CISM COURSES AND LECTURES NO. 470
INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

PHENOMENOLOGICAL AND MATHEMATICAL MODELLING OF STRUCTURAL INSTABILITIES

EDITED BY

MARCELLO PIGNATARO
VICTOR GIONCU

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PHENOMENOLOGICAL AND MATHEMATICAL MODELLING OF STRUCTURAL INSTABILITIES

EDITED BY

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PREFACE

The study of structural instability plays a role of primary importance in the field of applied mechanics. Despite the remarkable progresses made in the recent past years, the structural instability remains one of the most challenging topics in applied mechanics. Many problems have been solved in the last decades but still many others remain to be solved satisfactorily. The increasing number of papers published in journals and conferences organized by ECCS, SSRC, IUTAM, and EUROMECH strongly indicates the interest of scientists and engineers in the subject. A careful examination of these publications shows that they tend to fall into one of the two categories.

The first is that of **practical design direction** in which methods for analyzing specific stability problems related to some specific structural typologies are developed. The research works are restricted to determining the critical load, considering that it is sufficient to know the limits of stability range. These studies are invaluable since their aim is to provide solutions to practical problems, to supply the designer with data useful for design and prepare norms, specifications and codes.

The second direction is that of **theoretical studies**, aiming at a mathematical modeling of the instability problems, for a better understanding of the phenomena. In these studies, special emphasis is placed on the behavior of structures after the loss of stability in the post-critical range. This approach is less familiar to designers as its results have not yet become part of current structural design practice.

A wide range of researches has been developed in both directions: by mathematicians and engineers specialized in applied mathematics and mechanics, by engineers who have been working in the field of space and naval construction, etc... Each of these two directions has scored many remarkable achievements, but some incompatibilities exist between them because in the first direction mainly structural designers are involved, while in the second one essentially academic researchers alien to practice are working.

The purpose of the course "**Phenomenological and Mathematical Modeling in Structural Instabilities**" is to present some recent progress in the filed of structural instability, with regard both to practical applications and to the transfer of theoretical results to practice, in order to fill the gap existing between the accumulated theoretical knowledge and practical applications. The course progressively covers topics such as phenomenological, mathematical and numeric modeling of instability analysis, static and dynamic instabilities, structural instability and catastrophe theory.

The first section "**Mathematical Modelling of Instability Phenomena**", elaborated by Marcello Pignataro and Giuseppe Ruta, begins with the theory of motion and stability of equilibrium. For the continuous systems, the bifurcation and post-buckling analysis is presented and the effect of initial imperfections is evaluated. The examples refer to post-buckling of frames and thin walled compression members.

Section two "**Phenomenological Modelling of Instability Phenomena**", elaborated by Victor Gioncu, presents the main directions of research works in the field of structural stability, new phenomenological models of evolving systems, instability types and a phenomenological methodology for instability design.

Section three "**Modelling Buckling Interaction**", elaborated by Eduardo de Miranda Batista, refers to the light steel structures where many modes of stability are possible: flexural, torsional-flexural, local and distortional buckling modes. The paper presents the effect of interaction between these instability modes from theoretical and experimental point of view.

Section four "**Computational Asymptotic Post-buckling Analysis of Slender Elastic Structures**", elaborated by Raffaele Casciaro, introduces the computational treatment of asymptotic strategy for post-buckling analysis of elastic structures, using finite element method.

Section five "**Mechanical Models for the Subclasses of Catastrophes**" elaborated by Zsolt Gaspar presents the behavior of some mechanical models aiming to show the relation between theory of catastrophes and structural instability.

The lectures are addressed to post-graduate students (PhD and postdoc), to researchers as well as to civil, mechanical, naval and aeronautical engineers involved in structural design.

The coordinators of CISM course wish to thank warmly all the colleagues for the excellence of the work performed. Special thanks are also due to CISM Rector, Prof. M.G. Velarde, to the Editor of the Series, Prof C. Tasso, and to the entire CISM staff in Udine.

Marcello Pignataro
Victor Gioncu

CONTENTS

Mathematical Modelling of Instability Phenomena

M.Pignataro and G.C.Ruta
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Abstract Liapunov theory is first presented and discrete mechanical systems are in particular analysed. Then buckling and postbuckling analysis of continuous mechanical system using the general theory formulated by Koiter are discussed in some detail following Budiansky presentation. Finally, the influence of multiple interactive buckling modes on postbuckling behaviour is analysed in some detail for frames, thin-walled members and panels.

1 Theory of Stability of Motion

1.1 Introduction

In the development of the theory of differential equations, it is possible to distinguish two quite different approaches. The first is characterised by the search for a solution in closed form or through a process of approximation. The second can be distinguished from the first by the fact that information on the solution is sought without actually solving the problem. This qualitative analysis was introduced by Poincaré around 1880 (Poincaré, 1885) and developed in the following decades, especially in Russia.
The central problem in qualitative analysis is to investigate the relationship between the solution and its *neighbourhood*. A solution is a curve or a trajectory C in a certain space. The question is whether any D trajectory, which at the time $t=0$ starts *near* C, tends to remain near C or moves away from it. In the first case, the trajectory C is said to be *stable*, in the second *unstable*. Liapunov is credited with creating a systematic qualitative analysis, which is generally called the *theory of stability*. In 1892 he published the first of a series of fundamental papers "*General Problem on the Stability of Motion*" (Liapunov, 1966), in which he treated the problem of stability in two different ways. His so-called *first method* presupposes explicit knowledge of the solution and is applied only to a limited but important number of cases; the *second method*, or *direct method*, is altogether general and does not require knowledge of the solution.

1.2 Differential Equations

From a historical point of view differential equations were introduced by Newton through the laws of mechanics which define the motion of a body subjected to a system of forces. Subsequent developments in physics have shown how a wide range of problems in completely different fields is governed by laws which are altogether analogous to those of mechanics. Thus, it is desirable, as a first step, to describe the types of equations on which we shall be working and their properties.

The ordinary differential equations which are the basis of the problems we are to study are essentially of two types (La Salle and Lefschetz, 1961; Pontriaguine, 1969). The first is represented by an equation of n-th order

$$y^{(n)} = f(y, \dot{y}, ..., y^{(n-1)}; t) \tag{1.1}$$

where t is a variable and generally, but not necessarily, represents time, and $\dot{y}, ..., y^{(k)}$ represent the first,..., k-th derivative of y with respect to t. The second type is a system of n equations of the first order

$$\dot{y}_i = Y_i(y_j; t) \qquad (i, j = 1, 2, \ldots, n) \tag{1.2}$$

where, unless otherwise specified, the Latin indices are understood to vary from 1 to n.

The first type can be reduced to the second one if we introduce the new variables $y_1, y_2, ..., y_n$ defined by

$$y_i = y^{(i-1)} \tag{1.3}$$

In this case equation (1.1) is replaced by the system

$$\begin{aligned} \dot{y}_i &= y_{i+1} \qquad (i, j = 1, 2, \ldots, n-1) \\ \dot{y}_n &= f(y_j; t) \end{aligned} \tag{1.4}$$

As an example the well known equation of van der Pol

$$\ddot{y} + k(y^2 - 1)\dot{y} + y = 0 \tag{1.5}$$

can be replaced by the system

$$\begin{aligned} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -k(y_1^2 - 1)y_2 - y_1 \end{aligned} \tag{1.6}$$

If we consider $y_1, y_2, ..., y_n$ as components of a vector $\mathbf{y}$, and Y_1, Y_2,...,Y_n as components of a vector $\mathbf{Y}$, the system (1.2), can be written in the compact form

$$\dot{\mathbf{y}} = \mathbf{Y}(\mathbf{y}; t) \tag{1.7}$$

In many problems the variable t does not appear explicitly in (1.7). In this case, the system becomes

$$\dot{\mathbf{y}} = \mathbf{Y}(\mathbf{y}) \tag{1.8}$$

A system of this type is called *autonomous.* For example, the system deduced from van der Pol's equation is autonomous. A system of the type (1.7) is *non-autonomous.*

Once the solution $y_1 = f_1(t)$,$y_2 = f_2(t)$,..., $y_n = f_n(t)$ has been determined from (1.7) or (1.8) a curve, called the *integral curve*, in the space $E^{n+1}_{y,t}$ can be associated with it (Figure 1). The projection of this curve in the sub-space $E^n_y$of the $\mathbf{y}$ coordinates is defined as the *trajectory* or simple the *motion*, and the space E^n_y is the space of the *phases* (Figure 2). For the existence and uniqueness of the solution, the *Cauchy-Lipschitz* theorem holds.

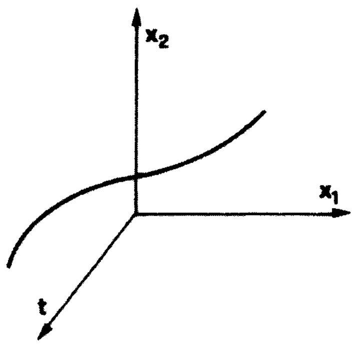

Figure 1. Integral curve

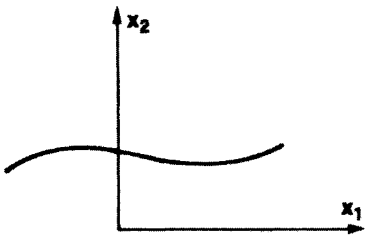

Figure 2. Trajectory

Theorem 1.1. *Let $E_{y,t}^{n+1}$ be the $n+1$ dimensions space y_i, t, and let Ω be a simply connected open region in such a space. Let the functions Y_i be continuous and admit partial derivatives $\partial Y_i / \partial y_h$ at each point of Ω. If $(\mathbf{y}_0, t_0)$ is a point in Ω there exists a unique solution of system (1.7) such that $\mathbf{y}(t_0) = \mathbf{y}_0$. Such a solution is a continuous solution of $(\mathbf{y}_0, t_0)$ as such a point varies in Ω.*

Let us consider a particular solution $\mathbf{v}(t)$ of system (1.7) and introduce new variables

$$\mathbf{x}(t) = \mathbf{y}(t) - \mathbf{v}(t) \tag{1.9}$$

By substituting (1.9) into (1.7) we obtain

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}, t) \tag{1.10}$$

where

$$\mathbf{X}(\mathbf{x}; t) = \mathbf{Y}(\mathbf{x} + \mathbf{v}; t) - \mathbf{Y}(\mathbf{v}; t) \tag{1.11}$$

Since in (1.11) $\mathbf{X}(\mathbf{0},\, t) = \mathbf{0}$, eqs. (1.10) admit as a solution $\mathbf{x}(t) = \mathbf{0}$, which is called *undisturbed motion* or *position of equilibrium* and furnish the differential equations of the *disturbed motion* $\mathbf{x}(t) \neq \mathbf{0}$. In the study of the stability of motion (eqs. (1.7), one can always refer to the study of stability of the undisturbed motion.

