3 is concerned with the derivation of the (linearized) incremental equations for elastostatics; it furnishes, in particular, expressions for the components of the elasticity tensor for the materials examined in Section 2. These expression are required for the analysis of the incremental equations of motion, which are summarized in Section 4. Also in Section 4 is a short account of the interpretation of the strong ellipticity condition in the context of homogeneous plane wave propagation. As a simple application of the incremental equations of motion some aspects of plane wave propagation are examined in Section 5. In particular, the influence of a pure homogeneous strain on the reflection of plane waves at the boundary of a half-space is analyzed briefly in Section 5 along with a corresponding analysis of the reflection and transmission of plane waves at the interface between two pure homogeneously strained half-spaces. Finally, Section 6 contains some concluding remarks.

2 Basic equations of finite deformation elastostatics

2.1 Kinematics

Consider a continuous body occupying a connected open subset of a three-dimensional Euclidean point space. Such a subset is referred to as a *configuration* of the body. A specific, but arbitrarily chosen, configuration is identified as a *reference configuration*, which is denoted $\mathcal{B}_r$, points of which are labelled by their position vectors $\mathbf{X}$ relative to some origin O (see Figure 1). The boundary of $\mathcal{B}_r$ is denoted $\partial\mathcal{B}_r$.

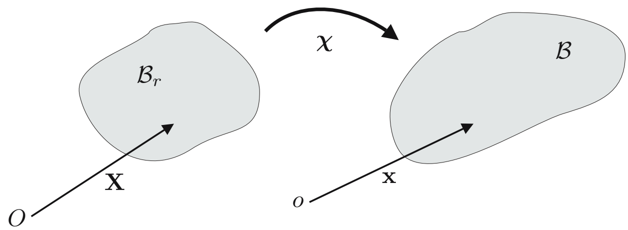

Figure 1. Depiction of the reference configuration $\mathcal{B}_r$ and the deformed configuration $\mathcal{B}$ under the deformation mapping $\boldsymbol{\chi}$. Material particles are labelled by the position vector $\mathbf{X}$ in $\mathcal{B}_r$ and located at position vector $\mathbf{x}$ in $\mathcal{B}$.

The body is deformed quasi-statically from $\mathcal{B}_r$ so that it occupies a new configuration, denoted $\mathcal{B}$, with boundary $\partial\mathcal{B}$. We refer to $\mathcal{B}$ as the *deformed configuration* of the body, and the deformation is represented by the mapping $\boldsymbol{\chi}:\mathcal{B}_r \rightarrow \mathcal{B}$ which takes points $\mathbf{X}$ in $\mathcal{B}_r$ to points $\mathbf{x}$ in $\mathcal{B}$. Thus,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}), \quad \mathbf{X} \in \mathcal{B}_r, \tag{2.1}$$

$\mathbf{x}$ being the position vector in $\mathcal{B}$ (relative to some origin o) of the material point $\mathbf{X}$. The deformation $\boldsymbol{\chi}$ and its inverse $\boldsymbol{\chi}^{-1}$ are required to be one-to-one and to satisfy appropriate regularity conditions.

Let $\mathbf{X}$ and $\mathbf{x}$ have rectangular Cartesian coordinates X_α and x_i, respectively, where α, $i \in \{1, 2, 3\}$, so that $x_i = \chi_i(X_\alpha)$, and we emphasize that Greek and Roman indices refer, respectively, to $\mathcal{B}_r$ and $\mathcal{B}$. The usual summation convention for repeated indices is also adopted. Throughout this chapter the word 'components' will always signify 'Cartesian components' (of vectors and tensors).

The *deformation gradient tensor* is

$$\mathbf{F} = \mathrm{Grad}\,\mathbf{x} \equiv \mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}) \tag{2.2}$$

with components $F_{i\alpha} = \partial x_i/\partial X_\alpha$, where Grad is the gradient operator in $\mathcal{B}_r$, and we adopt the usual convention that $J \equiv \det\mathbf{F} > 0$, thereby defining the notation J. The important role of J is that it is a local measure of change in material volume and it features in the mass conservation equation in the form $\rho_r = J\rho$, where ρ_r and ρ are the mass densities of the material in $\mathcal{B}_r$ and $\mathcal{B}$, respectively.

For an isochoric (volume preserving) deformation,

$$J = \det\mathbf{F} = 1. \tag{2.3}$$

For an incompressible material all deformations are constrained to be isochoric and (2.3) then forms the incompressibility constraint.

We record here the *polar decompositions*

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \tag{2.4}$$

of $\mathbf{F}$, where $\mathbf{R}$ is a proper orthogonal tensor and $\mathbf{U}$ and $\mathbf{V}$ are positive definite, symmetric tensors, the latter two referred to, respectively, as the *right* and *left stretch tensors*. In spectral form $\mathbf{U}$ and $\mathbf{V}$ can be decomposed as

$$\mathbf{U} = \sum_{i=1}^{3} \lambda_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \quad \mathbf{V} = \sum_{i=1}^{3} \lambda_i \mathbf{v}^{(i)} \otimes \mathbf{v}^{(i)}, \tag{2.5}$$

where $\lambda_i > 0$, $i \in \{1, 2, 3\}$, are the *principal stretches*, and $\mathbf{u}^{(i)}$ and $\mathbf{v}^{(i)}$, respectively, the (unit) eigenvectors of $\mathbf{U}$ and $\mathbf{V}$, are the *Lagrangian and Eulerian principal axes*, and $\otimes$ denotes the tensor product. We also note the connection

$$\mathbf{v}^{(i)} = \mathbf{R}\mathbf{u}^{(i)}, \quad i \in \{1, 2, 3\} \tag{2.6}$$

and that alternative expressions for $J = \det \mathbf{F}$ are provided by

$$J = \det \mathbf{U} = \det \mathbf{V} = \lambda_1 \lambda_2 \lambda_3. \tag{2.7}$$

The *right* and *left Cauchy-Green deformation tensors*, defined by

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{U}^2, \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{V}^2, \tag{2.8}$$

have important roles to play in the formation of constitutive laws, in particular through their principal invariants, defined by (for either $\mathbf{C}$ or $\mathbf{B}$)

$$I_1 = \mathrm{tr}\,\mathbf{C}, \quad I_2 = \tfrac{1}{2}[I_1^2 - \mathrm{tr}\,(\mathbf{C}^2)], \quad I_3 = \det \mathbf{C}. \tag{2.9}$$

In terms of the stretches, these are

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \quad I_2 = \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 + \lambda_1^2\lambda_2^2, \quad I_3 = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.10}$$

A useful alternative choice of invariants is provided by

$$i_1 = \mathrm{tr}\,\mathbf{U}, \quad i_2 = i_3 \mathrm{tr}\,(\mathbf{U}^{-1}), \quad i_3 = \det \mathbf{U}, \tag{2.11}$$

or, equivalently, in terms of the stretches

$$i_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad i_2 = \lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{2.12}$$

2.2 Stress tensors and equilibrium equations

In this chapter we use three stress tensors: the Cauchy stress $\boldsymbol{\sigma}$ (symmetric), the nominal stress tensor, denoted $\mathbf{S}$ (not in general symmetric) and related to $\boldsymbol{\sigma}$ by

$$\mathbf{S} = J\mathbf{F}^{-1}\boldsymbol{\sigma}, \tag{2.13}$$

and the Biot stress tensor (symmetric), denoted $\mathbf{T}$ and given by

$$\mathbf{T} = \tfrac{1}{2}(\mathbf{SR} + \mathbf{R}^T\mathbf{S}^T). \tag{2.14}$$

The symmetry

$$\mathbf{S}^T\mathbf{F}^T = \mathbf{FS} \tag{2.15}$$

is also noted.

