Advanced Structured Materials

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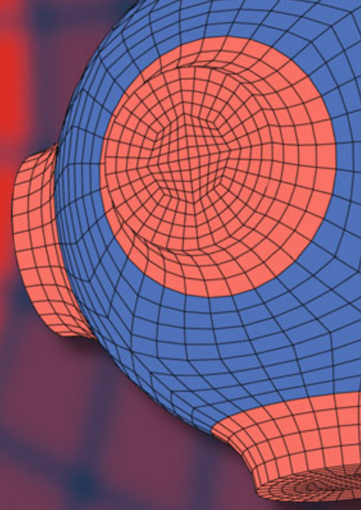

Generalized Continua as Models for Materials

With Multi-scale Effects or Under Multi-field Actions

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Preface

The Mechanics of Generalized Continua is an established research topic since the end of the 1950s–early 1960s of the last century. The starting point of this development was the monograph of the Cosserat brothers, published in 1909[1] , and some previous works of such famous scientists like Lord Kelvin, Duhem, Helmholtz among others. All these contributions were focussed on the fact that in a continuum one has to define translations and rotations independently (or in other words, one has to establish force and moment actions independently as it was done by Jakob Bernoulli and Euler). At the same time the continuum was not modeled as an infinite number of continuously distributed points with properties like the mass, but as an infinite number of continuously distributed infinitesimal small bodies with properties like the mass.

The reason for the revival in the mid of 1950s of the last century was that some effects of the mechanical behavior of solids and fluids could not be explained by the available classical continuum models. Examples of this are the turbulence of a fluid or the behavior of solids with a significant and very complex microstructure. Since the suggested new models fulfill all requirements from *Continuum Thermomechanics* (the balance laws were formulated and the general representation of the constitutive equations were given) the scientific community was satisfied for a while. At the same time real applicative developments were missed.

Indeed, for practical applications the developed models were not useful. The reason for this was the gap between the formulated constitutive equations and the possibilities to identify the material parameters. As is often the case one had much more parameters compared to classical models, but no facilities to measure all properties. In addition, computational progress and available machines in these times were limited.

During the last ten years the situation has drastically changed. More and more researches emerged, being kindled by the partly forgotten models. Now one has available much more computational possibilities and very complex problems can be simulated numerically. In addition, with the increased attention paid to a large

[1] E. et F. Cosserat: Cosserat, F.: Théorie des Corps Déformables, Hermann Editeurs, Paris, 1909 (Reprint, Gabay, Paris, 2008).

number of materials with complex microstructure and a deeper understanding of the meaning of the material parameters (scale effects) the identification becomes much more well founded. Thus we have contributions describing the micro- and macro-behavior, new existence and uniqueness theorems, the formulation of multi-scale problems, etc., and now it is time to ponder again[2] the state of matter and to discuss new trends and applications.

The main focus in this book will be directed on the following items:

- Modeling and simulation of materials with significant microstructure;
- Generalized continua as a result of multi-scale models;
- Multi-field actions on materials resulting in generalized material models; and
- Comparison with discrete modeling approaches.

This book contains selected papers submitted to the Second Trilateral Seminar *Generalized Continua as Models for Materials With Multi-scale Effects or Under Multifield Actions*, which held at the *Leucorea* (Lutherstadt Wittenberg, Germany) from September 26 upto 30, 2012.[3] Special thanks to Andreas Kutschke who took all duties connected with realization of the Seminar. In addition, we kindly acknowledge Dr. Christoph Baumann and Benjamin Feuchter (Springer Publisher) for the support of the book project. Last but not least it should be mentioned that the Seminar was sponsored by grants of the French National Center for Scientific Research (CNRS), the German Research Foundation (DFG) AL341/41-1, and the Russian Foundation for Basic Research 12-01-91260RFG.

Magdeburg, December 2012	Holm Altenbach
Paris	Samuel Forest
St. Petersburg	Anton Krivtsov

[2] There were two proceedings within the last years which should be mentioned here: Gérard A. Maugin, Andrei V. Metrikine (Eds) Mechanics of Generalized Continua - One Hundred Years After the Cosserats, Springer, 2010 (Advances in Mechanics and Mathematics, Vol. 21) and Holm Altenbach, Gérard Maugin, Vladimir Erofeev (Eds) Mechanics of Generalized Continua, Springer, 2011 (Advanced Structured Materials, Vol. 7).

[3] The First Trilateral French–German–Russian Seminar held also in Lutherstadt Wittenberg (Germany) August 9–11, 2010.

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Shells and Plates with Surface Effects

Holm Altenbach and Victor A. Eremeyev

Abstract The through-the-thickness integration procedure applied to a three-dimensional (3D) slender body leads to exact two-dimensional (2D) equations of plates and shells, see [36]. The procedure can be considered as a specific homogenization technique which results in a 2D generalized media—the non-linear theory of shells of Cosserat type. Within this theory the shell is described as a deformable surface each point of which has 3 translational and 3 rotational degrees of freedom similar to the 3D Cosserat continuum [15]. Below we discuss the through-the-thickness integration procedure applied to the non-classical problem of the theory of surface elasticity [21]. The theory can be applied to modeling of surface effects which are important in mechanics of nanostructured materials [11, 55]. Applying the through-the-thickness integration procedure we reduce 3D equations to 2D ones. The effective (apparent) stiffness properties of the shell are changed in comparison with the classical models of shells. Some examples of a plate bending are discussed taking into account surface effects.

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1 Introduction

Structures, which have a size of several μm and more, as usual are modeled within continuum mechanics taking into account only the properties of the bulk material. This statement is valid for any continuum theory—for the classical continuum mechanics in Cauchy's sense, Cosserat or micropolar theory and non-local theories among others. Special two-dimensional theories for plates and shells or one-dimensional theories for rods and beams can be introduced with the help of the through-the-thickness integration procedure or the through-the-cross-section integration procedure, respectively.

With respect to new technological developments an increasing miniaturization of devices and structural elements must be considered. Because of the changing surface-volume ratio in comparison with the classical sizes of the devices and structural elements the effects related to the surface phenomena have a significant influence on the mechanical behavior and should be taken into account. Here the possibilities to take into account these effects are demonstrated on plate- and shell-like structures.