Let us now consider the motion defined by the autonomous system (1.8) and assume that for $\mathbf{y} = \mathbf{c}$, with $\mathbf{c}$ constant, $\mathbf{Y}(\mathbf{c}) = \mathbf{0}$. If we replace $\mathbf{y}$ by $\mathbf{c}$ in (1.8) we can see that the system is satisfied, and consequently $\mathbf{y} = \mathbf{c}$ is a solution to the system. From a physical point of view this means that if the system is initially in $\mathbf{c}$ then it remains in this position, and therefore $\mathbf{c}$ is a configuration of equilibrium. The point $\mathbf{c}$ is defined as the *critical point* or *equilibrium point.* By introducing the change of coordinates

$$\mathbf{x}(t) = \mathbf{y}(t) - \mathbf{c} \tag{1.12}$$

one has from (1.8)

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}) \tag{1.13}$$

with

$$\mathbf{X}(\mathbf{x}) = \mathbf{Y}(\mathbf{x} + \mathbf{c}) \tag{1.14}$$

whence $\mathbf{X}(\mathbf{0}) = \mathbf{0}$, and therefore the equilibrium point of eqs. (1.13) coincides with the origin. From now on in studying stability of autonomous system, we can always refer to the origin as equilibrium point.

Let us now introduce the *norm* of vector $\mathbf{x}$ which is indicated with $||\mathbf{x}||$. The most common norm is the Euclidean vector length

$$\|\mathbf{x}\| = (\sum_{i=1}^{n} x_i^2)^{1/2} \tag{1.15}$$

Two other types of norms which are often encountered are

$$\|\mathbf{x}\| = max\,|x_i| \tag{1.16}$$

$$\|\mathbf{x}\| = \sum_{i=1}^{n} |x_i| \tag{1.17}$$

The concepts of stability and of asymptotic stability stated below have been introduced by Liapunov in 1893 and therefore we speak of stability in the Liapunov sense, even if other definitions have been introduced later.

Definition 1.2. *The solution* $\mathbf{x} = \mathbf{0}$ *of the system* (1.10) *or* (1.13) *is said to be stable in the Liapunov sense if for each positive number* ε *it is possible to find a positive number* $\delta(\varepsilon)$ *such that if*

$$\|\mathbf{x}(t_0)\| < \delta \tag{1.18}$$

holds, then

$$\|\mathbf{x}(t)\| < \varepsilon\,, \qquad \forall t > t_0 \tag{1.19}$$

Definition 1.3. *The solution* $\mathbf{x} = \mathbf{0}$ *of the system* (1.10) *or* (1.13) *is said asymptotically stable in the sense of Liapunov, if for each solution* $\mathbf{x}(t)$ *with initial conditions*

$$\|\mathbf{x}(t_0)\| < \delta \tag{1.20}$$

we have

$$\lim_{t\to\infty} \|\mathbf{x}\| = 0 \tag{1.21}$$

Definition 1.4. *The solution* $\mathbf{x} = \mathbf{0}$ *of the system* (1.10) *or* (1.13) *is said unstable in the sense of Liapunov if for each number* ε *and for a positive number* δ *however fixed, there exists at least a point* $\mathbf{x}(t_0)$ *with*

$$\| \mathbf{x}(t_0) \| < \delta \tag{1.22}$$

such that

$$\| \mathbf{x}(t) \| > \varepsilon, \qquad \forall t > t_0 \tag{1.23}$$

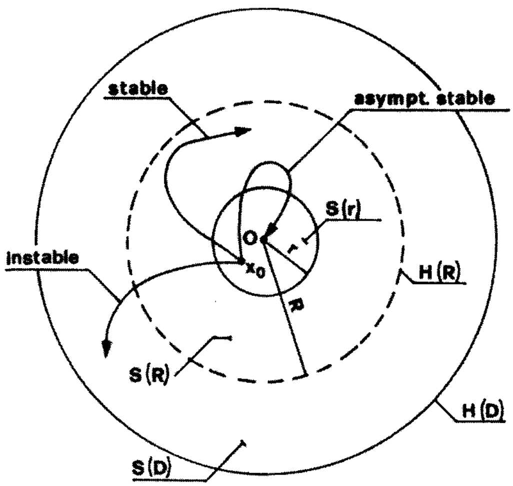

Figure 3. Types of equilibrium

Definition 1.5. *The domain of attraction of the solution of equilibrium* $\mathbf{x} = \mathbf{0}$ *of the system* (1.10) *or* (1.13) *is defined as the collection of the points* $\mathbf{x}(t_0)$, *such that motions starting from* $\mathbf{x}(t_0)$ *are asymptotically stable.*

Definitions 2, 3, 4 may be visualized in (Figure 3), where $S(D)$ is an open spherical region in Ω in which the conditions requested by the theorem of existence and uniqueness of the solution are satisfied and $H(D)$ is its boundary. In addition $S(R)$ is the spherical region defined by $||\mathbf{x}|| < R$ and $||\mathbf{x}|| = R$ is the spherical surface $H(R)$.Then we have

Definition 1.6. *A motion is stable if for every* $R < D$ *there exists* $r < R$ *such that a trajectory* $g(t)$ *with its origin at a point* $\mathbf{x}_0 \in S(r)$ *remains in the spherical region* $S(R)$ *when* t *increases; that is to say, a trajectory with origin in* $S(r)$ *never reaches the boundary* $H(R)$ *of* $S(R)$.

Definition 1.7. *A motion is asymptotically stable if it is stable and, besides, each trajectory* $g(t)$ *with origin in* $S(r)$ *tends to the origin for* $t \to \infty$.

Definition 1.8. *A motion is unstable if for a fixed $R < D$ and for any r, however small, there always exists a point $\mathbf{x}_0$ in $S(r)$, such that a $g(t)$ trajectory which originates in $\mathbf{x}_0$ reaches the boundary $H(R)$.*

The stability of motion (1.10) and of equilibrium (1.13) may depend on a certain number of parameters, besides the above mentioned disturbances. This is the case, for instance, of a rigid bar loaded by a vertical force N and connected to the ground by a hinge to which it is applied a linear elastic spring initially unloaded (Figure 4). Eq. (1.7) in this case becomes

$$\dot{\mathbf{y}} = \mathbf{Y}(\mathbf{y}; \gamma_k; t) \qquad (k = 1, 2, \ldots, m) \tag{1.24}$$

γ_k being a parameter. This problem is more complicated and can be treated through a perturbation analysis. By denoting with $\mathbf{v}(\gamma_k;t)$ a particular solution of (1.24), introducing the change of coordinates

$$\mathbf{x}(\gamma_k; t) = \mathbf{y}(\gamma_k; t) - \mathbf{v}(\gamma_k; t) \tag{1.25}$$

and replacing (1.25) into (1.24) the following equations are obtained

$$\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x}; \gamma_k; t) \tag{1.26}$$

having posed

$$\mathbf{X}(\mathbf{x}; \gamma_k; t) = \mathbf{Y}(\mathbf{x} + \mathbf{v}; \gamma_k; t) - \mathbf{Y}(\mathbf{v}; \gamma_k; t) \tag{1.27}$$

Since $\mathbf{X}(\mathbf{0};\gamma_k;t) = \mathbf{0}$, eqs. (1.26) admit as a solution $\mathbf{x}(\gamma_k;t) = \mathbf{0}$ which is the undisturbed motion, and permit to determine the disturbed motion $\mathbf{x}(\gamma_k;t) \neq \mathbf{0}$.

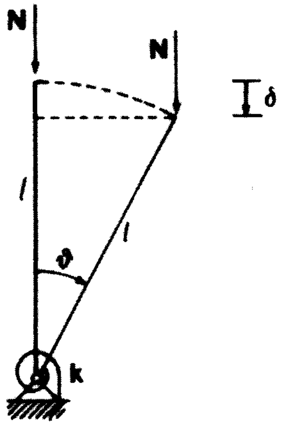

Figure 4. Stability depending on parameters

1.3 General Theorems on Stability

We shall give in this section few basic concepts regarding the *second method* or *direct method* of Liapunov on the qualitative response of a motion (eqs. (1.10) and (1.13)), without solving the relevant differential equations. We emphasize here that motion is any situation of a body defined by a set of state variables (chemical, electrical, mechanical processes). In particular, we shall refer first to motions defined by eqs. (1.13) and, following Liapunov, we introduce a scalar function $V(\mathbf{x})$ which is said to be *positive definite* in an open simply connected region Ω around the origin if the following conditions hold

(a) $V(\mathbf{x})$ together with its first partial derivatives is continuous in Ωg;
(b) $V(\mathbf{0}) = 0$;
(c) $V(\mathbf{x})$ has an isolated minimum at the origin.