In the present work we shall not be concerned with body forces, in which case the equilibrium equation can be written in either of the two equivalent forms

$$\mathrm{div}\,\boldsymbol{\sigma} = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{S} = \mathbf{0}, \tag{2.16}$$

where div and Div signify the divergence operator with respect to $\mathcal{B}$ and $\mathcal{B}_r$, respectively. In component form these are

$$\frac{\partial \sigma_{ij}}{\partial x_j} = 0, \quad \frac{\partial S_{\alpha i}}{\partial X_\alpha} = 0. \tag{2.17}$$

2.3 Elasticity

Here we consider the properties of an elastic material to be characterized by a *strain-energy function*, denoted $W = W(\mathbf{F})$, defined per unit volume on the space of deformation gradients. For an unconstrained material the nominal and Cauchy stress tensors are given by

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}, \quad \boldsymbol{\sigma} = J^{-1}\mathbf{F}\frac{\partial W}{\partial \mathbf{F}}, \tag{2.18}$$

and in components by

$$S_{\alpha i} = \frac{\partial W}{\partial F_{i\alpha}}, \quad \sigma_{ij} = J^{-1}F_{i\alpha}\frac{\partial W}{\partial F_{j\alpha}}. \tag{2.19}$$

We assume that W vanishes in $\mathcal{B}_r$ and that $\mathcal{B}_r$ is stress free, so that

$$W(\mathbf{I}) = 0, \quad \frac{\partial W}{\partial \mathbf{F}}(\mathbf{I}) = \mathbf{O}, \tag{2.20}$$

where $\mathbf{I}$ is the identity tensor.

On use of $(2.19)_1$ the equilibrium equation $(2.17)_2$ may be written as

$$\mathcal{A}_{\alpha i\beta j}\frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta} = 0, \tag{2.21}$$

where $\mathcal{A}_{\alpha i\beta j}$ are the components of the elasticity tensor $\boldsymbol{\mathcal{A}}$ and defined by

$$\mathcal{A}_{\alpha i\beta j} = \mathcal{A}_{\beta j\alpha i} = \frac{\partial^2 W}{\partial F_{i\alpha}\partial F_{j\beta}}. \tag{2.22}$$

Equations (2.21) form a quasi-linear system with the coefficients $\mathcal{A}_{\alpha i\beta j}$ being in general nonlinear functions of the components of $\mathbf{F}$. The system is strongly elliptic if the *strong ellipticity condition*

$$\mathcal{A}_{\alpha i\beta j} m_i m_j N_\alpha N_\beta > 0 \tag{2.23}$$

holds for all non-zero vectors $\mathbf{m}$ and $\mathbf{N}$, $\mathbf{m}$ being Eulerian with components m_i, $i \in \{1,2,3\}$, and $\mathbf{N}$ Lagrangian with components N_α, $\alpha \in \{1,2,3\}$.

For later reference we define $\mathbf{n}$ by $\mathbf{n} = \mathbf{F}^{-T}\mathbf{N}$ and $\boldsymbol{\mathcal{A}}_0$ to be the Eulerian elasticity tensor (the push forward of $\boldsymbol{\mathcal{A}}$) with components defined by

$$\mathcal{A}_{0piqj} = J^{-1} F_{p\alpha} F_{q\beta} \mathcal{A}_{\alpha i\beta j}, \quad \mathbf{F}^T\mathbf{n} = \mathbf{N}, \tag{2.24}$$

so that the strong ellipticity condition (2.23) may also be expressed as

$$\mathcal{A}_{0piqj} m_i m_j n_p n_q > 0 \tag{2.25}$$

for all non-zero vectors $\mathbf{m}$ and $\mathbf{n}$.

We now record the counterparts of the above equations for the case of an incompressible material. The modifications of (2.18) appropriate for incompressibility are

$$\mathbf{S} = \frac{\partial W}{\partial \mathbf{F}} - p\,\mathbf{F}^{-1}, \quad \boldsymbol{\sigma} = \mathbf{F}\frac{\partial W}{\partial \mathbf{F}} - p\mathbf{I}, \tag{2.26}$$

coupled with $\det \mathbf{F} = 1$, where p is a Lagrange multiplier associated with the incompressibility constraint and referred to as an *arbitrary hydrostatic pressure.*

The equilibrium $\operatorname{Div}\mathbf{S} = \mathbf{0}$ now takes on the component form

$$\mathcal{A}_{\alpha i\beta j}\frac{\partial^2 x_j}{\partial X_\alpha \partial X_\beta} - \frac{\partial p}{\partial x_i} = 0. \tag{2.27}$$

The strong ellipticity condition is unchanged in form and, in particular, (2.25) remains valid except that now $\mathbf{m}$ and $\mathbf{n}$ are subject to the restriction

$$\mathbf{m}\cdot\mathbf{n} = 0, \tag{2.28}$$

which is a consequence of the incompressibility constraint. The strong ellipticity condition will play an important part in the discussion of wave propagation in later sections.

Finally in this subsection we mention some boundary conditions that are to be appended to the equilibrium equations in the formulation of boundary-value problems. Some examples are the placement boundary condition

$$\mathbf{x} = \boldsymbol{\xi}(\mathbf{X}) \quad \text{on } \partial\mathcal{B}_r, \tag{2.29}$$

where $\boldsymbol{\xi}$ is a prescribed function of $\mathbf{X}$, and the traction boundary condition

$$\mathbf{S}^T\mathbf{N} = \boldsymbol{\tau}(\mathbf{F}, \mathbf{X}) \quad \text{on } \partial\mathcal{B}_r, \tag{2.30}$$

where $\boldsymbol{\tau}$ is a prescribed function of in general both $\mathbf{X}$ and $\mathbf{F}$. If $\boldsymbol{\tau}$ is independent of $\mathbf{F}$ the traction is said to be of *dead-load* type. In the case of *pressure loading*, with pressure P, $\boldsymbol{\tau}$ has the explicit form $\boldsymbol{\tau} = -JP\mathbf{F}^{-T}\mathbf{N}$, where $\mathbf{N}$ is the unit outward normal to $\partial\mathcal{B}_r$. We recall here Nanson's formula $\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{N}\mathrm{d}A$ connecting surface area elements $\mathrm{d}A$ and $\mathrm{d}a$ on $\partial\mathcal{B}_r$ and $\partial\mathcal{B}$, respectively, $\mathbf{n}$ being the unit outward normal to $\partial\mathcal{B}$.

2.4 Objectivity and material symmetry

The strain-energy function is required to be objective, which means that, for an arbitrary deformation gradient $\mathbf{F}$, it must satisfy

$$W(\mathbf{QF}) = W(\mathbf{F}) \tag{2.31}$$

for *all* proper orthogonal tensors (rotations) $\mathbf{Q}$. One implication of this follows on use of the polar decomposition (2.4) and the choice $\mathbf{Q} = \mathbf{R}^T$ in (2.31), leading to

$$W(\mathbf{F}) = W(\mathbf{U}). \tag{2.32}$$

Thus, W may therefore be defined on the class of positive definite symmetric tensors. Since the associated strain tensor $\mathbf{U} - \mathbf{I}$ is conjugate to the Biot stress tensor $\mathbf{T}$, we have

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{U}} \tag{2.33}$$

for an unconstrained material and, for an incompressible material,

$$\mathbf{T} = \frac{\partial W}{\partial \mathbf{U}} - p\mathbf{U}^{-1}, \quad \det \mathbf{U} = 1. \tag{2.34}$$

It is worth noting here that $\mathbf{U}$ is indifferent to rotations $\mathbf{Q}$ in $\mathcal{B}$ and hence, when expressed as a function of $\mathbf{U}$ (or, equivalently, $\mathbf{C}$) the strain energy is guaranteed to be objective.

Isotropic elasticity. For definiteness we now consider *isotropic elastic materials*, for which the strain-energy function is indifferent to rotations $\mathbf{Q}$ prior to deformation, i.e. in $\mathcal{B}_r$. This means that the symmetry group of the material is the *proper orthogonal group* and we have

$$W(\mathbf{FQ}) = W(\mathbf{F}) \tag{2.35}$$

for *all* rotations $\mathbf{Q}$ at any given deformation gradient $\mathbf{F}$. Note that the $\mathbf{Q}$'s in (2.32) and (2.35) are independent and the combination of these two equations therefore yields

$$W(\mathbf{QUQ}^T) = W(\mathbf{U}) \tag{2.36}$$

for all rotations $\mathbf{Q}$. Thus, W is an *isotropic function* of $\mathbf{U}$ (equivalently of $\mathbf{C}$). It follows from the spectral decomposition (2.5) that W depends on $\mathbf{U}$ only through the principal stretches $\lambda_1, \lambda_2, \lambda_3$ and is a symmetric function, i.e.

$$W(\lambda_1, \lambda_2, \lambda_3) = W(\lambda_1, \lambda_3, \lambda_2) = W(\lambda_2, \lambda_1, \lambda_3). \tag{2.37}$$

Equivalently, bearing in mind that the principal invariants of $\mathbf{C}$ are themselves symmetric functions of the stretches, we may regard W as a function of I_1, I_2, I_3.