1.1 Examples of Surface Phenomena

The influence of surface phenomena in deformable solid bodies is widely presented in the literature, see for example [6]. The main phenomena can be summarized as it follows:

- The development of nanotechnologies extends the field of application of the classical or non-classical theories of plates and shells towards the new thin-walled structures.
- In general, modern nanomaterials have physical properties which are different from the bulk material.
- The classical linear elasticity can be extended to the nanoscale by implementation of the theory of elasticity taking into account the surface stresses, cf. [11, 55].
- In particular, the surface stresses are responsible for the size-effect, that means the material properties of a specimen depend on its size. For example, Young's modulus of a cylindrical specimen increases significantly, when the cylinder diameter becomes very small [7, 9, 32, 39].
- The surface stresses are the generalization of the scalar surface tension which is a well-known phenomenon in the theory of capillarity.

The surface stresses which are the reason for the surface phenomena have influence on the following items:

- phase transitions (nucleation, crystal growth, etc.),
- fracture (Griffith criterion, effective surface energy density, line tension as a energy of a dislocation core),

- mechanics of porous media (nanoporous materials can be made stiffer than non-porous counterparts by surface modification) and
- other problems (surface diffusion, surface waves).

In various publications one reports on the changes of the mechanical properties in dependence on the size. Experimental results about the increasing Young's modulus with the decreasing size of nanowires made of ZnO are presented by Chen et al. [7]. Similar effects are described by Cuenot et al. [9] and Jing et al. [32] in the case of bending of nanobeams made of Ag and Pb. In [12] the behavior of nanoporous materials is discussed. In dependence on the size of the pores the material properties increase or decrease.

Taking into account only the elastic material behavior surface effects can be modeled within the classical theory of elasticity which was founded and influenced by French scientists in the $18^{th}/19^{th}$ century. The following contributions considering surface effects should be mentioned:

- First investigations of surface phenomena were initiated by Laplace [35], Young [58] and Gibbs [18].
- A modern treatise taking into account the surface stresses is given, for example, in the publications [21, 43, 44, 51]. Residual surface stresses are considered in [20, 24–26, 34, 53, 57].
- The treatment by the Finite Element Method or other numerical realizations is discussed in [27, 29–31].

For further reading about the history and the different approaches to modeling of the surface energy effects we recommend the reviews [11, 16, 41, 44, 47, 48, 55].

1.2 Basic Three-Dimensional Equations of Elasticity with Surface Effects

Let us summarize briefly the governing equations of the theory of elasticity with surface stresses in the sense of [21]. The reference configuration of the shell-like elastic body with surface stresses is shown in Fig. 1. The following equilibrium and boundary equations can be introduced

- Lagrangian equilibrium equation

$$\nabla_{\mathbf{X}} \cdot \mathbf{P} + \rho \mathbf{f} = \mathbf{0}, \tag{1}$$

- Equilibrium conditions on the upper and lower surfaces

$$(\mathbf{n} \cdot \mathbf{P} - \nabla_S \cdot \mathbf{S})|_{\Omega_S} = \mathbf{t}, \tag{2}$$

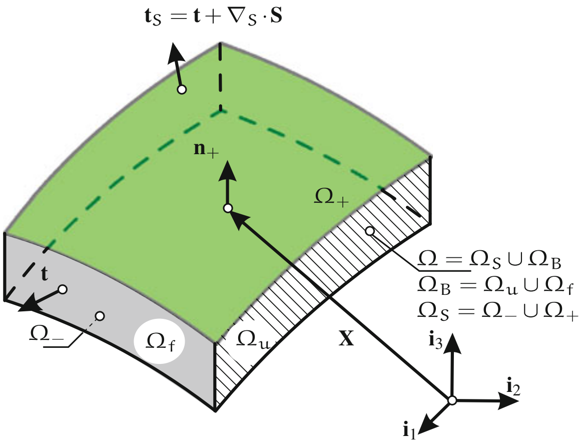

Fig. 1 Elastic body with surface effects

- Boundary conditions

$$\mathbf{u}|_{\Omega_u} = \mathbf{0}, \quad \mathbf{n} \cdot \mathbf{P}|_{\Omega_f} = \mathbf{t}. \tag{3}$$

Here $\mathbf{P}$ is the first Piola-Kirchhoff stress tensor, $\nabla_{\mathbf{X}}$ the 3D nabla operator, ∇_S the surface (2D) nabla operator, $\mathbf{S}$ the surface stress tensor of the first Piola-Kirchhoff type acting on the surfaces Ω_S, $\mathbf{u} = \mathbf{x} - \mathbf{X}$ the displacement vector, $\mathbf{x}$ and $\mathbf{X}$ are the position-vectors in the initial and actual configurations, respectively, $\mathbf{f}$ and $\mathbf{t}$ the body force and surface load vectors, respectively, and ρ the density. We assume that the part of the body surface Ω_u is fixed, while on Ω_f surface stresses $\mathbf{S}$ are absent. Equation (2) is the so-called generalized Young-Laplace equation describing the surface tension in solids.

The boundary-value problem (1)–(3) should be complemented by constitutive relations. For the bulk material we use the relation

$$\mathbf{P} = \frac{\partial W}{\partial \nabla_{\mathbf{X}} \mathbf{x}},$$

where W is the strain energy density. In the theory of Gurtin and Murdoch [21] the tensor $\mathbf{S}$ is similar to the membrane stress resultants defined as follows

$$\mathbf{S} = \frac{\partial U}{\partial \mathbf{F}},$$

where U is the surface strain energy density and $\mathbf{F} = \nabla_S \mathbf{x}$ the surface deformation gradient.

In the case of residual stresses we assume that W and $\mathbf{P}$ possess the properties

$$W(\mathbf{I}) = 0, \quad \mathbf{P}(\mathbf{I}) = \mathbf{0},$$

while there exist residual (initial) surface energy and surface stresses that is

$$\mathrm{U}(\mathbf{A}) = \mathrm{U}_0 \neq 0, \quad \mathbf{S}(\mathbf{A}) = \mathbf{S}_0 \neq \mathbf{0},$$

where $\mathbf{I}$ and $\mathbf{A} \equiv \mathbf{I} - \mathbf{N} \otimes \mathbf{N}$ are the 3D and the surface unit tensors, respectively. In other words, we assume that the reference placement for the bulk material is natural one while for the attached on Ω_S membranes we assume the non-natural reference placement.