If, in addition, dV/dt is non-positive in Ω along the trajectories of motion of system (1.13), that is

$$\dot{V} = V_{,i}\,\dot{x}_i = V_{,i}\,X_i(x_1, x_2, \dots, x_n) \leqslant 0 \tag{1.28}$$

the function $V(\mathbf{x})$ is called a *Liapunov function.* It is assumed that $X_i(x_j)$ and when necessary$\dot{X}_i(x_j)$are continuous. It follows that $\dot{V}$is a continuous function in Ω. In equation (1.28), a subscript preceded by a comma indicates differentiation with respect to the corresponding variable. Let us examine the quadratic form

$$V(\mathbf{x}) = a_{ij}x_ix_j \qquad (a_{ij} = a_{ji}\ , a_{ij} \in \mathbb{R}) \tag{1.29}$$

The necessary and sufficient condition for $V(\mathbf{x})$ to be positive definite is that the successive principal minors of the symmetrical matrix of the coefficients $[a_{ij}]$ have positive determinant (Sylvester).

Generally, the function $V(\mathbf{x})$ can be represented as a series of powers in $\mathbf{x}$ in the neighbourhood of the origin

$$V(\mathbf{x}) = V_p(\mathbf{x}) + V_{p+1}(\mathbf{x}) + \dots \tag{1.30}$$

where $V_p(\mathbf{x})$ is a homogeneous polynomial in $\mathbf{x}$ of degree p. A necessary condition for $V(\mathbf{x})$ to be positive definite is that the lowest degree p of the series of powers (1.30) is an even number. Such a condition, however, is not sufficient. In fact, for $p = 2$ the function

$$V_p(\mathbf{x}) = x_2^2 - x_1^2 \tag{1.31}$$

is positive definite for x_1= 0 and negative definite for $x_2 = 0$. If p is an odd number then $V(\mathbf{x})$ can never be a Liapunov function. Let us now pass to the enunciation of some basic theorems.

Theorem 1.9. ***Stability (Liapunov).*** *If in a certain neighbourhood Ω of the origin there exists a Liapunov function $V(\mathbf{x})$, then the origin is stable.*

Theorem 1.10. ***Asymptotic stability (Liapunov).*** *If there exists in Ω a Liapunov function $V(\mathbf{x})$ such that $\dot{V} < 0$ then the stability is asymptotic.*

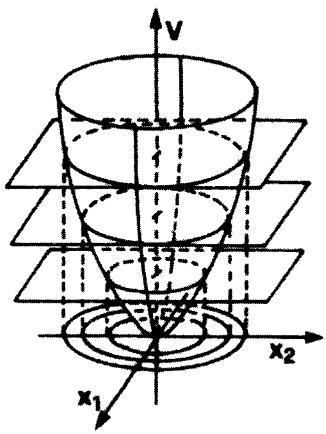

Figure 5. Liapunov function

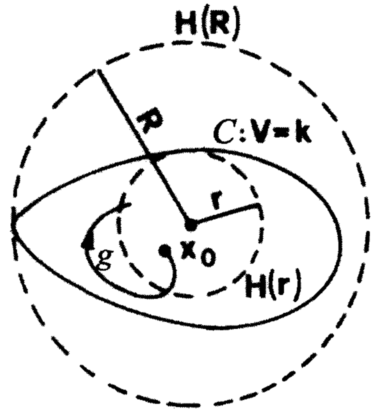

Figure 6. Plane of the phases

Let us first demonstrate the first theorem using the geometrical interpretation of a positive definite function $V(\mathbf{x})$, represented in (Figure 5) for $\mathbf{x} = \{x_1, x_2\}$. (Figure 6) the curve $V(\mathbf{x}) = k$ is represented by the solid line whilst the spheres $H(R)$ and $H(r)$ are indicated by dashes.

Given then $R < D$ (Figure 3) and $H(R)$, we can find a constant k such that the curve C defined by $V(\mathbf{x}) = k$ is contained in $H(R)$ and an $r > 0$ such that $H(r)$ is contained in C. Let us now consider a trajectory $g(\mathbf{x})$ with initial point $\mathbf{x}_0$ belonging to $S(r)$, the interior of $H(r)$. In $\mathbf{x}_0$ it is $V(\mathbf{x}_0) < k$. Furthermore, as $V(\mathbf{x})$ does not increase along the trajectories, $g(\mathbf{x})$ never reaches C and so will never reach $H(R)$. Therefore each trajectory with origin in $S(r)$ must remain in $S(R)$ and this implies stability.

The demonstration of the second theorem follows from the previous demonstration, since $\dot{V} < 0$ implies that the trajectory $g(\mathbf{x})$ which starts at $\mathbf{x}_0 \in S(r)$ tends to the origin as $t \to \infty$, and this implies asymptotic stability.

Theorem 1.11. ***Instability (Liapunov).*** *Let $V(\mathbf{x})$ with $V(\mathbf{0} = 0)$ have continuous first partial derivatives in Ω. Let $\dot{V}$ be positive definite and let $V(\mathbf{x})$ be able to assume positive values arbitrarily near the origin. Then the origin is unstable.*

The demonstration is omitted here. However, it is easy to guess that the condition $\dot{V} > 0$ implies that the trajectory $g(\mathbf{x})$ which starts from $\mathbf{x}_0 \in S(r)$ where $V(\mathbf{x}) > 0$ reaches C and therefore $H(R)$, and so we have instability.

Theorem 1.12. ***Instability (Chetayev).*** *Let Ω be a neighbourhood of the origin. Let $V(\mathbf{x})$ be a given function and Ω_1 a region in Ω with the following properties*

(a) *$V(\boldsymbol{x})$ and $\dot{V}$ are positive in Ω_1;*

(b) *$V(\boldsymbol{x})$ has continuous partial derivatives in Ω_1;*

(c) *at the boundary points of Ω_1 inside Ω $V(\mathbf{x}) = 0$;*

(d) *the origin is a point belonging to the boundary of* Ω_1.

Under these conditions the origin is unstable.

It is not difficult to see that any trajectory $g(\mathbf{x})$ starting from a point situated in Ω_1 must leave Ω since it cannot cross the boundary of Ω_1 inside Ω. As the origin is situated on the boundary of Ω_1, we can choose some points inside Ω_1 arbitrarily close to the origin from which trajectories $g(\mathbf{x})$ which start must leave Ω, and this implies instability.

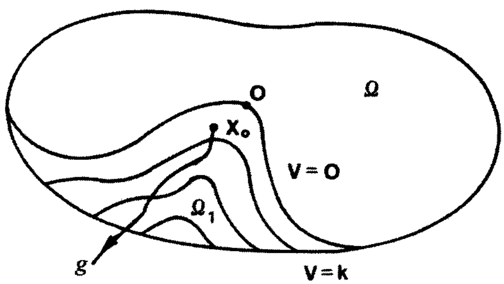

Figure 7. Representation of Chetayev's theorem on the plane of the phases

Example 1.13. Let us analyse the stability of the trivial solution to the system

$$\begin{aligned} \dot{x} &= -y - x^3 \\ \dot{y} &= x - y^3 \end{aligned} \tag{1.32}$$

The function $V(x, y) = x^2 + y^2$ satisfies the conditions of Liapunov's theorem on asymptotic stability. In fact

$$\begin{aligned} &\text{i)}\ \ V(x,y) =\geq 0, \qquad V(0,0) = 0 \\ &\text{ii)}\ \ \dot{V} = 2x(-y - x^3) + 2y(x - y^3) = -2(x^4 + y^4) \leq 0 \end{aligned} \tag{1.33}$$

At a point which is arbitrarily near the origin we have $\dot{V} < 0$, and so the origin is asymptotically stable.

Example 1.14. Analyse the stability of the equilibrium point $x = y = 0$ of the system of equations

$$\begin{aligned} \dot{x} &= y^3 + x^5 \\ \dot{y} &= x^3 + y^5 \end{aligned} \tag{1.34}$$

The function $V(x, y) = x^4 - y^4$ satisfies the conditions of Chetayev's theorem

$$\begin{aligned} &\text{i) } V(x,y) => 0, \qquad \text{for } |\,x\,|>|\,y\,| \\ &\text{ii) } \dot{V} = 4x^3(y^3+x^5) - 4y^3(x^3+y^5) = 4(x^8-y^8) > 0 \qquad \text{for } |\,x\,|>|\,y\,| \end{aligned} \tag{1.35}$$

In a neighbourhood of the origin and for $|\,x\,|>|\,y\,|$ we have $V > 0, \dot{V} > 0$; thus the equilibrium point $x = y = 0$ is unstable.

Liapunov theorems presented for autonomous systems (1.13) can be extended to non autonomous system (1.10). To this end we introduce a positive definite function $W(\mathbf{x})$. $V(\mathbf{x};t)$ is then *positive definite* if

$$\begin{array}{lll} \text{(a)} & V(\mathbf{0};t) = 0 & \text{for } t \geqslant 0 \\ \text{(b)} & V(\mathbf{x};t) \geqslant W(\mathbf{x}) & \text{for } t \geqslant 0 \text{ and } |\mathbf{x}| < r \end{array} \tag{1.36}$$

where r is a sufficiently small quantity (Figure 3). The function $V(\mathbf{x};t)$ is *negative definite* if, under condition (a)

$$V(\mathbf{x};t) \leqslant -W(\mathbf{x}) \qquad \text{for } t \geqslant 0 \text{ and } |\mathbf{x}| < r \tag{1.37}$$

For instance the function

$$V = t(x_1^2 + x_2^2) - 2x_1x_2\cos t \tag{1.38}$$

is positive definite for $t > 2$. In fact, by choosing $W = x_1^2 + x_2^2$ one has

$$V - W = (t-1)(x_1^2 + x_2^2) - 2x_1x_2\cos t > 0 \tag{1.39}$$

In a different example, the function $V = e^{-t}(x_1^2 + \ldots + x_n^2)$ is not positive definite in that $V \to 0$ when $t \to \infty$. In this case it is not possible to find any positive definite function W such that $V > W$. If, in addition to conditions (1.36), the function $V(\mathbf{x},t)$ satisfies the inequality $\dot{V} = \frac{\partial V}{\partial t} + V_{,i}\,X_i \leqslant 0$ along the trajectory of motion, we say that $V(\mathbf{x},t)$ is a *Liapunov function.*

The function $V(\mathbf{x};t)$ is said to be *decrescent* (or *uniformly small*) if it satisfies the condition

$$|V(\mathbf{x};t)| \leqslant W(\mathbf{x}) \qquad \text{for } t \geqslant 0 \text{ and } |\mathbf{x}| < r \tag{1.40}$$

where $W(\mathbf{x})$ is a positive definite function. For instance the function $V(\mathbf{x};t) =$ (sin $t)(x_1 + \ldots + x_n)$ is decrescent while the function $V(\mathbf{x};\, t) =$ sin $[t(x_1 + \ldots + x_n)]$ is not decrescent. In the following, theorems on stability for *non autonomous system* are presented without demonstration.