One consequence of isotropy is that $\mathbf{T}$ is *coaxial* with $\mathbf{U}$ (and $\boldsymbol{\sigma}$ with $\mathbf{V}$) and hence, in parallel with $(2.5)_1$, we have

$$\mathbf{T} = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{u}^{(i)}, \tag{2.38}$$

where t_i, $i \in \{1, 2, 3\}$ are the *principal Biot stresses*. For an unconstrained material,

$$t_i = \frac{\partial W}{\partial \lambda_i}, \tag{2.39}$$

where W is treated as a function of the stretches. The corresponding principal Cauchy stresses σ_i, $i \in \{1, 2, 3\}$, are given by

$$J\sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} \quad \text{(no sum over } i\text{)}. \tag{2.40}$$

If, by contrast, W is treated as a function of the principal invariants I_1, I_2, I_3 then the Cauchy stress is given by

$$I_3^{1/2}\boldsymbol{\sigma} = 2I_3 \frac{\partial W}{\partial I_3}\mathbf{I} + 2\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)\mathbf{B} - 2\frac{\partial W}{\partial I_2}\mathbf{B}^2. \tag{2.41}$$

This is sometimes referred to as the Rivlin representation. Note that for an isotropic material we have $J\mathbf{R}^T\boldsymbol{\sigma}\mathbf{R} = \mathbf{TU} = \mathbf{UT}$, $\mathbf{SR}$ is symmetric and we have the decomposition

$$\mathbf{S} = \mathbf{TR}^T = \sum_{i=1}^{3} t_i \mathbf{u}^{(i)} \otimes \mathbf{v}^{(i)}. \tag{2.42}$$

For an incompressible material equation (2.38) still holds but in this case the principal stresses are given by

$$t_i = \frac{\partial W}{\partial \lambda_i} - p\lambda_i^{-1}, \quad \sigma_i = \lambda_i \frac{\partial W}{\partial \lambda_i} - p, \quad \lambda_1\lambda_2\lambda_3 = 1, \tag{2.43}$$

while (2.41) is replaced by

$$\boldsymbol{\sigma} = -p\mathbf{I} + 2\left(\frac{\partial W}{\partial I_1} + I_1 \frac{\partial W}{\partial I_2}\right)\mathbf{B} - 2\frac{\partial W}{\partial I_2}\mathbf{B}^2, \tag{2.44}$$

with $I_3 = 1$.

Anisotropic elasticity with one or two preferred directions. The simplest form of anisotropy is that associated with a material that has a single preferred direction in the reference configuration, which may be associated with fibre reinforcement. Here, we illustrate the structure of the strain-energy function of an anisotropic elastic solid for such a material and also for the case of a material with two preferred directions. Some of the corresponding stress tensors are also provided.

For a single preferred direction (transverse isotropy) we take the preferred direction in the reference configuration to be characterized by the unit vector $\mathbf{M}$. The resulting material response is unaffected by an arbitrary rotation about the direction $\mathbf{M}$ or by reversal of $\mathbf{M}$. Thus, $W(\mathbf{QF}) = W(\mathbf{F})$ for all rotations $\mathbf{Q}$ such that $\mathbf{QM} = \pm\mathbf{M}$. The strain-energy function then depends on $\mathbf{F}$ and $\mathbf{M}$ through the tensors $\mathbf{C}$ and $\mathbf{M} \otimes \mathbf{M}$ (see, for example, Spencer (1972, 1984)). More specifically, W is an isotropic function of $\mathbf{C}$ and $\mathbf{M} \otimes \mathbf{M}$. As a result, W depends on just five invariants, namely

$$I_1,\ I_2,\ I_3,\ I_4 = \mathbf{M} \cdot (\mathbf{CM}),\ I_5 = \mathbf{M} \cdot (\mathbf{C}^2\mathbf{M}), \tag{2.45}$$

where I_1, I_2, I_3 are defined in (2.9).

For an unconstrained material the nominal stress tensor is then given by

$$\begin{aligned}\mathbf{S} = 2W_1\mathbf{F}^T + 2W_2(I_1\mathbf{I} - \mathbf{C})\mathbf{F}^T + 2I_3W_3\mathbf{F}^{-1} + 2W_4\mathbf{M} \otimes \mathbf{FM} \\ + 2W_5(\mathbf{M} \otimes \mathbf{FCM} + \mathbf{CM} \otimes \mathbf{FM}),\end{aligned} \tag{2.46}$$

where $W_i = \partial W/\partial I_i, i = 1, \ldots, 5$. The corresponding Cauchy stress is calculated via $J\boldsymbol{\sigma} = \mathbf{FS}$, while for an incompressible material the dependence on I_3 is omitted and the Cauchy stress tensor is given by

$$\begin{aligned}\boldsymbol{\sigma} = -p\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + 2W_4\mathbf{FM} \otimes \mathbf{FM} \\ + 2W_5(\mathbf{FM} \otimes \mathbf{BFM} + \mathbf{BFM} \otimes \mathbf{FM}),\end{aligned} \tag{2.47}$$

wherein $\mathbf{B}$ is the left Cauchy-Green tensor.

If there is a second preferred direction, defined by the unit vector field $\mathbf{M}'$, then further invariants are introduced. These are

$$I_6 = \mathbf{M}' \cdot (\mathbf{CM}'),\ I_7 = \mathbf{M}' \cdot (\mathbf{C}^2\mathbf{M}'),\ I_8 = \mathbf{M} \cdot (\mathbf{CM}'), \tag{2.48}$$

where I_6 and I_7 are the counterparts for $\mathbf{M}'$ of I_4 and I_5, and I_8 provides a coupling between the two preferred directions. The strain-energy function should in general depend on all the above invariants, but to ensure that the material response is indifferent to reversal of either $\mathbf{M}$ or $\mathbf{M}'$ we note that I_8 should appear in the combination $I_8\mathbf{M} \cdot \mathbf{M}'$. These are the only independent invariants for three-dimensional deformations, and their number reduces for restrictions to two-dimensional geometries.

For illustration we give here the expansion for the Cauchy stress tensor in the case of an incompressible material, which is

$$\begin{aligned}\boldsymbol{\sigma} = -p\mathbf{I} + 2W_1\mathbf{B} + 2W_2(I_1\mathbf{B} - \mathbf{B}^2) + 2W_4\mathbf{FM} \otimes \mathbf{FM} \\ + 2W_5(\mathbf{FM} \otimes \mathbf{BFM} + \mathbf{BFM} \otimes \mathbf{FM}) + 2W_6\mathbf{FM}' \otimes \mathbf{FM}' \\ + 2W_7(\mathbf{FM}' \otimes \mathbf{BFM}' + \mathbf{BFM}' \otimes \mathbf{FM}') + W_8(\mathbf{FM} \otimes \mathbf{FM}' + \mathbf{FM}' \otimes \mathbf{FM}),\end{aligned} \tag{2.49}$$

where the notation $W_i = \partial W/\partial I_i$ now applies for $i = 1, 2, 4, \ldots, 8$.

2.5 Homogeneous deformations

Homogeneous deformations have an important role in the understanding of basic wave propagation properties in pre-stressed materials and these are now considered here. A homogeneous deformation is one for which the deformation gradient **F** is independent of **X**. An example is *pure homogeneous strain*, which is described by the equations

$$x_1 = \lambda_1 X_1, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3, \tag{2.50}$$

where $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches, independent of **X**. The principal axes of **U** and **V** coincide with the Cartesian coordinate directions as the values of the stretches are varied.