1.3 Linearized Relations

In the case of infinitesimal strains of an isotropic body we have the following constitutive equations:

- For the stresses in the bulk material the Hooke law is valid

$$\mathbf{P} = 2\mu\varepsilon + \lambda\mathbf{I}\,\mathrm{tr}\,\varepsilon, \tag{4}$$

  where λ and μ are the Lamé elastic moduli.
- For the surface stresses one can assume

$$\mathbf{S} = \mathbf{S_0} + \mathbf{C}_S : \mathbf{e} + \mathbf{S_0} \cdot \nabla_S \mathbf{u}, \tag{5}$$

  Here the first part is related to the residual stresses, the second is similar to Hooke's law with the elasticity tensor $\mathbf{C}_S$, the last part has the origin in the linearization of the surface Piola–Kirchhoff stress tensor.

In the case of initial uniform surface tension we have $\mathbf{S}_0 = \mathrm{p}\mathbf{A}$ and

$$\mathbf{S} = \mathrm{p}\mathbf{A} + 2\mu_S\mathbf{e} + \lambda_S\mathbf{A}\,\mathrm{tr}\mathbf{e} + \mathrm{p}\nabla_S\mathbf{u}, \tag{6}$$

where p is the initial surface tension, λ_S and μ_S are the surface elastic moduli called also the surface Lamé moduli.

The linearized strain-displacement relations are given as it follows

$$\varepsilon = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right], \quad \mathbf{e} = \frac{1}{2}\left[\nabla_S\mathbf{v}_S \cdot \mathbf{A} + \mathbf{A} \cdot (\nabla_S\mathbf{v}_S)^T\right] \tag{7}$$

with

$$\mathbf{v}_S = \mathbf{u}\big|_{\Omega_S}.$$

The first Eq. (7) is valid for the bulk material, the second one for the surface contributions.

The number of material parameters is doubled in comparison with the classical isotropic case: instead of two we have now four. The requirement of the positive definiteness of the strain energy yields restrictions for λ, μ and λ_S, μ_S (see, for example, [2] and [28])

$$\mu > 0, \quad 3\lambda + 2\mu > 0; \quad \mu_S > 0, \quad \lambda_S + \mu_S > 0. \tag{8}$$

Note that $\mathbf{S}_0$ is an arbitrary second-order tensor, in general.

2 Two-Dimensional Theories of Nanosized Plates and Shells

The theory of elasticity with surface stresses was applied to the modifications of the two-dimensional theories of nano-sized plates and shells in [1, 3–5, 10, 14, 19, 22, 23, 37–39, 54, 57, 59], where various theories of plates and shells are formulated. The approaches can be classified, for example, by the starting point of the derivation. This can be the well-known three-dimensional continuum mechanics equations. In contrast, one can introduce à priory a two-dimensional deformable surface which is the basis for a more natural formulation of the two-dimensional governing equations. This so-called direct approach should be supplemented by the theoretical or experimental determination of the material parameters included in the constitutive equations.

Here we use the general theory of shells presented in [8, 15, 36] for the modification of the constitutive equations taking into account the surface stresses. We show that both the stress and the couple stress resultant tensors may be represented as a sum of two terms. The first term is the volume stress resultant while the second one determined by the surface stresses and the shell geometry. In the linear case this modification reduces to the add of new terms to the elastic stiffness parameters. The influence of these terms on the shell bending stiffness is discussed. We show that the surface elasticity makes the shell more stiffer in comparison with the shell without surface stresses.

2.1 Basic Equations of the 6-Parametric Elastic Shell Theory

The kinematics of the shell can be presented in the actual configuration by the position-vector $\mathbf{r}$ and a triad of three orthogonal vectors $\mathbf{d}_k$

$$\{\mathbf{r}(q^1, q^2, t); \mathbf{d}_k(q^1, q^2, t)\}; \qquad \mathbf{d}_k \cdot \mathbf{d}_m = \delta_{km}, \quad k, m = 1, 2, 3,$$

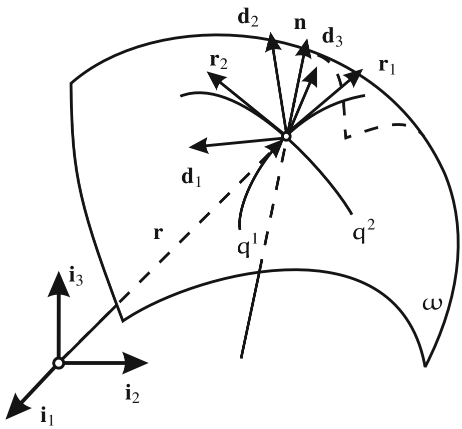

Fig. 2 Actual configuration of the shell

where δ_{km} is the Kronecker symbol, see Fig. 2. For the reference configuration one has the position-vector $\mathbf{R}$ and the triad $\mathbf{D}_k$

$$\{\mathbf{R}(q^1, q^2);\ \mathbf{D}_k(q^1, q^2)\}; \quad \mathbf{D}_k \cdot \mathbf{D}_m = \delta_{km}.$$

Here q^1, q^2 are the Gaussian coordinates used for both configurations.

The quality of any continuum theory (three- or lower dimensional) depends significantly on the correct formulation of the corresponding constitutive equations that is in the case of elastic material the strain energy function. Let us assume

$$\mathcal{W} = \mathcal{W}(\mathbf{F}, \mathbf{Q}, \nabla_S \mathbf{Q}) \tag{9}$$

with

$$\mathbf{F} \stackrel{\triangle}{=} \nabla_S \mathbf{r}, \quad \mathbf{Q} \stackrel{\triangle}{=} \mathbf{D}^k \otimes \mathbf{d}_k,$$

$$\nabla_S(\ldots) \stackrel{\triangle}{=} \mathbf{R}^\alpha \partial(\ldots)/\partial q^\alpha.$$

The base vectors are defined as

$$\mathbf{R}^\alpha \cdot \mathbf{R}_\beta = \delta^\alpha_\beta, \quad \mathbf{R}^\alpha \cdot \mathbf{N} = 0, \quad \mathbf{R}_\alpha = \partial \mathbf{R}/\partial q^\alpha, \quad \alpha,\beta = 1, 2.$$