Theorem 1.15. *Stability (Liapunov).* *The equilibrium is stable if there exists a positive definite function $V(\mathbf{x};t)$ such that its total derivative $\dot{V}$ along the trajectory of motion (1.10) is not positive.*

Theorem 1.16. ***Asymptotic stability (Liapunov).*** *The equilibrium is asymptotically stable if a positive definite and decrescent function $V(\mathbf{x};t)$ exists such that its total derivative along the trajectory of motion (1.10) is negative.*

Theorem 1.17. ***Instability (Liapunov).*** *The equilibrium is unstable if a decrescent function $V(\mathbf{x};t)$ exists having the same sign of $\dot{V}$ along the trajectory of motion (1.10).*

It is important to remember that the existence of a Liapunov function is a sufficient but not necessary condition for stability.

1.4 Analysis of the Stability of Equilibrium by Linear Approximation

Let us consider the autonomous system (1.13) $\dot{\mathbf{x}} = \mathbf{X}(\mathbf{x})$ with $\mathbf{X}(\mathbf{0}) = \mathbf{0}$. If the functions X_i are derivable in a neighbourhood of the origin of coordinates, then the second member of system (1.13) can be substituted by a series expansion

$$\dot{x}_i = a_{ij}x_j + R_i(x_1, \ldots, x_n) \tag{1.41}$$

where $a_{ij} = (\partial X_i/\partial x_j)_{x=0}$ and $||\mathbf{R}||$ is small with respect to $||\mathbf{x}||$, that is to say

$$\lim_{\mathbf{x}\to 0} \frac{||\mathbf{R}||}{||\mathbf{x}||} = \mathbf{0} \quad \Leftrightarrow ||\mathbf{R}|| = o(||\mathbf{x}||) \tag{1.42}$$

Instead of investigating the stability of the equilibrium point $\mathbf{x} = \mathbf{0}$ of system (1.41), the stability of the same point of the linear system

$$\dot{x}_i = a_{ij}x_j \tag{1.43}$$

is analysed. System (1.43) is called a *system of equations of linear approximation* with respect to system (1.41). The conditions of stability of this system were examined by Liapunov and successively generalised by Malkin, Chetayev and others.

The analysis of stability of the system of equations of linear approximation is a much simpler problem than the study of the original system. In this regard there are two useful theorems of great practical importance.

Let us suppose that the characteristic roots λ_i of the matrix of coefficients $[a_{ij}]$ are real and distinct, and let us apply to system (1.4.1) the linear transformation of coordinates $\mathbf{y} = \mathbf{Px}$ with $\mathbf{P}$ non-singular and time independent. As $(d/dt)\mathbf{Px} = P\dot{x}$, by making use of (1.41) we can write

$$\dot{\mathbf{y}} = \mathbf{P}\dot{\mathbf{x}} = \mathbf{PAP}^{-1}\mathbf{y} + \mathbf{PR} \tag{1.44}$$

We now choose the matrix $\mathbf{P}$ in such a way that $\mathbf{PAP}^{-1} = \text{diag}(\lambda_1,\ \lambda_2, \ldots, \lambda_n)$ and assume $\mathbf{PR} = \mathbf{R}^*$. Then, the system (1.44) is written as

$$\dot{\mathbf{y}} = diag(\lambda_1, \lambda_2, ..., \lambda_n)\mathbf{y} + \mathbf{R}^* \tag{1.45}$$

where it can easily be shown that

$$||\mathbf{R}^*(\mathbf{y})|| = o(||\mathbf{y}||) \tag{1.46}$$

The transformation of system (1.41) into system (1.45) is useful for demonstrating the following theorems.

Theorem 1.18. *A sufficient condition for the origin of the non-linear system (1.45) to be asymptotically stable is that the characteristic roots are all negative. If there is a single positive characteristic root, then the origin is unstable.*

Two cases can be distinguished in the demonstration.

(a) *The λ_h roots are all negative*

The following Liapunov function is assumed

$$V = y_1^2 + y_2^2 + \ldots + y_n^2 \tag{1.47}$$

from which

$$\dot{V} = 2(\lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2) + r(\mathbf{y}) \tag{1.48}$$

where r is small with respect to the terms in parenthesis. In a sufficiently small region Ω around the origin V and -$\dot{V}$ are positive definite functions, and so the origin is asymptotically stable.

(b) *Some of the λ_h roots, for example λ_1, λ_2,...,λ_p (p ¡ n) are positive and the rest negative.*

This time we take

$$V = y_1^2 + \ldots + y_p^2 - y_{p+1}^2 - \ldots - y_n^2 \tag{1.49}$$

from which

$$\dot{V} = 2(\lambda_1 y_1^2 + \ldots + \lambda_p y_p^2 - \lambda_{p+1} y_{p+1}^2 - \ldots - \lambda_n y_n^2) + r_1(\mathbf{y}) \tag{1.50}$$

where, as before, the r_1 term is small with respect to those in parenthesis. At some points which are arbitrarily near to the origin (those for which $y_{p+1} = \ldots = y_n = 0$) V is positive. As for $\dot{V}$, since $\lambda_1, \lambda_2, \ldots, \lambda_p > 0$, it is positive definite in that Ω_1 region in Ω where V is positive definite and therefore, according to the Chetayev theorem, the origin is unstable.

Let us now suppose that some of the λ_h are complex. For example, let λ_1,...,λ_p be real and $\lambda_{p+1}, \bar{\lambda}_{p+1}, \ldots, \lambda_{p+m}, \bar{\lambda}_{p+m}$ be complex with $p+2m = n$. If λ_1,...,λ_p are negative and $\lambda_{p+h}, \bar{\lambda}_{p+h}$ have negative real part, then we can choose the following Liapunov function

$$V = y_1^2 + \ldots + y_p^2 + y_{p+1}\bar{y}_{p+1} + \ldots + y_{p+m}\bar{y}_{p+m} \tag{1.51}$$

and everything proceeds as in case (a), with the origin asymptotically stable. If, on the other hand, some of the λ_1,...,λ_p are positive or some of the λ_{p+h} have a positive real part, then we proceed exactly as in case (b) and we find that the origin is unstable. We can therefore enunciate the following theorem.

Theorem 1.19. *A sufficient condition for the origin of the non-linear system (1.45) to be asymptotically stable is that the characteristic roots all have negative real parts. If there is a characteristic root with positive real part, then the origin is unstable.*

If a certain number of characteristic roots vanish or have a purely imaginary value, results from the analysis of the linear approximation system cannot be extended to the nonlinear system, as the nonlinear terms R_i influence the stability of the system.

Example 1.20. Analyse the stability of the equilibrium point $x = y = 0$ of the system

$$\begin{aligned} \dot{x} &= 2x + 8\sin y \\ \dot{y} &= 2 - e^x - 3y - \cos y \end{aligned} \tag{1.52}$$

By expanding $\sin y$,$\cos y$ and e^x in a Taylor series around the origin we can write the system in the form

$$\begin{aligned} \dot{x} &= 2x + 8y + R_1 \\ \dot{y} &= -x - 3y + R_2 \end{aligned} \tag{1.53}$$

where $R_1 = -4y^3/3 + \ldots$ and $R_2 = (y^2 - x^2)/2 + \ldots$. Since the limitations (1.42) are satisfied we can analyse the stability of equilibrium point of the linear system

$$\begin{aligned} \dot{x} &= 2x + 8y \\ \dot{y} &= -x - 3y \end{aligned} \tag{1.54}$$

The roots of the characteristic equation $\lambda^2 + \lambda + 2 = 0$ are $\lambda_{1,2} = -1/2 \pm i\sqrt{7/4}$; therefore the equilibrium point $x = y = 0$ of system (1.54) and (1.52) is asymptotically stable.

Example 1.21. Let us consider the system

$$\begin{aligned} \dot{x} &= y - xf(x,y) \\ \dot{y} &= -x - yf(x,y) \end{aligned} \tag{1.55}$$

and suppose that the nonlinear terms $xf(x,y)$ and $yf(x,y)$ satisfy condition (1.42) and that $f(0,0) = 0$. The characteristic roots of the linear system are $\lambda_{1,2} = \pm i$ and therefore the analysis of the stability of equilibrium point $x = y = 0$ of system (1.55) depends on nonlinear terms. In fact, let us choose as Liapunov function $V = (x^2 + y^2)/2$, from which

$$\dot{V} = -(x^2 + y^2)f(x,y) \tag{1.56}$$

Three case can occur

$f \geq 0$ in an arbitrary vicinity of the origin, the origin is stable;
$f < 0$ in an arbitrary vicinity of the origin, the origin is unstable;
f is positive definite within a certain vicinity of the origin, the origin is asymptotically stable.

Note that the system of equations studied in Example(1.13) is of the same type as system (1.55). In fact, as the characteristic roots of the linearized system are $\lambda_{1,2} = \pm i$, the stability of the equilibrium point has been decided by non linear terms.

1.5 Criterion of Negative Real Parts of all the Roots of a Polynomial

In the previous section the problem of the stability of the trivial solution to a wide class of systems of differential equations was reduced to the analysis of the sign of the real parts of the roots of the characteristic equation.