For an unconstrained isotropic elastic material the associated principal Cauchy stresses are given by (2.40). For an incompressible material we have the constraint

$$\lambda_1\lambda_2\lambda_3 = 1 \tag{2.51}$$

and the principal Cauchy stresses are given by $(2.43)_2$. Since then only two stretches can be varied independently it is convenient to express the strain energy as a function of two independent stretches. We therefore define

$$\hat{W}(\lambda_1, \lambda_2) = W(\lambda_1, \lambda_2, \lambda_1^{-1}\lambda_2^{-1}). \tag{2.52}$$

On elimination of p from equations $(2.43)_2$ we obtain

$$\sigma_1 - \sigma_3 = \lambda_1 \frac{\partial \hat{W}}{\partial \lambda_1}, \quad \sigma_2 - \sigma_3 = \lambda_2 \frac{\partial \hat{W}}{\partial \lambda_2}. \tag{2.53}$$

Another homogeneous deformation of interest is that of *simple shear*. This may be defined by

$$x_1 = X_1 + \gamma X_2, \quad x_2 = X_2, \quad x_3 = X_3, \tag{2.54}$$

where γ is the *amount of shear*, which is independent of **X**. The invariants (2.10) for this plane deformation reduce to $I_1 = I_2 = 3 + \gamma^2, I_3 = 1$. From (2.44) the components of $\boldsymbol{\sigma}$ for an incompressible material are then easily calculated as

$$\sigma_{11} = -p + 2(1+\gamma^2)W_1 + 2(2+\gamma^2)W_2, \quad \sigma_{22} = -p + 2W_1 + 4W_2, \tag{2.55}$$

$$\sigma_{12} = 2\gamma(W_1 + W_2), \quad \sigma_{33} = -p + 2W_1 + 2(2+\gamma^2)W_2, \quad \sigma_{13} = \sigma_{23} = 0. \tag{2.56}$$

For simple shear the orientations of the principal axes of **U** and **V** are different and depend on the magnitude of γ. In particular, in the (X_1, X_2)-plane the Eulerian principal axes $\mathbf{v}^{(1)}, \mathbf{v}^{(2)}$ are given by

$$\mathbf{v}^{(1)} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2, \quad \mathbf{v}^{(2)} = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2, \tag{2.57}$$

where $\mathbf{e}_1, \mathbf{e}_2$ are the Cartesian axes and the angle ϕ is given by

$$\tan 2\phi = 2/\gamma. \tag{2.58}$$

The associated principal stretches are $\lambda \equiv \lambda_1, \lambda_2 = \lambda^{-1}, \lambda_3 = 1$, and λ is related to γ by

$$\lambda - \lambda^{-1} = \gamma, \tag{2.59}$$

where we have taken $\lambda \geq 1$ to correspond to $\gamma \geq 0$.

In view of the dependence of I_1 and I_2 on γ we may treat W as a function of γ and define

$$\bar{W}(\gamma) = \hat{W}(\lambda, \lambda^{-1}), \tag{2.60}$$

and it follows that

$$\sigma_{12} = \bar{W}'(\gamma), \quad \sigma_{11} - \sigma_{22} = \gamma\sigma_{12}. \tag{2.61}$$

The homogeneous deformations of pure strain and simple shear may also be considered for the anisotropic materials discussed in Section 2.4. For an incompressible material, for example, the components of $\boldsymbol{\sigma}$ can be read off from (2.47) or (2.49) for appropriate choices of $\mathbf{M}$ and $\mathbf{M}'$. For details we refer to Ogden (2001), for example.

3 Incremental deformations and stresses

Let $\boldsymbol{\chi}$, with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, be a known finite deformation and let $\boldsymbol{\chi}'$, with $\mathbf{x}' = \boldsymbol{\chi}'(\mathbf{X})$, be a second finite deformation that is 'close' to $\boldsymbol{\chi}$. The displacement, which can be thought of as a perturbation of $\boldsymbol{\chi}$, is written

$$\dot{\mathbf{x}} = \mathbf{x}' - \mathbf{x} = \boldsymbol{\chi}'(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{X}) \equiv \dot{\boldsymbol{\chi}}(\mathbf{X}) \tag{3.1}$$

and its gradient is

$$\mathrm{Grad}\,\dot{\boldsymbol{\chi}} = \mathrm{Grad}\,\boldsymbol{\chi}' - \mathrm{Grad}\,\boldsymbol{\chi} \equiv \dot{\mathbf{F}}. \tag{3.2}$$

This expression is exact, no approximation having been made. In order to examine the incremental constitutive laws and equilibrium equations, however, it will be necessary to use linear approximations in terms of the incremental deformation $\dot{\mathbf{x}}$ and its gradient $\dot{\mathbf{F}}$. Such linear approximations will be indicated by a superposed dot.

For example, for an unconstrained material the associated nominal stress difference is

$$\dot{\mathbf{S}} = \mathbf{S}' - \mathbf{S} = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}') - \frac{\partial W}{\partial \mathbf{F}}(\mathbf{F}), \tag{3.3}$$

which has the linear approximation

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}}, \tag{3.4}$$

where $\boldsymbol{\mathcal{A}}$ is the elasticity tensor with components defined by (2.22). The component form of (3.4) is

$$\dot{S}_{\alpha i} = \mathcal{A}_{\alpha i \beta j}\dot{F}_{j\beta}, \tag{3.5}$$

which provides the convention for defining the product appearing in (3.4).

To obtain the corresponding expression for an incompressible material, we take the increment of equation $(2.26)_1$ and obtain

$$\dot{\mathbf{S}} = \boldsymbol{\mathcal{A}}\dot{\mathbf{F}} - \dot{p}\,\mathbf{F}^{-1} + p\,\mathbf{F}^{-1}\dot{\mathbf{F}}\mathbf{F}^{-1}, \tag{3.6}$$

where $\dot{p}$ is the incremental form of p. This equation is coupled with the incremental form of the incompressibility constraint (2.3), which has the form

$$\mathrm{tr}\,(\dot{\mathbf{F}}\mathbf{F}^{-1}) = 0. \tag{3.7}$$

The components of $\boldsymbol{\mathcal{A}}$ are again given by (2.22), but now subject to the constraint (2.3).

3.1 Incremental equilibrium equations

From the equilibrium equation $(2.16)_1$ and its counterpart for $\boldsymbol{\chi}'$, we obtain, by using (3.3),

$$\mathrm{Div}\,\dot{\mathbf{S}} = \mathbf{0}, \tag{3.8}$$

which does not involve approximation. In its linear approximation $\dot{\mathbf{S}}$ is replaced by either (3.4), or (3.6) with (3.7), as appropriate.

The incremental versions of the boundary conditions (2.29) and (2.30) are written, respectively, as

$$\dot{\mathbf{x}} = \dot{\boldsymbol{\xi}} \quad \text{on } \partial\mathcal{B}_r, \tag{3.9}$$

$$\dot{\mathbf{S}}^T\mathbf{N} = \dot{\boldsymbol{\tau}} \quad \text{on } \partial\mathcal{B}_r, \tag{3.10}$$

where $\dot{\boldsymbol{\xi}}$ and $\dot{\boldsymbol{\tau}}$ are the prescribed data for the incremental deformation $\dot{\boldsymbol{\chi}}$.

In component form equation (3.8) may be expanded out for an unconstrained material as

$$\mathcal{A}_{\alpha i\beta j}\dot{x}_{j,\alpha\beta} + \mathcal{A}_{\alpha i\beta j,\alpha}\dot{x}_{j,\beta} = 0, \tag{3.11}$$

and if the underlying finite deformation is homogeneous this reduces to

$$\mathcal{A}_{\alpha i\beta j}\dot{x}_{j,\alpha\beta} = 0. \tag{3.12}$$

The counterpart of (3.12) for an incompressible material is

$$\mathcal{A}_{\alpha i\beta j}\dot{x}_{j,\alpha\beta} - \dot{p}_{,i} = 0. \tag{3.13}$$

In dealing with incremental deformations it is often convenient to use the deformed configuration $\mathcal{B}$ as the reference configuration instead of the initial configuration $\mathcal{B}_r$ and to treat all incremental quantities as functions of $\mathbf{x}$ instead of $\mathbf{X}$. For this purpose we define the notations

$$\mathbf{u}(\mathbf{x}) = \dot{\boldsymbol{\chi}}(\boldsymbol{\chi}^{-1}(\mathbf{x})), \quad \boldsymbol{\Gamma} = \dot{\mathbf{F}}\mathbf{F}^{-1}, \quad \boldsymbol{\Sigma} = J^{-1}\mathbf{F}\dot{\mathbf{S}}, \tag{3.14}$$

the latter being the push-forward of $\dot{\mathbf{S}}$ motivated by the connection $\boldsymbol{\sigma} = J^{-1}\mathbf{FS}$.

For an incompressible material the incompressibility constraint then takes the form

$$\mathrm{tr}\,\boldsymbol{\Gamma} \equiv \mathrm{div}\,\mathbf{u} = 0, \tag{3.15}$$

and the updated incremental stress is

$$\boldsymbol{\Sigma} = \boldsymbol{\mathcal{A}}_0\boldsymbol{\Gamma} + p\boldsymbol{\Gamma} - \dot{p}\mathbf{I}, \tag{3.16}$$

where the components of $\boldsymbol{\mathcal{A}}_0$ are defined as in $(2.24)_1$ with $J = 1$.