$\mathbf{Q}$ is an orthogonal tensor called the microrotation tensor, and $\mathbf{N}$ is the unit normal to the surface Ω in the reference configuration. After application of the principle of the frame indifference $\mathcal{W}$ takes the form

$$\mathcal{W} = \mathcal{W}(\mathbf{E}, \mathbf{K}) \tag{10}$$

with the strain measures which are given by

$$\mathbf{E} \stackrel{\triangle}{=} \mathbf{F}\cdot\mathbf{Q}^{\mathsf{T}} - \mathbf{A}, \quad \mathbf{K} \stackrel{\triangle}{=} \frac{1}{2}\mathbf{R}^{\alpha}\otimes\left(\frac{\partial\mathbf{Q}}{\partial q^{\alpha}}\cdot\mathbf{Q}^{\mathsf{T}}\right)_{\times}. \tag{11}$$

Here $(\ldots)_{\times}$ denotes the vectorial invariant of the second-order tensor. In particular, for a diad it is given by

$$(\mathbf{a}\otimes\mathbf{b})_{\times} = \mathbf{a}\times\mathbf{b}.$$

The Lagrangian equilibrium equations are formulated as it follows

$$\nabla_{S}\cdot\mathbf{T} + \mathbf{q} = \mathbf{0}, \quad \nabla_{S}\cdot\mathbf{M} + \left[\mathbf{F}^{\mathsf{T}}\cdot\mathbf{T}\right]_{\times} + \mathbf{c} = \mathbf{0}. \tag{12}$$

The force and the moment tensors **T** and **M** can be computed from

$$\mathbf{T} \stackrel{\triangle}{=} \frac{\partial\mathcal{W}}{\partial\mathbf{E}}\cdot\mathbf{Q}, \quad \mathbf{M} \stackrel{\triangle}{=} \frac{\partial\mathcal{W}}{\partial\mathbf{K}}\cdot\mathbf{Q}. \tag{13}$$

They are the resultant tensors of the first Piola-Kirchhoff type on the deformable surface, while **q** and **c** are the external surface force and moment vectors, respectively. The equilibrium equations (12) are the exact consequence of three-dimensional equilibrium equations, see [8, 36]. Within the framework of the theory the approximation error is localized in the constitutive equation (10) only. The strain measures **E** and **K** are work-conjugate to the respective stress measures.

2.2 Plates and Shells with Surface Stresses

Applying the through-the-thickness integration technique described in [36] to shell-like bodies with surface stresses, we obtain the following 2D constitutive equations for nano-sized plates and shells, see [1]:

$$\mathbf{T}^{*} = \mathbf{T} + \mathbf{T}_{S}, \quad \mathbf{M}^{*} = \mathbf{M} + \mathbf{M}_{S}, \tag{14}$$

where

- **T**,**M** are classical resultant tensors given by

$$\mathbf{T} = \int \mathbf{G}\cdot\mathbf{P}\,d\zeta, \quad \mathbf{M} = -\int \mathbf{G}\cdot\mathbf{P}\times\mathbf{z}\,d\zeta \tag{15}$$

  with

$$\int(\ldots)\,d\zeta = \int_{h_-}^{h_+}(\ldots)\,d\zeta$$

  and

- $\mathbf{T}_S$,$\mathbf{M}_S$ are the resultant tensors induced by surface stresses $\mathbf{S}_\pm$ acting on the shell faces $\Omega_\pm$

$$\mathbf{T}_S = G_+\mathbf{S}_+ + G_-\mathbf{S}_-\,,$$
$$\mathbf{M}_S = -\frac{h}{2}\left[G_+(\mathbf{S}_+)\times\mathbf{z}_+ - G_-(\mathbf{S}_-)\times\mathbf{z}_-\right]. \tag{16}$$

Here $\mathbf{z}$ is the base reference deviation and $\mathbf{G} \equiv -\mathbf{N}\times(\mathbf{A} - \zeta\nabla_S\mathbf{N})\times\mathbf{N}$ is the geometrical tensor, $G \equiv \det(\mathbf{A} - \zeta\nabla_S\mathbf{N})$ is the geometric scale factor defined in [36], ζ is the coordinate along the unit normal $\mathbf{N}$ in the reference placement, $G_\pm = G\big|_{\zeta=\pm h/2}$, and h is the shell thickness, see Fig. 3.

3 Linear Theory of Plates with Surface Stresses

The theory can be significantly simplified for plates and infinitesimal strains. In this case the shell strain energy density is given by

$$\begin{aligned} 2\mathcal{W} &= \alpha_1\mathrm{tr}^2\mathbf{E}_\| + \alpha_3\mathrm{tr}\left(\mathbf{E}_\|\cdot\mathbf{E}_\|^T\right) + \alpha_4\mathbf{N}\cdot\mathbf{E}^T\cdot\mathbf{E}\cdot\mathbf{N} \\ &+\beta_1\mathrm{tr}^2\mathbf{K}_\| + \beta_3\mathrm{tr}\left(\mathbf{K}_\|\cdot\mathbf{K}_\|^T\right) + \beta_4\mathbf{N}\cdot\mathbf{K}^T\cdot\mathbf{K}\cdot\mathbf{N} \end{aligned} \tag{17}$$

with $\mathbf{E}_\| = \mathbf{E}\cdot\mathbf{A}$ and $\mathbf{K}_\| = \mathbf{K}\cdot\mathbf{A}$. α_i and β_i are elastic parameters

$$\begin{aligned} &\alpha_1 = C\nu,\ \alpha_3 = C(1-\nu),\ \alpha_4 = \alpha_s C(1-\nu)\,, \\ &\beta_1 = D\nu,\ \beta_3 = D(1-\nu),\ \beta_4 = \alpha_t D(1-\nu) \end{aligned} \tag{18}$$

with

$$C = \frac{Eh}{1-\nu^2},\quad D = \frac{Eh^3}{12(1-\nu^2)}\,.$$

E and ν are Young's modulus and Poisson's ratio of the bulk material, α_s and α_t are dimensionless coefficients, while h is the shell thickness. α_s is similar to the shear correction factor introduced by Reissner [45] ($\alpha_s = 5/6$) and Mindlin [40] ($\alpha_s = \pi^2/12$). The value $\alpha_t = 7/10$ was proposed in [42].