If the characteristic equation is a polynominal of high degree, then its solution is very difficult, and therefore the methods which allow us to determine whether the roots do or do not have negative real parts are of great importance. With regard to this, we have the following

Theorem 1.22. ***(Hurwitz).*** *The necessary and sufficient condition for the real parts of all the roots of the polynomial*

$$p(z) \quad = z^n + a_1 z^{n-1} + \ldots + a_{n-1} z + a_n \tag{1.57}$$

with positive real coefficients to be negative is that each principal minor of the Hurwitz matrix

$$\begin{pmatrix} a_1 & 1 & 0 & 0 & \ldots & 0 \\ a_3 & a_2 & a_1 & 1 & \ldots & 0 \\ a_5 & a_4 & a_3 & a_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & a_n \end{pmatrix} \tag{1.58}$$

is positive.

Example 1.23. Let us consider the polynominal

$$p(z) \quad = z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 \tag{1.59}$$

The Hurwitz conditions require

$$a_1 > 0, \quad a_1 a_2 - a_3 > 0, \quad (a_1 a_2 - a_3) a_3 - a_4 a_1^2 > 0, \quad a_4 > 0 \tag{1.60}$$

to be satisfied.

2 Equilibrium of Mechanical System

2.1 Stability of Equilibrium of Discrete Mechanical Systems. Lagrange and Hamilton Equations of Motion

Lagrange has demonstrated (Gantmacher, 1970) that the differential equations of motion of a system with n degrees of freedom can be written immediately if we know the *kinetic potential* or *Lagrange function* defined by

$$L = K - \Phi \tag{2.1}$$

where K is the kinetic energy and Φ is the potential energy of the forces acting on the system.

Let $q_1, q_2, \ldots, q_n$ be generalized coordinates with which it is possible to define the configuration of the discrete system, and suppose that the q_i $(i = 1, 2, \ldots, n)$ are chosen in such a way that in the position of equilibrium we have $q_i = 0$. Indicating the position vector by $\mathbf{r} = \mathbf{r}(\mathbf{q};\, t)$, the kinetic energy of a system of N particles is expressed by the relation

$$K = \frac{1}{2}\sum_{i=1}^{N} m_i(\dot{r}_i)^2 = \frac{1}{2}\sum_{i=1}^{N} m_i\left(\frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t}\right)^2 \tag{2.2}$$

which can be written in the form

$$K = \frac{1}{2}a_{ij}\dot{q}_i\dot{q}_j + a_i\dot{q}_i + a_0 = K_2 + K_1 + K_0 \tag{2.3}$$

for *reonomous* systems or $K = K_2$ for *scleronomous systems.* In eq. (2.3) the summation convention with respect to repeated indices has been adopted.

The coefficientsa_{ij}, a_i, a_0 are function of $\mathbf{q}$ and t in the first case, while a_{ij} are function of $\mathbf{q}$, only, in the second case. We shall always refer, in the future, to scleronomous systems. The external forces Q_i are supposed to be *conservative,* i.e. derivable from a potential $\Phi = \Phi(\mathbf{q})$

$$Q_i = -\Phi_{,i} \tag{2.4}$$

The Lagrange equations of motions are then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2.5}$$

which are of the type

$$\ddot{\mathbf{q}} = \mathbf{f}\left(\mathbf{q}; \dot{\mathbf{q}}\right) \tag{2.6}$$

which, according to notations of sect. 1.2, can always be reduced to the form

$$\dot{\mathbf{x}} = \mathbf{S}(\mathbf{x}) \tag{2.7}$$

The kinetic potential L from which the equations (2.5) have been deduced depends on the variables $\mathbf{q}, \dot{\mathbf{q}}$ which are called *Lagrange variables.* Hamilton proposed to assume as basic variables the quantities $\mathbf{q}$ and $\mathbf{p}$, where $\mathbf{p}$ is the generalized linear momentum defined by

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = a_{ij}(q)\dot{q}_j \tag{2.8}$$

The quantities $\mathbf{q}$, $\mathbf{p}$ are called *Hamilton variables.* By simple steps it is possible to express the kinetic energy as a function of $\mathbf{q}$ and $\mathbf{p}$, arriving at the expression

$$K = \frac{1}{2} l_{ij}(\mathbf{q}) p_i p_j \tag{2.9}$$

The potential energy $\Phi(\mathbf{q})$ in terms of the new variables remains unchanged.

By introducing the *Hamiltonian*

$$H(\mathbf{p};\mathbf{q}) = \Phi(\mathbf{q}) + K(\mathbf{p};\mathbf{q}) + C \tag{2.10}$$

with C arbitrary constant, it is easily demonstrated that the following equations hold

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \end{aligned} \tag{2.11}$$

Equations (2.11) constitute the Hamilton equations of motion and it is possible to use them as an alternative to (2.5) in order to study the stability of equilibrium. These equations are of the type (2.7).

2.2 Stability of Equilibrium According to Liapunov

Let us consider a system in the state of equilibrium $\mathbf{q} = \mathbf{0}$, $\dot{\mathbf{q}} = \mathbf{0}$ and suppose that we apply at the instant $t = 0$ a perturbation characterised by

$$\mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \tag{2.12}$$

We now introduce a norm ρ which measures the *distance* between the state of equilibrium and the current state and endow ρ with the following properties

$$\begin{aligned} &\rho(\mathbf{q};\dot{\mathbf{q}}) > 0 && \text{for } \mathbf{q} \neq \mathbf{0}, \dot{\mathbf{q}} \neq \mathbf{0} \\ &\rho(\mathbf{q}_1 + \mathbf{q}_2; \dot{\mathbf{q}}_1 + \dot{\mathbf{q}}_2) \leqslant \rho(\mathbf{q}_1;\dot{\mathbf{q}}_1) + \rho(\mathbf{q}_2;\dot{\mathbf{q}}_2) && \text{(triangle inequality)} \\ &\rho(\alpha\mathbf{q};\alpha\dot{\mathbf{q}}) = |\alpha|\,\rho(\mathbf{q};\dot{\mathbf{q}}) && (\alpha \text{ real}) \end{aligned} \tag{2.13}$$

Definition 2.1. Liapunov. *The configuration of equilibrium $\mathbf{q} = \mathbf{0}$, $\dot{\mathbf{q}} = \mathbf{0}$ is stable if, for every positive number ε, there exists a second positive number $\delta(\varepsilon)$ with the property*

$$\rho\,[\mathbf{q}(t);\dot{\mathbf{q}}(t)] \leqslant \varepsilon \tag{2.14}$$

for any $t > 0$ and for any motion with initial conditions which satisfy

$$\rho_0 = \rho\left(\mathbf{q}_0; \dot{\mathbf{q}}_0\right) \leqslant \delta(\varepsilon) \tag{2.15}$$

Expressions of ρ which are suitable for the solution to mechanical problems are, for example

$$\rho = \sqrt{q_i q_i + \dot{q}_i \dot{q}_i} \tag{2.16}$$

$$\rho = \max|q_i| + \max|\dot{q}_i| \tag{2.17}$$

2.3 Lagrange-Dirichlet Theorem

In this section we demonstrate the Lagrange-Dirichlet theorem by following the presentation furnished in La Salle and Lefschetz (La Salle and Lefschetz, 1961). We notice from (2.10) that along the motion

$$\dot{H} = \frac{\partial H}{\partial q_i}\dot{q}_i + \frac{\partial H}{\partial p_i}\dot{p}_i = 0 \tag{2.18}$$

having made use of (2.11). Relation (2.18) shows that during motion the sum of the kinetic energy and of the potential energy remains constant. The theorem is therefore enunciated as follows

Theorem 2.2. ***(Lagrange-Dirichlet).*** *If the potential energy $\Phi(\mathbf{q})$ of a conservative system is positive definite in the neighbourhood $\Omega : ||\mathbf{q}|| < D$ of an equilibrium configuration, then the configuration of equilibrium is stable.*

Let us assume that the origin $\mathbf{q} = \mathbf{0}$ is a configuration of equilibrium and consider the motion arising from the perturbation (2.12) impressed at the instant $t = 0$. We choose the constant C, which appears in (2.10), so that $H(\mathbf{0}; \mathbf{0}) = 0$. Of the two terms $\Phi(\mathbf{q})$ and $K(\mathbf{p}; \mathbf{q})$ forming the Hamiltonian $H(\mathbf{p}; \mathbf{q})$, the kinetic energy is always positive definite. If *the potential energy has an isolated minimum by correspondence with the configuration of equilibrium* $\mathbf{q} = \mathbf{0}$, then $H(\mathbf{p}; \mathbf{q})$ is also positive definite, and as $\dot{H} = 0$ in conservative systems, the function H is a Liapunov function. Therefore in accordance with the Liapunov theorem in section 1.3, the position of equilibrium is stable.

The Lagrange-Dirichlet theorem on the stability of equilibrium does not give any information on the behaviour of a mechanical system when the potential energy corresponding to a configuration of equilibrium does not exhibit a minimum. There are two theorems regarding this, accredited to Liapunov and Chetayev, respectively, which are enunciated here without proof.

Theorem 2.3. ***Theorem on instability (Liapunov).*** *If the potential energy $\Phi(\mathbf{q})$ of a conservative system has an isolated maximum corresponding to a configuration of equilibrium, then the configuration of equilibrium is unstable.*

Theorem 2.4. ***Theorem on instability (Chetayev).*** *If the potential energy $\Phi(\mathbf{q})$ of a conservative system is a homogeneous function of the coordinates $\mathbf{q}$ and if, corresponding to a configuration of equilibrium, $\Phi(\mathbf{q})$ does not have a minimum, then the configuration of equilibrium is unstable.*

Example 2.5. Let us suppose that the potential energy of a system is of the type $\Phi(\mathbf{q}) = A q_1 q_2 \dots q_n$, with A positive real constant, and that $\mathbf{q} = \mathbf{0}$ is a configuration of equilibrium. The aim is to examine the type of equilibrium of such a configuration. According to the Chetayev theorem, we can assert that the configuration of equilibrium $\mathbf{q} = \mathbf{0}$ of the system is unstable.