The equilibrium equation (3.13) updates to $\operatorname{div}\boldsymbol{\Sigma} = \mathbf{0}$, or, in components,

$$\mathcal{A}_{0piqj}u_{j,pq} - \dot{p}_{,i} = 0, \tag{3.17}$$

and $\dot{p}$ can be eliminated by applying the curl operator:

$$\operatorname{curl}(\operatorname{div}\boldsymbol{\Sigma}) = \mathbf{0}, \quad \varepsilon_{rst}\mathcal{A}_{0piqj}u_{j,pqs} = 0. \tag{3.18}$$

In the latter ε_{rst} is the alternating symbol.

If $\mathbf{n}$ is the unit outward normal to $\partial\mathcal{B}$ then the incremental traction per unit area of $\partial\mathcal{B}$ is $\boldsymbol{\Sigma}^T\mathbf{n}$, or, in components,

$$\Sigma_{ji}n_j = (\mathcal{A}_{0jilk} + p\delta_{jk}\delta_{il})u_{k,l}n_j - \dot{p}n_i. \tag{3.19}$$

Components of the elasticity tensor. If W depends on the invariants $I_1, I_2, \ldots, I_N$, where $N = 2, 3, 5$, or 8 depending on whether it is isotropic or anisotropic with one or two preferred directions, compressible or incompressible, then we have

$$\frac{\partial W}{\partial \mathbf{F}} = \sum_{i=1}^{N} W_i \frac{\partial I_i}{\partial \mathbf{F}}, \tag{3.20}$$

and

$$\boldsymbol{\mathcal{A}} = \frac{\partial^2 W}{\partial \mathbf{F}\partial \mathbf{F}} = \sum_{i=1}^{N}\sum_{j=1}^{N} W_{ij}\frac{\partial I_i}{\partial \mathbf{F}} \otimes \frac{\partial I_j}{\partial \mathbf{F}} + \sum_{i=1}^{N} W_i \frac{\partial^2 I_i}{\partial \mathbf{F}\partial \mathbf{F}}, \tag{3.21}$$

where $W_i = \partial W/\partial I_i$, $W_{ij} = \partial^2 W/\partial I_i \partial I_j$ for $i, j \in \{1, 2, \ldots, N\}$, with the index 3 omitted in the case of an incompressible material. Then $\boldsymbol{\sigma}$ is calculated from $(2.18)_2$ and the components of $\boldsymbol{\mathcal{A}}_0$ from $(2.24)_1$.

The above expressions require calculation of the first and second derivatives of the invariants with respect to $\mathbf{F}$. The components of these derivatives for a selection of invariants are given as follows. First derivatives:

$$\frac{\partial I_1}{\partial F_{i\alpha}} = 2F_{i\alpha}, \quad \frac{\partial I_2}{\partial F_{i\alpha}} = 2(c_{\gamma\gamma}F_{i\alpha} - c_{\alpha\gamma}F_{i\gamma}), \quad \frac{\partial I_3}{\partial F_{i\alpha}} = 2I_3 F^{-1}_{\alpha i}, \tag{3.22}$$

$$\frac{\partial I_4}{\partial F_{i\alpha}} = 2M_\alpha(F_{i\gamma}M_\gamma), \quad \frac{\partial I_5}{\partial F_{i\alpha}} = 2(F_{i\gamma}M_\gamma c_{\alpha\beta}M_\beta + F_{i\gamma}c_{\gamma\beta}M_\beta M_\alpha); \tag{3.23}$$

second derivatives:

$$\frac{\partial^2 I_1}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}\delta_{\alpha\beta}, \tag{3.24}$$

$$\frac{\partial^2 I_2}{\partial F_{i\alpha}\partial F_{j\beta}} = 2(2F_{i\alpha}F_{j\beta} - F_{i\beta}F_{j\alpha} + c_{\gamma\gamma}\delta_{ij}\delta_{\alpha\beta} - b_{ij}\delta_{\alpha\beta} - c_{\alpha\beta}\delta_{ij}), \tag{3.25}$$

$$\frac{\partial^2 I_3}{\partial F_{i\alpha}\partial F_{j\beta}} = 4I_3F^{-1}_{\alpha i}F^{-1}_{\beta j} - 2I_3F^{-1}_{\alpha j}F^{-1}_{\beta i}, \tag{3.26}$$

$$\frac{\partial^2 I_4}{\partial F_{i\alpha}\partial F_{j\beta}} = 2\delta_{ij}M_\alpha M_\beta, \tag{3.27}$$

$$\frac{\partial^2 I_5}{\partial F_{i\alpha}\partial F_{j\beta}} = 2[\delta_{ij}(c_{\alpha\gamma}M_\gamma M_\beta + c_{\beta\gamma}M_\gamma M_\alpha) + \delta_{\alpha\beta}F_{i\gamma}M_\gamma F_{j\delta}M_\delta \tag{3.28}$$

$$+ F_{i\gamma}M_\gamma F_{j\alpha}M_\beta + F_{j\gamma}M_\gamma F_{i\beta}M_\alpha + b_{ij}M_\alpha M_\beta]. \tag{3.29}$$

The derivatives of I_6 and I_7 are obtained from those for I_4 and I_5, respectively, by replacing $\mathbf{M}$ by $\mathbf{M}'$, while the first and second derivatives of I_8 are

$$\frac{\partial I_8}{\partial F_{i\alpha}} = F_{i\gamma}(M'_\alpha M_\gamma + M_\alpha M'_\gamma), \quad \frac{\partial^2 I_8}{\partial F_{i\alpha}\partial F_{j\beta}} = \delta_{ij}(M_\alpha M'_\beta + M'_\alpha M_\beta). \tag{3.30}$$

Of course, the resulting expressions for $\boldsymbol{\mathcal{A}}$ are quite lengthy in general. However, in the special case of isotropy, an alternative and somewhat more compact representation for the components of $\boldsymbol{\mathcal{A}}$ referred to the principal axes of $\mathbf{U}$ and $\mathbf{V}$ can be given. The only non-zero components are

$$\mathcal{A}_{iijj} = W_{ij}, \tag{3.31}$$

$$\mathcal{A}_{ijij} - \mathcal{A}_{ijji} = \frac{W_i + W_j}{\lambda_i + \lambda_j} \quad i \neq j, \tag{3.32}$$

$$\mathcal{A}_{ijij} + \mathcal{A}_{ijji} = \frac{W_i - W_j}{\lambda_i - \lambda_j} \quad i \neq j,\ \lambda_i \neq \lambda_j, \tag{3.33}$$

$$\mathcal{A}_{ijij} + \mathcal{A}_{ijji} = W_{ii} - W_{ij} \quad i \neq j,\ \lambda_i = \lambda_j, \tag{3.34}$$

where now $W_i = \partial W/\partial\lambda_i$, $W_{ij} = \partial^2 W/\partial\lambda_i\partial\lambda_j$, $i,j \in \{1,2,3\}$, and no summation is implied by the repetition of indices. Note that in (3.31)–(3.34) the convention of using Greek letters for indices relating to Lagrangian components has been dropped. For details of the derivation of these components we refer to Ogden (1997). These equations are valid for both compressible and incompressible materials subject, in the latter case, to the constraint (2.51). Corresponding expressions for the components of $\boldsymbol{\mathcal{A}}_0$ can be obtained by use of (2.24) with (3.31)–(3.34) to give

$$\begin{aligned}
J\mathcal{A}_{0iijj} &= \lambda_i\lambda_j W_{ij}, && (3.35)\\
J\mathcal{A}_{0ijij} &= \frac{\lambda_i W_i - \lambda_j W_j}{\lambda_i^2 - \lambda_j^2}\lambda_i^2 \quad i \neq j,\, \lambda_i \neq \lambda_j, && (3.36)\\
J\mathcal{A}_{0ijji} &= \frac{\lambda_j W_i - \lambda_i W_j}{\lambda_i^2 - \lambda_j^2}\lambda_i\lambda_j \quad i \neq j,\, \lambda_i \neq \lambda_j, && (3.37)\\
J\mathcal{A}_{0ijij} &= \frac{1}{2}\left(\lambda_i^2 W_{ii} - \lambda_i\lambda_j W_{ij} + \lambda_i W_i\right) \quad i \neq j,\, \lambda_i = \lambda_j, && (3.38)\\
J\mathcal{A}_{0ijji} &= \frac{1}{2}\left(\lambda_i^2 W_{ii} - \lambda_i\lambda_j W_{ij} - \lambda_i W_i\right) \quad i \neq j,\, \lambda_i = \lambda_j, && (3.39)
\end{aligned}$$

and $J = 1$ for an incompressible material.