Considering the surface stress tensors $\mathbf{S}_\pm$ we assume $p = 0$ in (6). So we have

$$\begin{aligned} &\mathbf{S}_\pm = \lambda_S^\pm\mathbf{A}\mathrm{tr}\mathbf{e}_\pm + 2\mu_S^\pm\mathbf{e}_\pm\,, \\ &2\mathbf{e}_\pm = \nabla\mathbf{u}_\pm\cdot\mathbf{A} + \mathbf{A}\cdot(\nabla\mathbf{u}_\pm)^T\,, \end{aligned} \tag{19}$$

$\mathbf{u}_\pm = \mathbf{u}\big|_{\zeta=\pm h/2}$. For the sake of simplicity we consider the symmetric case with $\lambda_S^\pm = \lambda_S$ and $\mu_S^\pm = \mu_S$. Taking into account (19) we obtain the stiffness parameters

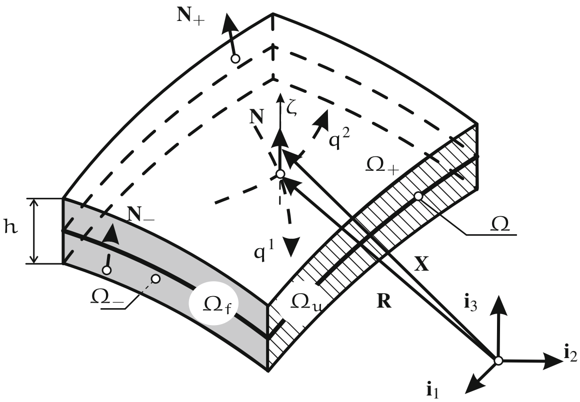

Fig. 3 Geometry (reference configuration) of the shell-like body

for the plate with surface stresses, see [4, 14]

$$\begin{aligned}
&\alpha_1 = C\nu + 2\lambda_S, \quad \alpha_3 = C(1-\nu) + 4\mu_S, \\
&\beta_1 = D\nu + h^2\lambda_S/2, \quad \beta_3 = D(1-\nu) + h^2\mu_S, \\
&C^* = C + 4\mu_S + 2\lambda_S, \\
&D^* = D + h^2\mu_S + h^2\lambda_S/2.
\end{aligned} \tag{20}$$

C^* and D^* are the effective in-plane and bending stiffness of the plate with surface stresses. $C^* > C$ and $D^* > D$, i.e. the plate with surface stresses is stiffer. α_4 and β_4 do not depend on λ_S and μ_S.

As an example let us consider a nanoplate made of aluminium. Using the data presented in [11] the dependence D^* versus the plate thickness h is shown in Fig. 4. Here $\mu = 34.7$ GPa, $\nu = 0.3$, $\lambda_S = -3.48912$ N/m, $\mu_S = 6.2178$ N/m, where μ is the shear modulus. For these values of λ_S and μ_S the influence of the surface stresses is significant if $h \le 20$ nm.

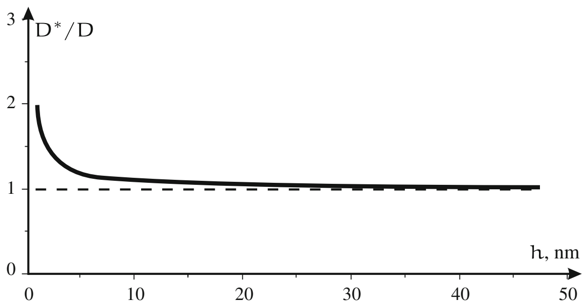

Fig. 4 Bending stiffness versus plate thickness

4 Viscoelastic Case

In most of contributions on the surface stresses the elastic medium is considered. On the other hand, dissipative processes in the vicinity of the surface are observed. These surface phenomena are related to the higher mobility of molecules near the surface, surface imperfections, absorbates, etc., see e.g. [50]. As a special case of inelastic behavior the surface viscoelasticity exists for both liquids and solids. The experimental methods of the surface viscoelasticity are different than in the case of bulk material, in general. One can use various types of microscopies, light scattering, etc., see e.g. [13, 17, 33, 49, 52, 56]. For the description of the surface dissipation of nanosized beams, Ru [46] proposed the one-dimensional constitutive law that is similar to the model of the standard viscoelastic solids. In [5] we extended Ru's model to the case of two-dimensional surface stresses.

The simplest case of analyzing viscoelastic material behavior is based on the correspondence principle. This principle states that if an elastic solution is known, the corresponding viscoelastic solution can be obtained by substituting the elastic quantities in the Laplace transforms of the unknown functions. In other words, one can use the solution of the boundary-value-problem (BVP) for the elastic material behavior as the solution of BVP for the viscoelastic material but given in terms of Laplace transforms. According to this principle we use the results of 3D to 2D reduction procedure for the elastic shell-like body [4].

Let us introduce the Laplace transform

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt.$$

Applying the Laplace transform to the viscoelastic constitutive equations at the surface we obtain the relation

$$\overline{\mathbf{S}} = 2s\bar{\mu}_S(s)\bar{\mathbf{e}} + s\bar{\lambda}_S(s)(\mathrm{tr}\,\bar{\mathbf{e}})\mathbf{A}, \tag{21}$$

which coincides formally with the surface Hooke law (6) but with two surface relaxation functions $\mu_S(t)$ and $\lambda_S(t)$. In addition, we establish the constitutive equations for the shell considering viscoelastic behavior in the form, see [5]

$$\begin{aligned}
\mathbf{T} &= \int_{-\infty}^{t} \left[C_1(t-\tau)\dot{\mathbf{e}}(\tau) + C_2(t-\tau)\mathbf{A}\,\mathrm{tr}\,\dot{\mathbf{e}}(\tau)\right]\,d\tau + \Gamma\boldsymbol{\gamma}\otimes\mathbf{N}, \\
\mathbf{M} &= -\int_{-\infty}^{t} \left[D_1(t-\tau)\dot{\boldsymbol{\kappa}}(\tau) + D_2(t-\tau)\mathbf{A}\,\mathrm{tr}\,\dot{\boldsymbol{\kappa}}(\tau)\right]\,d\tau\times\mathbf{N},
\end{aligned}$$

where $\boldsymbol{\epsilon}$, $\boldsymbol{\kappa}$, and $\boldsymbol{\gamma}$ are the surface strain measures expressed via the translation and rotation vectors $\mathbf{w}$ and $\boldsymbol{\vartheta}$ by

$$\begin{aligned}
\boldsymbol{\epsilon} &= \frac{1}{2}\left(\nabla_S \mathbf{w}\cdot\mathbf{A} + \mathbf{A}\cdot(\nabla_S\mathbf{w})^T\right),\\
\boldsymbol{\kappa} &= \frac{1}{2}\left(\nabla_S \boldsymbol{\vartheta}\cdot\mathbf{A} + \mathbf{A}\cdot(\nabla_S\boldsymbol{\vartheta})^T\right),\\
\boldsymbol{\gamma} &= \nabla_S(\mathbf{w}\cdot\mathbf{N}) - \boldsymbol{\vartheta}.
\end{aligned}$$