3 Stability of Equilibrium of Mechanical Autonomous Systems and Postcritical Behaviour

3.1 Discrete Systems

We have furnished in sect. 1.3 the necessary and sometimes sufficient conditions for the Liapunov function to be positive definite and therefore for the equilibrium of a general system to be stable. The same arguments hold true for a mechanical system when,according to Lagrange theorem, the total potential energy $\Phi(\mathbf{q})$ is employed instead of $V(\mathbf{x})$. More in general, by assuming that $\Phi(\mathbf{q})$ is a continuous regular function we write the series expansion

$$\begin{aligned}\Phi(\mathbf{q}) &= \Phi_2(\mathbf{q}) + \Phi_3(\mathbf{q}) + \Phi_4(\mathbf{q}) + \dots \\ &= \frac{1}{2}\left(\frac{\partial^2\Phi}{\partial q_i \partial q_j}\right)_{\mathbf{q}=\mathbf{0}} q_i q_j + \frac{1}{6}\left(\frac{\partial^3\Phi}{\partial q_i \partial q_j \partial q_h}\right)_{\mathbf{q}=\mathbf{0}} q_i q_j q_h \\ &+ \frac{1}{24}\left(\frac{\partial^4\Phi}{\partial q_i \partial q_j \partial q_h \partial q_k}\right)_{\mathbf{q}=\mathbf{0}} q_i q_j q_h q_k + \dots \qquad (i,j,h,k = 1,\dots,n)\end{aligned} \tag{3.1}$$

where derivatives are evaluated at the origin and the first order derivative term vanishes because of the equilibrium at that point. In alternative form, eq. (3.1) is written as

$$\Phi(\mathbf{q}) = C_{ij} q_i q_j + C_{ijh} q_i q_j q_h + C_{ijhk} q_i q_j q_h q_k + \dots (i,j,h,k = 1,\dots,n) \tag{3.2}$$

The matrix $[C_{ij}]$ is called *stiffness matrix* in the configuration of equilibrium. If the quadratic form $C_{ij} q_i q_j$is positive definite, negative definite or indefinite, then it prevails on higher order terms and consequently the equilibrium is stable in the first case or unstable in the second and third case. If it is positive semidefinite (positive definite in all directions except in one direction $\bar{\mathbf{q}}$ where $\Phi_2(\bar{\mathbf{q}}) = 0$), then higher order terms must be analysed. The total potential energy is then positive definite if

$$\Phi_3(\bar{\mathbf{q}}) = 0 \tag{3.3}$$

$$\Phi_4(\bar{\mathbf{q}}) > 0 \tag{3.4}$$

where (3.3) is a *necessary and sufficient condition* and (3.4) is a *necessary condition only.*

For (3.4) to be also sufficient, $\Phi_4(\bar{\mathbf{q}})$ must be "*sufficiently larger*" than zero. As an example, let us consider the function

$$\Phi(\mathbf{q}) = q_2^2 + q_1^2 q_2 + c q_1^4 \tag{3.5}$$

of a two degrees of freedom system where c is a constant. Along the direction $q_2 = 0$ the quadratic and cubic terms vanish and besides $\Phi(\mathbf{q}) > 0$ if $c > 0$. This condition is only necessary but not sufficient. To show this, we observe that (3.5) can be rewritten in the form

$$\Phi(\mathbf{q}) = (q_2 + \frac{1}{2}q_1^2)^2 + (c - \frac{1}{4})q_1^4 \tag{3.6}$$

Along the curve

$$q_2 = -\frac{1}{2}q_1^2 \tag{3.7}$$

the energy is positive definite if $c > 1/4$ and negative definite if $c < 1/4$. Therefore the sufficient condition for stability is $c > 1/4$. The results of this analysis are represented in (Figure 8).

The variational equation

$$\delta\Phi_2(\mathbf{q})\delta(\mathbf{q}) = 0 \tag{3.8}$$

is an eigenvalue problem which furnishes the *bifurcation points* along the *fundamental path* and one or more coincident or nearly coincident *buckling modes*. In addition, initial imperfections may by present in the structures as geometric imperfections, loads eccentricity and so on.

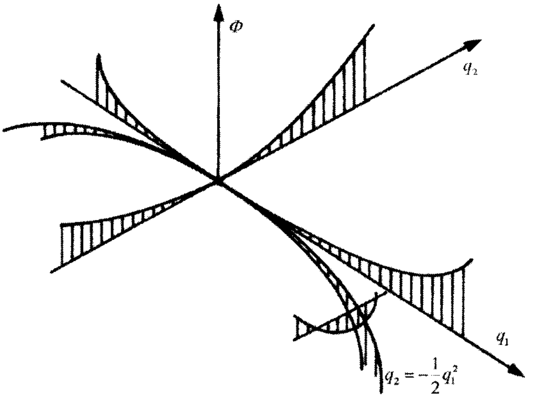

Figure 8. Representation of the function (3.6) for $c < 1/4$

The entire analysis regarding the solution to eq. (3.8), the evaluation of all equilibrium paths for perfect structures, the effect of the interaction between several simultaneous buckling modes in the presence or without initial imperfections, will be analysed in detail in dealing with continuous systems. Actually, the theory of buckling and postbuckling behaviour for discrete and continuous systems follows parallel directions and is based on perturbation theory.

4 Equilibrium of Mechanical Continuous Systems

4.1 Introduction

In this chapter we present a resumé of Koiter general theory of elastic stability (Koiter, 1945) in the form reformulated by Budiansky (Budiansky, 1974), focusing in particular our attention on interactive buckling.

The subject of interactive buckling has received a great deal of attention in the last decades, after Koiter and Skaloud (Koiter and Skaloud, 1963) have pointed out the danger of naive optimization without due regard to imperfection sensitivity. Van der Neut (Van der Neut, Springer Verlag, Berlin 1969) formulated a simple mechanical model to investigate the behaviour of a thin-walled column. Graves-Smith (Graves-Smith, 1967) investigated the full range behaviour of a locally buckled box column including the interaction of the overall mode as well as plasticity effects.

After these pioneering works there has been a spread of studies on this subject. With regard to framed structures we mention the works by Pignataro and Rizzi (Pignataro and Rizzi, 1983; Rizzi and Pignataro, 1982) who investigated symmetric and asymmetric structures. Interaction between two and three overall buckling modes in thin walled members was studied by Grimaldi and Pignataro (Grimaldi and Pignataro, 1979). Stiffened panels have been analysed by Tveergaard (Tvergaard, 1973) and successively by Koiter and Pignataro (Koiter and Pignataro, 1976). Axially stiffened cylindrical shells have been investigated by Byskov and Hutchinson (Byskov and Hutchinson, 1977). All these works furnish an analitycal solution to the problem.

More recently, in order to override mathematical difficulties, a semianalytical approach has been utilised by many researchers who have employed the finite strip method to study local-overall interaction in plated structures such as thin-walled members. Among these authors we mention Hancock (Hancock, 1981), Bradford and Hancock (Bradford and Hancock, 1984), Sridharan et al. (Benito and Sridharan, 1984-1985; Sridharan, 1983; Sridharan and Ali, 1985; Sridharan and Benito, 1984), Pignataro et al. (Pignataro and Luongo, 1987; Pignataro, Luongo et al., 1985). The problem of the interaction of infinitely many buckling modes has been finally studied by Byskov (Byskov, 1986) and Luongo and Pignataro (Luongo and Pignataro, 1988) who have confirmed the occurrence of localization phenomena previously pointed out by Tvergaard and Needleman (Tvergaard and Needleman, 1980) and Potier-Ferry (Potier-Ferry, 1984) after the experimental results obtained by Moxham (Moxham, 1971).

A few of the previously listed works make use of the direct equilibrium method while most of them utilize the Koiter / Budiansky perturbation theory to calculate the postbuckling equilibrium paths. For asymmetric structures, the analysis is usually carried out up to third order terms in order to evaluate the slope of the bifurcated paths. This is in general sufficient to describe the postcritical behaviour of the systems. If the system is symmetric, then the analysis is more involved since the evaluation of the curvature of the bifurcated paths is necessary. There are however a few cases in which the slope of the bifurcated paths, even if different from zero, is so small that the evaluation of the curvature is necessary.

4.2 Bifurcation and Postbuckling Analysis

Let $\Phi[\mathbf{u};\lambda]$ be the total potential energy of a hyperelastic body subjected to conservative loads, where $\mathbf{u}$ is the displacement field measured from the stress free configuration and λ a parameter governing the external force field. The equilibrium condition is obtained by requiring the functional Φ $[\mathbf{u};\lambda]$ to be stationary with respect to all kinematically admissible displacement fields, that is

$$\Phi'[\mathbf{u};\lambda]\delta\mathbf{u} = 0 \qquad \forall\, \delta\mathbf{u} \tag{4.1}$$

where a prime denotes Fréchet differentiation with respect to $\mathbf{u}$. Eq. (4.1) furnishes all possible equilibrium paths $\mathbf{u} = \mathbf{u}(\lambda)$. In stability theory it is usually assumed that an equilibrium path $\mathbf{u}_0 = \mathbf{u}_0(\lambda)$ is known (*fundamental path*). Then, a second bifurcated equilibrium path is detected by writing

$$\mathbf{u}(\lambda) = \mathbf{u}_0(\lambda) + \mathbf{v}(\lambda) \tag{4.2}$$

$\mathbf{v}(\lambda)$ being an additional displacement measured from the fundamental configuration (Figure 9). By replacing eq. (4.2) into eq. (4.1) and performing the series expansion with respect to $\mathbf{v}$, we have

$$\Phi_0''\mathbf{v}\delta\mathbf{u} + \frac{1}{2}\Phi_0'''\mathbf{v}^2\delta\mathbf{u} + \ldots = 0 \tag{4.3}$$

where $\Phi_0'' \equiv \Phi''[\mathbf{u}_0(\lambda);\lambda], \ldots$and use has been made of eq. (4.1). From eq. (4.3), by expanding each term with respect to λ starting from the bifurcation value $\lambda = \lambda_c$ we obtain

$$\begin{aligned}&[\Phi_c'' + (\lambda-\lambda_c)\,\dot{\Phi}''_c + \frac{1}{2}(\lambda-\lambda_c)^2\ddot{\Phi}_c'' + \ldots]\mathbf{v}\delta\mathbf{u} + \frac{1}{2}[\Phi_c''' + (\lambda-\lambda_c)\dot{\Phi}_c''' + \ldots]\mathbf{v}^2\delta\mathbf{u}\\ &+ \frac{1}{6}[\Phi_c^{IV} + \ldots]\mathbf{v}^3\delta\mathbf{u} = 0\end{aligned} \tag{4.4}$$