When evaluated in the stress-free reference configuration the components of $\boldsymbol{\mathcal{A}}$ have the compact classical form

$$\mathcal{A}_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{3.40}$$

where λ and μ are the Lamé moduli of elasticity and δ_{ij} is the Kronecker delta. When evaluated for $\lambda_i = 1$ for $i \in \{1, 2, 3\}$, we have $W_i = 0, W_{ii} = \lambda + 2\mu$, $W_{ij} = \lambda$, $i \neq j$. For an incompressible material there is an element of non-uniqueness in the components of $\boldsymbol{\mathcal{A}}_0$ since they depend on the point at which the incompressibility condition is applied during the differentiations. The counterpart of (3.40) in this case is

$$\mathcal{A}_{iiii} = \mathcal{A}_{ijij} = \mu, \quad \mathcal{A}_{iijj} = \mathcal{A}_{ijji} = 0 \quad i \neq j, \tag{3.41}$$

and $W_{ii} = W_i = \mu, W_{ij} = 0$, where μ is the *shear modulus* in $\mathcal{B}_r$. The differences between (3.41) and any alternative expressions are absorbed by the incremental Lagrange multiplier $\dot{p}$ in (3.16).

4 Elastodynamics

4.1 Time-dependent deformations

Again $\mathcal{B}_r$ denotes a fixed (time independent) reference configuration and let $t \in \mathcal{I} \subset \mathbb{R}$ denote time, where $\mathcal{I}$ is an appropriate interval of time. Time $t \in \mathcal{I}$ is used to parametrize the deformed configuration of the body, now denoted $\mathcal{B}_t$, which is assumed to evolve continuously with t. The (one-parameter) family of configurations $\{\mathcal{B}_t : t \in \mathcal{I}\}$ is referred to as a *motion* of the body. As in Section 2, a point of $\mathcal{B}_r$ is labelled by its position vector $\mathbf{X}$. Let $\mathbf{x}$ be its position vector in the *current configuration* $\mathcal{B}_t$.

Since the deformation now depends on t, we write

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \quad \text{for all } \mathbf{X} \in \mathcal{B}_r,\, t \in \mathcal{I}, \tag{4.1}$$

where, for each t, $\boldsymbol{\chi}$ has the same properties as in Section 2, with, additionally, sufficient regularity in t.

The *velocity* $\mathbf{v}$ and *acceleration* $\mathbf{a}$ of a material point $\mathbf{X}$ are defined by

$$\mathbf{v} \equiv \mathbf{x}_{,t} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t), \quad \mathbf{a} \equiv \mathbf{v}_{,t} \equiv \mathbf{x}_{,tt} = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(\mathbf{X}, t), \tag{4.2}$$

respectively. Thus, $\partial/\partial t$ is the *material time derivative*, which, for brevity, is denoted by $_{,t}$ and it is then to be understood that the independent variables are $\mathbf{X}$ and t. Of course, any scalar, vector or tensor field may be changed between the Eulerian description (with $\mathbf{x}$ and t as independent variables) and the Lagrangian description (with $\mathbf{X}$ and t as independent variables) by means of the motion (4.1) or its inverse.

It will sometimes be convenient to treat the velocity $\mathbf{v}$ as a function of $\mathbf{x}$ and t, and we then define the (Eulerian) *velocity gradient tensor*, denoted $\mathbf{L}$, as

$$\mathbf{L} = \operatorname{grad}\mathbf{v}, \quad L_{ij} = \frac{\partial v_i}{\partial x_j}. \tag{4.3}$$

Then

$$\operatorname{Grad}\mathbf{x}_{,t} = \mathbf{F}_{,t} = (\operatorname{grad}\mathbf{v})\mathbf{F} = \mathbf{LF}, \tag{4.4}$$

wherein $\mathbf{F}$ is the deformation gradient, defined as in (2.2) but now with dependence on t. This is similar to the formula $\dot{\mathbf{F}} = \boldsymbol{\Gamma}\mathbf{F}$ for (static) incremental deformations obtained from $(3.14)_2$.

With the notation $J = \det\mathbf{F}$ and the standard relation

$$(\det\mathbf{F})_{,t} = (\det\mathbf{F})\operatorname{tr}(\mathbf{F}^{-1}\mathbf{F}_{,t}), \tag{4.5}$$

and (4.4) we have

$$J_{,t} \equiv (\det\mathbf{F})_{,t} = (\det\mathbf{F})\operatorname{tr}(\mathbf{L}) = J\operatorname{div}\mathbf{v}. \tag{4.6}$$

Since J is a measure of volume change, this shows that $\operatorname{div}\mathbf{v}$ is a measure of the rate at which volume changes during the motion. If the motion is isochoric then $J \equiv 1$ and (4.6) reduces to

$$\operatorname{div}\mathbf{v} = 0. \tag{4.7}$$

This is similar to the incompressibility condition (3.15) arising in the *linearized* incremental theory. However, (4.7) is exact in the dynamic context whereas (3.15) is a linear approximation.

For a non-isochoric motion, with ρ the mass density in $\mathcal{B}_t$, we have

$$\rho_{,t} + \rho\operatorname{div}\mathbf{v} = 0, \tag{4.8}$$

which is the rate form of the mass conservation equation $\rho_r = J\rho$.

4.2 Equations of motion

The equation of motion analogous to the equilibrium equation $(2.16)_2$ is

$$\operatorname{Div}\mathbf{S} = \rho_r\mathbf{a} \equiv \rho_r\mathbf{x}_{,tt}, \tag{4.9}$$

again in the absence of body forces, where $\mathbf{S}$ is given by $(2.18)_1$ for an unconstrained material and $(2.26)_1$ for an incompressible material, with $\mathbf{F}$ and p now depending on t. We recall that ρ_r is the mass density in $\mathcal{B}_r$. For an unconstrained material equation (4.9) has the component form

$$\mathcal{A}_{\alpha i\beta j}x_{j,\alpha\beta} = \rho_r x_{i,tt}, \tag{4.10}$$

and its incompressible counterpart is

$$\mathcal{A}_{\alpha i\beta j}x_{j,\alpha\beta} - p_{,i} = \rho_r x_{i,tt}, \tag{4.11}$$

the latter coupled with $\det(x_{i,\alpha}) = 1$, in which case $\rho = \rho_r$. Note that in (4.10) and (4.11) the coefficients $\mathcal{A}_{\alpha i\beta j}$ are in general nonlinear functions of the components $x_{i,\alpha}$ of the deformation gradient and the equations are to be solved for $x_i\, i = 1, 2, 3$, as a function of X_α, $\alpha = 1, 2, 3$, and t subject to suitable boundary and initial conditions, which we do not specify here. Such problems are very difficult to solve in general and very few exact solutions are available in the literature. We refer to the review by Ogden (2001) for a list of references. Some simplifications occur when the motion is considered to be of small amplitude so that the equations can be linearized. In the following section therefore we examine the problem of incremental motions superimposed on a known finite motion and its specialization to a known static deformation.