The relaxation functions are given by

$$\begin{aligned}
&C_1(t) = 2C_{22} + 4\mu_S(t), \quad C_2(t) = C_{11} - C_{22} + 2\lambda_S(t),\\
&D_1(t) = 2D_{22} + h^2\mu_S(t),\ D_2(t) = D_{33} - D_{22} + \frac{h^2}{2}\lambda_S(t),
\end{aligned}$$

$$\begin{aligned}
C_{11} &= \frac{Eh}{2(1-\nu)}, \quad C_{22} = \frac{Eh}{2(1+\nu)},\\
D_{22} &= \frac{Eh^3}{24(1+\nu)},\ D_{33} = \frac{Eh^3}{24(1-\nu)}, \quad \Gamma = k\mu h,\\
E &= 2\mu(1+\nu), \quad \nu = \frac{\lambda}{2(\lambda+\mu)}.
\end{aligned}$$

E and ν are the Young modulus and Poisson ratio of the bulk material, Γ is the transverse shear stiffness, and k the transverse shear correction factor. The tangential and bending relaxation functions are given by

$$C = \frac{Eh}{1-\nu^2} + 4\mu_S(t) + 2\lambda_S(t), \qquad D = \frac{Eh^3}{12(1-\nu^2)} + \frac{h^2}{2}\left[2\mu_S(t) + \lambda_S(t)\right]. \quad (22)$$

Let us note that the surface stresses do not influence the transverse shear stiffness.

5 Conclusions

In this paper we discussed the two-dimensional equilibrium equations for plates and shells taking into account the surface stresses. We presented the expressions for effective stiffness parameters of plates and shells. In particular, the bending stiffness is bigger for the shells with surface stresses than for shells without surface elasticity. Elastic case is extended to viscoelastic behavior.

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Geometrical Picture of Third-Order Tensors

Nicolas Auffray

Abstract Because of its strong physical meaning, the decomposition of a symmetric second-order tensor into a deviatoric and a spheric part is heavily used in continuum mechanics. When considering higher-order continua, third-order tensors naturally appear in the formulation of the problem. Therefore researchers had proposed numerous extensions of the decomposition to third-order tensors. But, considering the actual literature, the situation seems to be a bit messy: definitions vary according to authors, improper uses of denomination flourish, and, at the end, the understanding of the physics contained in third-order tensors remains fuzzy. The aim of this paper is to clarify the situation. Using few tools from group representation theory, we will provide an unambiguous and explicit answer to that problem.

1 Introduction

In classical continuum mechanics [28, 29], only the first displacement gradient is involved and all the higher-order displacement gradients are neglected in measuring the deformations of a body. This usual kinematical framework turns out not to be rich enough to describe a variety of important mechanical and physical phenomena. In particular, the size effects and non-local behaviors due to the discrete nature of matter at a sufficiently small scale, the presence of microstructural defects or the existence of internal constraints cannot be captured by classical continuum mechanics [2, 18, 24]. The early development of higher-order (or generalized) continuum theories of elasticity was undertaken in the 1960s and marked with the major contributions of [5, 19–21, 26]. For the last two decades, the development and application of high-order continuum theories have gained an impetus, owing to a growing interest in modeling

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and simulating size effects and non-local behaviors observed in a variety of materials, such as polycrystalline materials, geomaterials, biomaterials and nanostructured materials (see, e.g., [7, 17, 22]), and in small size structures. In order to take into account size-effects, the classical continuum mechanics has to be generalized. To construct such an extension there are, at least, two options:

- Higher-order continua:
  In this approach the set of degrees of freedom is extended; a classical example is the micromorphic theory [6, 11, 20];
- Higher-grade continua:
  In this approach the mechanical state is described using higher-order gradients of the displacement field; a classical example is the strain-gradient theory [19].

In the following section the linear formulation of micromorphic and strain-gradient theory we will be detailed. The aim is to anchor the analysis that will be made on third-order tensors into a physical necessity for the understanding of those models.

2 Some Generalized Continua

2.1 Micromorphic Elasticity

Let us begin with the micromorphic approach. In this theory the set of degrees of freedom (DOF) is extended in the following way

$$\mathrm{DOF} = \{\underline{u}, \underset{\sim}{\chi}\} \quad ; \quad (\underline{u}, \underset{\sim}{\chi}) \in \mathbb{R}^3 \times \otimes^2\mathbb{R}^3,$$

where $\otimes^k\mathbb{V}$ stands for the k-th order tensorial power of $\mathbb{V}$. In this formulation the second-order tensor $\underset{\sim}{\chi}$ is generally not symmetric. This micro-deformation tensor encodes the generally incompatibility deformation of the microstructure. As a consequence, the set of primary state variables (PSV) now becomes

$$\mathrm{PSV} = \{\underline{u} \otimes \underline{\nabla}, \underset{\sim}{\chi} \otimes \underline{\nabla}\},$$

where $\underline{\nabla}$ is the classical nabla vector, i.e.

$$\underline{\nabla}^T = \left(\frac{\partial}{\partial x}\ \frac{\partial}{\partial y}\ \frac{\partial}{\partial z}\right)$$

It can be observed that, despite being of higher-degree, the obtained model is still a 1st-grade continuum. The model is defined by the following set of strain measures:

- $\underset{\sim}{\varepsilon} = \varepsilon_{(ij)}$ is the strain tensor;

- $\underset{\sim}{e} = u \otimes \underline{\nabla} - \underset{\sim}{\chi}$ is the relative strain tensor;
- $\underset{\simeq}{\kappa} = \underset{\sim}{\chi} \otimes \underline{\nabla}$ is the micro-strain gradient tensor;

where the notation $(..)$ indicates symmetry under in parentheses permutations. The first strain measure is the classical one and is, as usually, described by a symmetric second-order tensor. The relative strain tensor measures how the micro-deformation differs from the displacement gradient, this information is encoded into a non-symmetric second-order tensor. Finally, we have the third-order non-symmetric micro strain-gradient tensor. By duality the associated stress tensors can be defined:

- $\underset{\sim}{\sigma} = \sigma_{(ij)}$ is the Cauchy stress tensor;
- $\underset{\sim}{s} = s_{ij}$ is the relative stress tensor;
- $\underset{\simeq}{S} = S_{ijk}$ is the double-stress tensor.