In eq. (4.4) $\dot{\Phi}''_c \equiv (d/d\lambda)\Phi_0''|_{\lambda=\lambda_c}, \ldots$etc. It is now convenient to express the dependence of $\mathbf{v}$ on λ through a parameter ξ

$$\lambda = \lambda(\xi) \qquad \mathbf{v} = \mathbf{v}(\xi) \tag{4.5}$$

Then under the assumption of regularity and keeping in mind that we are looking for an asymptotic solution to our problem, we write eqs. (4.5) as series expansions from $\xi = 0$

$$\begin{aligned}\lambda &= \lambda_c + \lambda_1\xi + \frac{1}{2}\lambda_2\xi^2 + \ldots\\ \mathbf{v} &= \mathbf{v}_1\xi + \frac{1}{2}\mathbf{v}_2\xi^2 + \frac{1}{6}\mathbf{v}_3\xi^3 + \ldots\end{aligned} \tag{4.6}$$

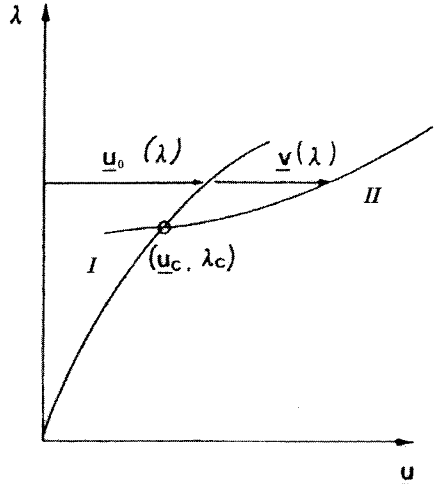

Figure 9. Equilibrium paths in a structure

where $\lambda(0) = \lambda_c$ and $\mathbf{v}(0) = \mathbf{0}$. Besides $\lambda_n = d^n\lambda/d\xi^n|_{\lambda=\lambda_c}$ and $v_n = d^n v/d\xi^n|_{\lambda=\lambda_c}$. By replacing (4.6) into (4.4) and collecting terms with equal power of ξ we have

$$\begin{aligned}&\xi\ \{\Phi_c''\mathbf{v}_1\}\ \delta\mathbf{u} + \frac{1}{2}\xi^2\ \{\ \Phi_c''\mathbf{v}_2 + 2\lambda_1\dot{\Phi}_c''\mathbf{v}_1 + \Phi_c'''\mathbf{v}_1^2\ \}\ \delta u + \frac{1}{6}\xi^3 \left\{ \Phi_c''\mathbf{v}_3 + 3\lambda_1\dot{\Phi}_c''\mathbf{v}_2 \right. \\ &\left. + 3\lambda_2\dot{\Phi}_c''\mathbf{v}_1 + 3\lambda_1^2\ddot{\Phi}_c''\mathbf{v}_1 + 3\Phi_c'''\mathbf{v}_1\mathbf{v}_2 + 3\lambda_1\dot{\Phi}_c'''\mathbf{v}_1^2 + \Phi_c^{IV}\mathbf{v}_1^3\ \right\}\ \delta\mathbf{u} + \ldots = 0\end{aligned} \tag{4.7}$$

whence the first, second and third order perturbation equations are obtained by equating separately to zero terms with equal power of ξ

$$\begin{aligned}\Phi_c''\mathbf{v}_1\delta\mathbf{u} &= 0 \\ \Phi_c''\mathbf{v}_2\delta\mathbf{u} &= -\ \{\ 2\lambda_1\dot{\Phi}_c''\mathbf{v}_1 + \Phi_c'''\mathbf{v}_1^2\ \}\ \delta\mathbf{u} \\ \Phi_c''\mathbf{v}_3\delta\mathbf{u} &= -\ 3\,\{\ \lambda_1\dot{\Phi}_c''\mathbf{v}_2 + \lambda_2\dot{\Phi}_c''\mathbf{v}_1 + \lambda_1^2\ddot{\Phi}_c''\mathbf{v}_1 \\ &\qquad + \Phi_c'''\mathbf{v}_1\mathbf{v}_2 + \lambda_1\dot{\Phi}_c'''\mathbf{v}_1^2 + \frac{1}{3}\Phi_c^{IV}\mathbf{v}_1^3\ \}\ \delta\mathbf{u}\end{aligned} \tag{4.8}$$

By denoting with ($\dot{}$) partial differentiation with respect to λ eqs. (4.8) change into

$$
\begin{aligned}
&\Phi_c''\mathbf{v}_1\delta\mathbf{u} = 0 \\
&\Phi_c''\mathbf{v}_2\delta\mathbf{u} = -\left\{\ \Phi_c'''\mathbf{v}_1^2 + 2\lambda_1\Phi_c'''\hat{\mathbf{u}}_c\mathbf{v}_1 + 2\lambda_1\hat{\Phi}_c''\mathbf{v}_1\ \right\}\delta\mathbf{u} \\
&\Phi_c''\mathbf{v}_3\delta\mathbf{u} = -\,3\ \{\ \Phi_c'''\ \mathbf{v}_1\mathbf{v}_2 + \frac{1}{3}\Phi_c^{IV}\mathbf{v}_1^3 \\
&\qquad + \lambda_1\left[\ \Phi_c'''\hat{\mathbf{u}}_c\mathbf{v}_2 + \hat{\Phi}_c''\mathbf{v}_2 + \Phi_c^{IV}\hat{\mathbf{u}}_c\mathbf{v}_1^2 + \hat{\Phi}_c'''\mathbf{v}_1^2\ \right] \\
&\qquad + \lambda_1^2\left[\ \Phi_c^{IV}\hat{\mathbf{u}}_c^2\mathbf{v}_1 + 2\hat{\Phi}_c'''\hat{\mathbf{u}}_c\mathbf{v}_1 + \Phi_c'''\hat{\mathbf{u}}_c\mathbf{v}_1 + \hat{\hat{\Phi}}{}_c''\mathbf{v}_1\ \right] \\
&\qquad + \lambda_2\left[\Phi_c'''\ \hat{\mathbf{u}}_c\mathbf{v}_1 + \hat{\Phi}_c''\mathbf{v}_1\ \right]\ \}\,\delta\mathbf{u}
\end{aligned}
\tag{4.9}
$$

If Φ is bilinear in $\mathbf{u}$ and λ and besides $\mathbf{u}$ is a linear function of λ along the fundamental path as in most practical problems, it results $\hat{\Phi}_c'' = \hat{\Phi}_c''' = \hat{\hat{\Phi}}{}_c'' = \hat{u}_c = 0$ and eqs. (4.9) simplify into

$$
\begin{aligned}
&\Phi_c''\mathbf{v}_1\delta\mathbf{u} = 0 \\
&\Phi_c''\mathbf{v}_2\delta\mathbf{u} = -\left\{\ \Phi_c'''\mathbf{v}_1^2 + 2\lambda_1\Phi_c'''\hat{\mathbf{u}}_c\mathbf{v}_1\ \right\}\delta\mathbf{u} \\
&\Phi_c''\mathbf{v}_3\delta\mathbf{u} = -\,3\ \{\ \Phi_c'''\ \mathbf{v}_1\mathbf{v}_2 + \frac{1}{3}\Phi_c^{IV}\mathbf{v}_1^3 + \lambda_1\left[\Phi_c'''\hat{\mathbf{u}}_c\mathbf{v}_2 + \Phi_c^{IV}\hat{\mathbf{u}}_c\mathbf{v}_1^2\right] \\
&\qquad\qquad \lambda_1^2\Phi_c^{IV}\hat{\mathbf{u}}_c^2\mathbf{v}_1 + \lambda_2\Phi_c'''\ \hat{\mathbf{u}}_c\mathbf{v}_1\}\ \delta\mathbf{u}
\end{aligned}
\tag{4.10}
$$

Eq. $(4.8)_1$ or $(4.9)_1$ is an eigenvalue problem whose solution furnishes the critical load λ_c and the buckling mode $\mathbf{v}_1$. Suppose that the solution to this problem has several linearly independent eigenmodes $\mathbf{v}_{1i}$ $(i = 1, \dots, m)$ all associated with the lowest eigenvalue λ_c. Then the most general solution to eq. $(4.8)_1$ can be expressed as a linear combination

$$
\mathbf{v}_1 = \nu_i\mathbf{v}_{1i} \qquad (i = 1, 2, \dots, m) \tag{4.11}
$$

where repeated indices denote summation from 1 to m and ν_i are arbitrary parameters. Without loss of generality these modes can be orthonormalised according to

$$
\Pi_2''\mathbf{v}_{1i}\mathbf{v}_{1j} = \delta_{ij} \qquad (i, j = 1, 2, \dots, m) \tag{4.12}
$$

where δ_{ij} is the Kronecker delta and Π_2 collects all quadratic terms of the series expansion around the stress-free configuration of the elastic energy. By requesting $\Pi_2''\mathbf{v}_1^2 = 1$, the condition

$$
\nu_i\nu_i = 1 \tag{4.13}
$$

follows.