4.3 Incremental motions

First consider incremental motions superimposed on a finite motion. Let

$$\dot{\mathbf{x}} = \dot{\boldsymbol{\chi}}(\mathbf{X}, t) \tag{4.12}$$

be the time-dependent counterpart of the increment defined in (3.1). We emphasize that a superposed dot represents an increment, whereas a material time derivative is represented by $_{,t}$. The corresponding dynamic counterpart of the incremental equilibrium equation (3.8) is

$$\mathrm{Div}\,\dot{\mathbf{S}} = \rho_r \dot{\mathbf{x}}_{,tt}. \tag{4.13}$$

When this equation is linearized in the incremental quantities it becomes

$$\mathrm{Div}\,(\boldsymbol{\mathcal{A}}\dot{\mathbf{F}}) = \rho_r \dot{\mathbf{x}}_{,tt} \tag{4.14}$$

in the case of an unconstrained material, or, in components,

$$\mathcal{A}_{\alpha i\beta j}\dot{x}_{j,\alpha\beta} + \mathcal{A}_{\alpha i\beta j,\alpha}\dot{x}_{j,\beta} = \rho_r \dot{x}_{i,tt}. \tag{4.15}$$

This equation applies whether the increment is superimposed on a finite motion or a static finite deformation. If this motion (or deformation) is homogeneous (independent of $\mathbf{X}$) then equation (4.15) reduces to

$$\mathcal{A}_{\alpha i\beta j}\dot{x}_{j,\alpha\beta} = \rho_r \dot{x}_{i,tt}, \tag{4.16}$$

where the coefficients $\mathcal{A}_{\alpha i\beta j}$ now depend only on time. Henceforth, however, we confine our attention to incremental motions superimposed on a static homogeneous finite deformation so that the coefficients $\mathcal{A}_{\alpha i\beta j}$ are constants that involve material constants and the (uniform and constant) components of $\mathbf{F}$, the gradient of the underlying deformation $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. For this purpose we define the Eulerian form of the incremental displacement by

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\boldsymbol{\chi}(\mathbf{X}), t) \equiv \dot{\boldsymbol{\chi}}(\mathbf{X}, t) \tag{4.17}$$

and update the reference configuration to the configuration $\mathcal{B}$ associated with $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ to give

$$\mathcal{A}_{0piqj} u_{j,pq} = \rho u_{i,tt}. \tag{4.18}$$

For an incompressible material the corresponding equation is

$$\mathcal{A}_{0piqj} u_{j,pq} - \dot{p}_{,i} = \rho u_{i,tt}, \tag{4.19}$$

coupled with the incompressibility condition

$$u_{i,i} = 0. \tag{4.20}$$

4.4 Incremental plane waves

We conclude the present section by considering the propagation of incremental plane waves of the form

$$\mathbf{u} = \mathbf{m} f(\mathbf{n} \cdot \mathbf{x} - ct), \tag{4.21}$$

where $\mathbf{m}$ is a unit vector referred to as the *polarization vector*, c is the *wave speed* and f is a twice continuously differentiable function. For homogeneous plane waves the unit vector $\mathbf{n}$ is real and defines the *direction of propagation* of the wave (see Figure 2). In general, however, $\mathbf{n}$ (and $\mathbf{m}$ and f) may be complex, and the wave is referred to as an *inhomogeneous plane wave.*

In respect of the incremental displacement (4.21) equation (4.18) yields

$$\mathbf{Q}_0(\mathbf{n})\mathbf{m} = \rho c^2 \mathbf{m}, \tag{4.22}$$

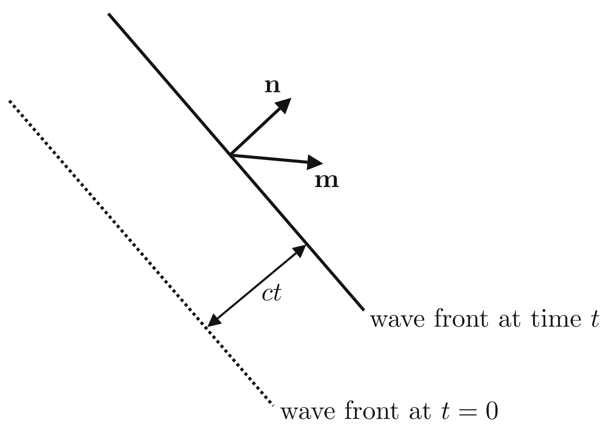

Figure 2. Depiction of a plane wave with unit normal $\mathbf{n}$ and polarization $\mathbf{m}$.

where the so-called *acoustic tensor* $\mathbf{Q}_0(\mathbf{n})$ has been introduced. It depends on $\mathbf{n}$ and is defined (in component form) by

$$[\mathbf{Q}_0(\mathbf{n})]_{ij} = \mathcal{A}_{0piqj} n_p n_q. \tag{4.23}$$

Equation (4.22) is called the *propagation condition.* For a given direction $\mathbf{n}$ it determines possible waves speeds and polarizations corresponding to plane waves propagating in that direction. The wave speeds are determined by the *characteristic equation*

$$\det[\mathbf{Q}_0(\mathbf{n}) - \rho c^2 \mathbf{I}] = 0, \tag{4.24}$$

where $\mathbf{I}$ is again the identity tensor. Since $\mathbf{Q}_0(\mathbf{n})$ is symmetric its eigenvalues are real, but not necessarily positive. For the wave speeds to be real the eigenvalues ρc^2 must be positive. If this is the case then for the $\mathbf{n}$ in question three real plane waves exist. Now, from (4.22) and (4.23), we obtain

$$\rho c^2 = [\mathbf{Q}_0(\mathbf{n})\mathbf{m}] \cdot \mathbf{m} = \mathcal{A}_{0piqj} n_p n_q m_i m_j, \tag{4.25}$$

and we see, with reference to (2.25), that if the strong ellipticity condition holds then $\rho c^2 > 0$ for all directions of propagation $\mathbf{n}$, i.e. strong ellipticity guarantees that the speeds of homogeneous plane waves are real.

Turning next to incompressible materials, in addition to (4.21) we need to take $\dot{p}$ to have a similar form, specifically $\dot{p} = qf'(\mathbf{n} \cdot \mathbf{x} - ct)$. Then, substitution into equations (4.19) and (4.20) yields

$$\mathbf{Q}_0(\mathbf{n})\mathbf{m} - q\mathbf{n} = \rho c^2 \mathbf{m}, \quad \mathbf{m} \cdot \mathbf{n} = 0. \tag{4.26}$$

By taking the scalar product of $(4.26)_1$ with $\mathbf{n}$ we obtain $q = [\mathbf{Q}_0(\mathbf{n})\mathbf{m}] \cdot \mathbf{n}$ and hence q can be eliminated from $(4.26)_1$ to leave

$$\mathbf{Q}_0^*(\mathbf{n})\mathbf{m} = \rho c^2 \mathbf{m}, \quad \mathbf{m} \cdot \mathbf{n} = 0, \tag{4.27}$$

where we have defined

$$\mathbf{Q}_0^*(\mathbf{n}) = \mathbf{Q}_0(\mathbf{n}) - \mathbf{n} \otimes \mathbf{Q}_0(\mathbf{n})\mathbf{n}. \tag{4.28}$$

Note, in particular, that $\mathbf{Q}_0^*(\mathbf{n})$ is not in general symmetric. Moreover, it is singular, with a zero eigenvalue corresponding to the left eigenvector $\mathbf{n}$. This is a consequence of the incompressibility constraint, which ensures that there can be at most two real plane waves, both of them being transverse.

By taking the scalar product of $(4.26)_1$ with $\mathbf{m}$ we obtain (4.25), as in the unconstrained case. Thus, strong ellipticity again guarantees that homogeneous plane waves (when they exist) have real speeds.

5 Plane incremental motions

In this section we apply the general equations in the foregoing section to particular examples in order to illustrate the influence of pre-stress as compared with the classical theory in the absence of pre-stress. For definiteness we restrict attention to plane incremental motions in the (x_1, x_2) plane with displacement $\mathbf{u}$ having components

$$u_1(x_1, x_2, t), \quad u_2(x_1, x_2, t), \quad u_3 = 0. \tag{5.1}$$

Moreover, we consider only incompressible materials. Then, referred to the principal axes of $\mathbf{V}$ in the considered plane, equations (4.19) reduce to the two equations

$$\mathcal{A}_{01111}u_{1,11} + (\mathcal{A}_{01122} + \mathcal{A}_{02112})u_{2,12} + \mathcal{A}_{02121}u_{1,22} - \dot{p}_{,1} = \rho u_{1,tt}, \tag{5.2}$$

$$\mathcal{A}_{01212}u_{2,11} + (\mathcal{A}_{01122} + \mathcal{A}_{02112})u_{1,12} + \mathcal{A}_{02222}u_{2,22} - \dot{p}_{,2} = \rho u_{2,tt}, \tag{5.3}$$

and the incompressibility condition (4.20) reduces to

$$u_{1,1} + u_{2,2} = 0. \tag{5.4}$$

From (5.4) we deduce the existence of a scalar stream-like function $\psi(x_1, x_2, t)$ such that

$$u_1 = \psi_{,2}, \quad u_2 = -\psi_{,1}, \tag{5.5}$$

and on substitution of these expressions into (5.2) and (5.3) followed by elimination of $\dot{p}$ by cross differentiation (a special case of (3.18)), we obtain an equation for ψ, namely

$$\alpha\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} = \rho(\psi_{,11tt} + \psi_{,22tt}), \tag{5.6}$$

which involves the three parameters α, β, γ defined by

$$\alpha = \mathcal{A}_{01212}, \quad 2\beta = \mathcal{A}_{01111} + \mathcal{A}_{02222} - 2\mathcal{A}_{01122} - 2\mathcal{A}_{01221}, \quad \gamma = \mathcal{A}_{02121}. \tag{5.7}$$