If we suppose that the relation between strain and stress tensors is linear, the following constitutive law is obtained:

$$\begin{cases} \underset{\sim}{\sigma} = \underset{\approx}{A} : \underset{\sim}{\varepsilon} + \underset{\approx}{B} : \underset{\sim}{e} + \underset{\approxeq}{C} \therefore \underset{\simeq}{\kappa} \\ \underset{\sim}{s} = \underset{\approx}{B}^{T} : \underset{\sim}{\varepsilon} + \underset{\approx}{D} : \underset{\sim}{e} + \underset{\approxeq}{E} \therefore \underset{\simeq}{\kappa} \\ \underset{\simeq}{S} = \underset{\approxeq}{C}^{T} : \underset{\sim}{\varepsilon} + \underset{\approxeq}{E}^{T} : \underset{\sim}{e} + \underset{\approx\atop\sim}{F} \therefore \underset{\simeq}{\kappa} \end{cases}$$

The behavior is therefore defined by

- three fourth-order tensors having the following index symmetries: $A_{\underline{(ij)}\ \underline{(lm)}}$; $B_{(ij)lm}$; $D_{\underline{ij}\ \underline{lm}}$;
- two fifth-order tensors having the following index symmetries: $C_{(ij)klm}$; E_{ijklm};
- one sixth-order tensor having the following index symmetries: $F_{\underline{ijk}\ \underline{lmn}}$,

where $\underline{.}\ \underline{.}$ indicates symmetry under block permutations.

2.2 *Strain-Gradient Elasticity*

In the strain-gradient elasticity the set of degrees of freedom is the usual one, but the primary state variables are extended to take the second gradient of $\underline{u}$ into account:

$$\text{PSV} = \{\underline{u} \otimes \underline{\nabla}, \underline{u} \otimes \underline{\nabla} \otimes \underline{\nabla}\}$$

We therefore obtain a second-grade continuum defined by the following set of strain measures:

- $\underset{\sim}{\varepsilon} = \varepsilon_{(ij)}$ is the strain tensor;
- $\underset{\simeq}{\eta} = \underset{\sim}{\varepsilon} \otimes \underline{\nabla} = \eta_{(ij),k}$ is the strain-gradient tensor.

By duality, we obtain the related stress tensors:

- $\underset{\sim}{\sigma} = \sigma_{(ij)}$ is the Cauchy stress tensor;
- $\underset{\simeq}{\tau} = \tau_{(ij)k}$ is the hyper-stress tensor.

Assuming a linear relation between these two sets we obtain:

$$\begin{cases} \underset{\sim}{\boldsymbol{\sigma}} = \underset{\approx}{A} : \underset{\sim}{\varepsilon} + \underset{\approxeq}{C} \therefore \underset{\simeq}{\eta} \\ \underset{\simeq}{\boldsymbol{\tau}} = \underset{\approxeq}{C}^{T} : \underset{\sim}{\varepsilon} + \underset{\underset{\sim}{\approx}}{F} \therefore \underset{\simeq}{\eta} \end{cases}$$

The strain-gradient and hyperstress tensors are symmetric under permutation of their two first indices. The constitutive tensors verify the following index permutation symmetry properties:

$$C_{\underline{(ij)}\ \underline{(lm)}} \quad ; \quad M_{(ij)(kl)m} \quad ; \quad A_{\underline{(ij)k}\ \underline{(lm)n}}$$

2.3 Synthesis

Those two models are distinct but under the kinematic constraint $\underset{\sim}{\chi} = \underline{u} \otimes \underline{\nabla}$ strain-gradient elasticity is obtained from the micromorphic model. In the first case, the micro strain-gradient is element of:

$$\mathbb{T}_{ijk} = \{\underset{\simeq}{T} | \underset{\simeq}{T} = \sum_{i,j,k=1}^{3} T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k\}$$

Assuming that we are in a 3D physical space, $\mathbb{T}_{ijk}$ is 27-dimensional and constructed as $\mathbb{T}_{ijk} = \otimes^3\mathbb{R}^3$. For the strain-gradient theory, strain-gradient tensors belong to the following subspace of $\mathbb{T}_{ijk}$:

$$\mathbb{T}_{(ij)k} = \{\underset{\simeq}{T} | \underset{\simeq}{T} = \sum_{i,j,k=1}^{3} T_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \ \ T_{ijk} = T_{jik}\}$$

which is 18-dimensional and constructed as[1] $\mathbb{T}_{(ij)k} = (\mathbb{R}^3 \otimes^S \mathbb{R}^3) \otimes \mathbb{R}^3$. Therefore, as it can be seen, the structure of the third-order tensors changes according to the considered theory.

Facing this kind of non-conventional model, a natural question is to ask what kind of information is encoded in these higher-order strain measures. In classical elasticity the physical content of symmetric second-order tensors is well-known through the

[1] The notation $\otimes^S$ indicates the symmetric tensor product.

physical meaning of its decomposition into a deviatoric (distorsion) and a spheric (dilatation) part. But the same result for third-order tensors is not so well-known, and its physical content has to be investigated. In the literature some results concerning the strain-gradient tensors can be found, but the situation seems to be fuzzy. In mechanics,[2] third-order tensor orthogonal decomposition was first investigated in the context of strain-gradient plasticity. According to the authors and the modeling assumptions the number of components varies from 2 to 4. In the appendices of [25] the authors introduced a first decomposition of the strain-gradient tensors under an incompressibility assumption, and expressed the decomposition into the sum of 3 mutually orthogonal parts. This decomposition was then used in [7, 8]. In [17] the situation is analyzed more in depth, and a decomposition into four parts is proposed. In some other works, it is said that strain-gradient can be divided into two parts. Therefore the following questions are raised:

- What is the right generalization of the decomposition of a tensor into deviatoric parts ?
- In how many orthogonal parts a third-order tensor can be split in a irreducible way ?
- Is this decomposition canonical ?

The aim of this paper is to answer these questions. These points will be investigated using the geometrical language of group action.

3 Harmonic Space Decomposition

To study the orthogonal decomposition of third-order tensors, and following the seemingly work of Georges Backus [3], an extensive use of harmonic tensors will be made. This section is thus devoted to formally introduce the concept of harmonic decomposition. After a theoretical introduction, the space of third-order tensors identified in the first section will be decomposed into a sum of harmonic tensor spaces. This $\mathrm{O}(3)$-irreducible[3] decomposition is the higher-order generalization of the well-known decomposition of $\mathbb{T}_{(ij)}$ into a deviatoric ($\mathbb{H}^2$) and spherical ($\mathbb{H}^0$) spaces.

3.1 The Basic Idea

Before studying decomposition of third-order tensors, let us get back for a while on the case of second-order symmetric ones. It is well known that any $\mathrm{T}_{(ij)} \in \mathbb{T}_{(ij)}$ admits the following decomposition:

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[2] In field of condensed matter physics this decomposition is known since, at least, the 70' [15].

[3] $\mathrm{O}(3)$: the orthogonal group, i.e. the group of all isometries of $\mathbb{R}^3$ i.e. if $\mathrm{Q} \in \mathrm{O}(3)$ $\det(\mathrm{Q}) \pm 1$ and $\mathrm{Q}^{-1} = \mathrm{Q}^{\mathrm{T}}$.

$$T_{(ij)} = H^2_{(ij)} + \frac{1}{3}H^0\delta_{ij} = \phi(H^2_{(ij)}, H^0),$$

where $H^2 \in \mathbb{H}^2$ and $H^0 \in \mathbb{H}^0$ are, respectively, the 5-D deviatoric and 1-D spheric part of $T_{(ij)}$ and are defined by the following formula:

$$H^0 = T_{ii} \quad ; \quad H^2_{(ij)} = T_{(ij)} - \frac{1}{3}H^0\delta_{ij}$$

In fact ϕ, defined by the expression (3.1), is an isomorphism between $\mathbb{T}_{(ij)}$ and the direct sum of $\mathbb{H}^2$ and $\mathbb{H}^0$

$$\mathbb{T}_{(ij)} \cong \mathbb{H}^2 \oplus \mathbb{H}^0$$

The main property of this decomposition is to be O(3)-invariant, or expressed in another way the components $(H^0, \underset{\sim}{H}^2)$ are covariant with $\underset{\sim}{T}$ under O(3)-action, i.e.

$$\forall \underset{\sim}{Q} \in O(3),\ \forall \underset{\sim}{T} \in \mathbb{T}_{(ij)}, \quad \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T = \phi(\underset{\sim}{Q}\underset{\sim}{H}^2\underset{\sim}{Q}^T, H^0)$$

Irreducible tensors satisfying this property are called harmonic. By irreducible we mean that those tensors can not be split into other tensors satisfying this property. In a certain way harmonic tensors are the elementary gears of the complete tensor. Let now give a more precise and general definition of this decomposition.

3.2 Harmonic Decomposition

The O(3)-irreducible decomposition of a tensor is known as its harmonic decomposition. Such a decomposition is well-known in group representation theory. It allows to decompose any finite order tensor into a sum of irreducible ones [3, 14, 30]. Consider a n-th order tensor T belonging to $\mathbb{T}$ then its decomposition can be written [14]:

$$T = \sum_{k,\tau} H^{k,\tau},$$

where the tensors $H^{k,\tau}$ are components[4] of the irreducible decomposition, k denotes the order of the harmonic tensor embedded in H and τ separates the same order terms. This decomposition defines an isomorphism between $\mathbb{T}$ and a direct sum of harmonic tensor spaces $\mathbb{H}^k$ [10] as

[4] To be more precise, $H^{k,\tau}$ is the embedding of the τth irreducible component of order k into a n-th order tensor.

$$\mathbb{T} \cong \bigoplus_{k,\tau} \mathbb{H}^{k,\tau}$$

but, as explained in [12], this decomposition is not unique. Alternatively, the O(3)-isotypic decomposition, where same order spaces are grouped, is unique:

$$\mathbb{T} \cong \bigoplus_{k=0}^{n} \alpha_k \mathbb{H}^k,$$

where α_k is the multiplicity of $\mathbb{H}^k$ in the decomposition, i.e. the number of copies of the space $\mathbb{H}^k$ in the decomposition. Harmonic tensors are totally symmetric and traceless. In $\mathbb{R}^3$, the dimension of their vector space $\dim \mathbb{H}^k = 2k+1$. For $k = 0$ we obtain the space of scalars, $k = 1$ we obtain the space of vectors, $k = 2$ we obtain the space of deviators, and for $k > 2$ we obtain spaces of k-th order deviators. The family $\{\alpha_k\}$ is a function of the tensor space order and the index symmetries. Various methods exist to compute this family [1, 14, 30]. In $\mathbb{R}^3$ a very simple method based on the Clebsch-Gordan decomposition can be used.

In the next section this construction is introduced. It worths noting that we obtain the harmonic structure of the space under investigation modulo an unknown isomorphism. The construction of an isomorphism making this decomposition explicit is an ulterior step of the process. Furthermore, according to the nature of the sought information, the explicit knowledge of the isomorphism might by unnecessary. As an example, the determination of the set of symmetry classes of a constitutive tensor space does not require such a knowledge[5] [16, 23].

3.3 Computation of the Decomposition

The principle is based on the tensorial product of group representations. More details can be found in [1, 14]. The computation rule is simple. Consider two harmonic tensor spaces $\mathbb{H}^i$ and $\mathbb{H}^j$, whose product space is noted $\mathbb{G}^{i+k} := \mathbb{H}^i \otimes \mathbb{H}^j$. This space, which is GL(3)-invariant, admits the following O(3)-invariant decomposition:

$$\mathbb{G}^{i+j} = \bigoplus_{k=|i-j|}^{i+j} \mathbb{H}^k$$

For example, consider $\mathbb{H}^1_a$ and $\mathbb{H}^1_b$ two different first-order harmonic spaces. Elements of such spaces are vectors. According the above formula the O(3)-invariant decomposition of $\mathbb{G}^2$ is:

[5] Even if some authors explicitly construct this isomorphism [10, 13] this step is useless.