On specializing (2.28) to the considered two-dimensional situation we may write $m_1 = n_2, m_2 = -n_1$, and the strong ellipticity condition (2.25) then simplifies to

$$\alpha n_1^4 + 2\beta n_1^2 n_2^2 + \gamma n_2^4 > 0 \tag{5.8}$$

for all (two-dimensional) unit vectors $\mathbf{n} = (n_1, n_2, 0)$. It follows that necessary and sufficient conditions for (5.8) to hold are simply

$$\alpha > 0, \quad \gamma > 0, \quad \beta > -\sqrt{\alpha\gamma}. \tag{5.9}$$

Next, we consider the incremental traction $\mathbf{\Sigma}^T\mathbf{n}$ given by (3.19) on an Eulerian principal plane of the underlying homogeneous deformation. We take this to be the plane $x_2 = 0$, for which $n_1 = 0, n_2 = 1$. It is easy to show that the only non-vanishing components of $\mathbf{\Sigma}^T\mathbf{n}$ are Σ_{21} and Σ_{22}, and that

$$\Sigma_{21} = \mathcal{A}_{02121}u_{1,2} + (\mathcal{A}_{02112} + p)u_{2,1}, \tag{5.10}$$

and hence, after a short calculation,

$$\Sigma_{21} = \gamma\psi_{,22} + (\gamma - \sigma_2)\psi_{,11}. \tag{5.11}$$

Also,

$$\Sigma_{22} = \mathcal{A}_{01122}u_{1,1} + (\mathcal{A}_{02222} + p)u_{2,2} - \dot{p}. \tag{5.12}$$

By differentiating this with respect to x_1 and then using (5.2) to eliminate $\dot{p}_{,1}$ we arrive at the expression

$$\Sigma_{22,1} = \rho\psi_{,2tt} - (2\beta + \gamma - \sigma_2)\psi_{,112} - \gamma\psi_{,222}. \tag{5.13}$$

5.1 Application to plane harmonic waves

We now focus on plane harmonic waves and write ψ in the form

$$\psi = A\exp[\mathrm{i}k(x_1\cos\theta + x_2\sin\theta - ct)], \tag{5.14}$$

where we have set $n_1 = \cos\theta, n_2 = \sin\theta$ and k is the wave number. Substitution of (5.14) into equation (5.6) leads to the propagation condition

$$(\alpha + \gamma - 2\beta)\cos^4\theta + 2(\beta - \gamma)\cos^2\theta + \gamma = \rho c^2. \tag{5.15}$$

In the classical theory of incompressible isotropic elasticity we have $\alpha = \beta = \gamma = \mu$, where μ is the shear modulus identified in (3.41), and (5.15) reduces to $\rho c^2 = \mu$ independently of the direction of propagation. This gives the speed of a classical shear wave. On inspection of (5.15) we see that in the special case for which the material properties and/or the state of deformation satisfy $2\beta = \alpha + \gamma$ the propagation condition simplifies to

$$\rho c^2 = \alpha\cos^2\theta + \gamma\sin^2\theta. \tag{5.16}$$

For either (5.15) or (5.16) a shear wave can propagate along a principal axis, with $\rho c^2 = \alpha$ corresponding to the x_1 axis and $\rho c^2 = \gamma$ to the x_2 axis. In the case of (5.16), for a general (in-plane) direction of propagation, ρc^2 necessarily lies between the values α and $\gamma \neq \alpha$. This need not be so in respect of (5.15), for which the extreme values of ρc^2 are α, γ and $(\beta^2 - \alpha\gamma)/(2\beta - \alpha - \gamma)$. The latter is greater than either of α or γ if $2\beta > \alpha + \gamma$ and less than either if $2\beta < \alpha + \gamma$.

For any direction of propagation in the considered plane the wave speed is given by (5.15) or (5.16). On the other hand, for a given wave speed there will only exist an associated real direction of propagation if the wave speed lies within permissible bounds. If this is the case, then (5.16) yields two (in general distinct) directions, symmetric with respect to the axes. By contrast, for (5.15) there may be two pairs of distinct directions of propagation. For more detailed discussion we refer to Ogden and Sotiropoulos (1997), and to Ogden and Sotiropoulos (1998) for corresponding discussion in the case of unconstrained materials.

Reflection of a plane wave from the boundary of a half-space. We now consider a wave of the form (5.14) propagating in the material half-space $x_2 < 0$ subject to the pure homogeneous strain with in-plane principal axes of deformation parallel and normal to the boundary $x_2 = 0$. The boundary $x_2 = 0$ is taken to be free of incremental traction but subject to an underlying normal stress σ_2. From (5.11) and (5.13) we obtain the appropriate boundary conditions in terms of ψ, namely

$$\gamma\psi_{,22} + (\gamma - \sigma_2)\psi_{,11} = 0 \quad \text{on } x_2 = 0, \tag{5.17}$$

$$\rho\psi_{,2tt} - (2\beta + \gamma - \sigma_2)\psi_{,112} - \gamma\psi_{,222} = 0 \quad \text{on } x_2 = 0. \tag{5.18}$$

The wave is incident on the boundary $x_2 = 0$ and, depending on the material properties and the state of deformation, generates one or two reflected waves and/or a surface

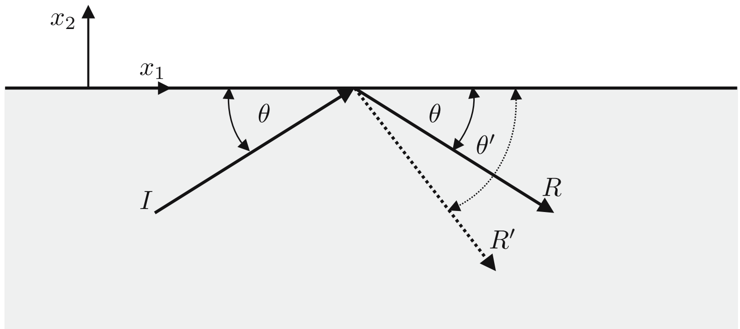

Figure 3. Incompressible isotropic elastic half-space subject to pure homogeneous strain ($x_2 < 0$). Plane wave I incident on the boundary $x_2 = 0$ with direction of propagation at an angle θ to the boundary; Reflected waves R and R' with directions of propagation at angles θ and θ' to the boundary.

wave (see Figure 3). The total solution for ψ consisting of the incident wave and two reflected waves is written in the form

$$\psi = A\mathrm{e}^{\mathrm{i}k(n_1x_1+n_2x_2-ct)} + AR\mathrm{e}^{\mathrm{i}k(n_1x_1-n_2x_2-ct)} + AR'\mathrm{e}^{\mathrm{i}k'(n_1'x_1+n_2'x_2-c't)}, \qquad (5.19)$$

where R and R' are the reflection coefficients and k' and c' are the wave number and wave speed associated with the second reflected wave. The first reflected wave has the same speed as the incident wave and is reflected at angle θ to the boundary, while the angle of reflection of the second reflected wave is θ'. Thus, $n_1 = \cos\theta, n_2 = \sin\theta, n_1' = \cos\theta', n_2' = \sin\theta'$. For compatibility of the three waves we must have $kc = k'c'$, i.e. they have the same frequency. Furthermore, application of the boundary conditions leads to $kn_1 = k'n_1'$ and hence

$$c'n_1 = cn_1', \qquad (5.20)$$

which is a statement of Snell's law.

The two cases $2\beta = \alpha + \gamma$ and $2\beta \neq \alpha + \gamma$ need to be treated separately. Results for the first case are broadly similar to those for the classical theory in that at most one reflected wave is possible, although the wave speed does depend on the direction of propagation. We shall not discuss this case here but refer the reader to Ogden and Sotiropoulos (1997) for details. The second case is more interesting and reveals several features that distinguish it from the classical theory. Thus, we now give some attention to this.

The propagation condition for the incident wave is given by (5.15). This equation also governs one of the reflected waves (θ is replaced by $-\theta$). The propagation condition for the second reflected wave is

$$(\alpha + \gamma - 2\beta)\cos^4\theta' + 2(\beta - \gamma)\cos^2\theta' + \gamma = \rho c'^2. \qquad (5.21)$$