To evaluate the second coefficient $\mathbf{v}_2$ of the series expansion $(4.6)_2$ we use the differential equation

$$
\Phi_c''\mathbf{v}_2\delta\mathbf{u} = -\left\{2\lambda_1\dot{\Phi}_c''\nu_i\mathbf{v}_{1i} + \Phi_c'''(\nu_i\mathbf{v}_{1i})^2\right\}\delta\mathbf{u} \tag{4.14}
$$

obtained from $(4.8)_2$ by replacing $\mathbf{v}_1$ with $\nu_i \mathbf{v}_{1i}$. By assuming $\delta\mathbf{u} = \mathbf{v}_{11}, \mathbf{v}_{12}, \dots, \mathbf{v}_{1m}$ successively, eq. (4.14) in conjunction with eq. $(4.8)_1$ yields a set of m equations of the type (Fredholm orthogonality conditions)

$$A_{ijk}\nu_i\nu_j + \lambda_1 B_{ik}\nu_i = 0 \qquad (i, j, k = 1, \dots, m) \tag{4.15}$$

where

$$A_{ijk} = \Phi_c''' \mathbf{v}_{1i}\mathbf{v}_{1j}\mathbf{v}_{1k}, \qquad B_{ik} = 2\dot{\Phi}_c'' \mathbf{v}_{1i}\mathbf{v}_{1k} \tag{4.16}$$

Eqs. (4.15) together with eq. (4.13) permit the evaluation of the $m+1$ coefficients ν_i and λ_1. Since eqs. (4.15) are nonlinear it is shown that, unless all the A_{ijk}'s vanish, there are at the most $2^m - 1$ essentially different real solutions and at least one real solution each one describing a bifurcated path (Van der Waerden, 1950). In the following we shall distinguish the case in which all the A_{ijk} vanish ($\lambda_1 = 0$, symmetric postbuckling behaviour) from the case in which one at least of these coefficients is different from zero ($\lambda_1 \neq 0$, asymmetric behaviour).

4.3 Asymmetric Postcritical Behaviour

The general integral of the differential equation (4.14) corresponding to any of the r solutions of eqs. (4.15) can be written as

$$\mathbf{v}_2 = \beta_i \mathbf{v}_{1i} + \mathbf{v}_{2p} \tag{4.17}$$

where $\mathbf{v}_{2p}$ denotes a particular integral and $\beta_i \mathbf{v}_{1i}$ is the general solution of the homogeneous equation with β_i arbitrary constants. If the orthogonality condition

$$\Pi_2'' \mathbf{v}_1 \mathbf{v}_2 = 0 \tag{4.18}$$

is imposed, then by using (4.11), (4.12), (4.17) the following condition on the coefficients β_i is obtained

$$\nu_i(\Pi_2'' \mathbf{v}_{1i}\mathbf{v}_{2p} + \beta_i) = 0 \tag{4.19}$$

In order to evaluate the constants β_i and the second load rate coefficient λ_2 corresponding to each of the r bifurcated paths we make use of the equation

$$\begin{aligned} \Phi_c'' \mathbf{v}_3 \delta\mathbf{u} = -3\Big\{ & \lambda_1 \dot{\Phi}_c'' \left(\mathbf{v}_{2p} + \beta_i \mathbf{v}_{1i}\right) + \lambda_2 \dot{\Phi}_c'' \nu_i \mathbf{v}_{1i} + \lambda_1^2 \ddot{\Phi}_c'' \nu_i \mathbf{v}_{1i} \\ & + \Phi_c'''(\nu_i \mathbf{v}_{1i}\mathbf{v}_{2p} + \nu_i\beta_j \mathbf{v}_{1i}\mathbf{v}_{1j}) + \lambda_1 \dot{\Phi}_c'''(\nu_i \mathbf{v}_{1i})^2 \\ & + \frac{1}{3}\Phi_c^{IV}(\nu_i \mathbf{v}_{1i})^3 \ \Big\} \ \delta\mathbf{u} \end{aligned} \tag{4.20}$$

obtained by replacing eqs. (4.11), (4.17) in $(4.8)_3$. Note that there are r equations (4.20) each one corresponding to a particular solution ν_i, λ_1 of eqs. (4.15). By successively identifying $\delta\mathbf{u}$ with $\mathbf{v}_{11}, \dots, \mathbf{v}_{1m}$ and by imposing the orthogonality condition on the right hand member of each of these equations we obtain

$$\begin{aligned}(2A_{ijk}\nu_j + \lambda_1 B_{ik})\beta_i + \lambda_2 B_{ik}\nu_i = &- 2\breve{A}_{ijkl}\nu_i\nu_j\nu_l \\ &- \lambda_1(2B_{ijk}\nu_i\nu_j + B_{pk}) \\ &- \lambda_1^2 C_{ik}\nu_i - 2A_{pik}\nu_i \\ &(i,\, j,\, k,\, l = 1, \ldots, m)\end{aligned} \tag{4.21}$$

where A_{ijk} andB_{ik} are given by (4.16) and

$$\begin{aligned}\breve{A}_{ijkl} &= \frac{1}{3}\Phi_c^{IV}\mathbf{v}_{1i}\mathbf{v}_{1j}\mathbf{v}_{1k}\mathbf{v}_{1l} \\ B_{ijk} &= \dot{\Phi}_c'''\mathbf{v}_{1i}\mathbf{v}_{1j}\mathbf{v}_{1k} \\ C_{ik} &= 2\ddot{\Phi}_c''\mathbf{v}_{1i}\mathbf{v}_{1k}\end{aligned} \tag{4.22}$$

Besides

$$\begin{aligned}A_{pik} &= \Phi_c'''\mathbf{v}_{2p}\mathbf{v}_{1i}\mathbf{v}_{1k} \\ B_{pk} &= 2\dot{\Phi}_c''\mathbf{v}_{2p}\mathbf{v}_{1k}\end{aligned} \tag{4.23}$$

Eqs. (4.19), (4.21) are r linear nonhomogeneous systems each one containing $m + 1$ equations (m orthogonality conditions plus a constraint equation) and $m + 1$ unknowns β_i, λ_2. For each bifurcated path the coefficients $\mathbf{v}_2$ and λ_2 of the series expansions (4.6) can thus be evaluated.

4.4 Symmetric Postcritical Behaviour

Symmetric postcritical behaviour arises in the particular case in which all coefficients A_{ijk}'s previously examined vanish. In this situation from eqs. (4.15) the solution $\lambda_1 = 0$ is obtained and the coefficients ν_i's, which are undetermined at this level, must be evaluated from the third order perturbation equation. Eq. (4.14) admits now the particular solution

$$\mathbf{v}_{2p} = \nu_i\nu_j\mathbf{v}_{2ij} \tag{4.24}$$

where the $\mathbf{v}_{2ij}$'s satisfy the equations

$$\Phi_c''\mathbf{v}_{2ij}\delta\mathbf{u} = -\Phi_c'''\mathbf{v}_{1i}\mathbf{v}_{1j}\delta\mathbf{u} \tag{4.25}$$

Also, due to the arbitrariness of the ν_i's, the orthogonality condition (4.19) furnishes all β_i's as function of ν_i's

$$\beta_i = p_{ijk}\nu_j\nu_k \tag{4.26}$$

where

$$p_{ijk} = -\Pi_2''\mathbf{v}_{1i}\mathbf{v}_{2jk} \tag{4.27}$$

By replacing (4.24) into eq. (4.20) and accounting for $\lambda_1 = 0$, we have the Fredholm orthogonality conditions

$$2A_{ijkl}\nu_i\nu_j\nu_l + \lambda_2 B_{ik}\nu_i = 0 \qquad (i,\ j,\ k,\ l = 1,\dots,m) \tag{4.28}$$

where the B_{ik}'s are given by $(4.16)_2$ and

$$A_{ijkl} = \Phi_c'''\mathbf{v}_{1i}\mathbf{v}_{1l}\mathbf{v}_{2jk} + \Phi_c^{IV}\mathbf{v}_{1i}\mathbf{v}_{1j}\mathbf{v}_{1k}\mathbf{v}_{1l} \tag{4.29}$$

Note that the solution to eqs. (4.28) is unaffected by the β_i's because of the assumption $A_{ijk} = 0$.

Eqs. (4.28) together with condition (4.13) are a set of $m + 1$ equations in the $m + 1$ unknowns ν_i's and λ_2 and therefore, according to Bézout (Van der Waerden, 1950), they furnish at the most $3^m - 1$ essentially different real solutions and at least one real solution. By correspondence with each set ν_i, the corresponding β_i's are determined from eq. (4.26) and consequently $\mathbf{v}_2$ from eqs. (4.17), (4.24) and (4.25).

If the second hand member in eq. (4.25) vanishes for any $\delta\mathbf{u}$, then $\mathbf{v}_{2ij} = \mathbf{0}$. Consequently $\beta_i = 0$ from eqs. (4.26), (4.27), $\mathbf{v}_{2p} = \mathbf{0}$ from eq. (4.24) and $\mathbf{v}_2 = \mathbf{0}$ follows from eq. (4.17).

4.5 Single Buckling Mode

If a single buckling mode $\mathbf{v}_1$ occurs for $\lambda = \lambda_c$, explicit expressions for the first and second load rate λ_1 and λ_2 are obtained by solving the Fredholm orthogonality conditions relative to eqs. $(4.8)_2$ and $(4.8)_3$, respectively. It is found that

$$\lambda_1 = -\frac{1}{2}\frac{\Phi_c'''\mathbf{v}_1^3}{\dot{\Phi}_c''\mathbf{v}_1^2} \tag{4.30}$$

$$\lambda_2 = -\frac{\frac{1}{3}\Phi_c^{IV}\mathbf{v}_1^4 + \Phi_c'''\mathbf{v}_1^2\mathbf{v}_2 + \lambda_1(\dot{\Phi}_c''\mathbf{v}_1\mathbf{v}_2 + \dot{\Phi}_c'''\mathbf{v}_1^3 + \lambda_1\ddot{\Phi}_c''\mathbf{v}_1^2)}{\dot{\Phi}_c''\mathbf{v}_1^2} \tag{4.31}$$

which for $\lambda_1 = 0$ reduces to

$$\lambda_2 = -\frac{\frac{1}{3}\Phi_c^{IV}\mathbf{v}_1^4 + \Phi_c'''\mathbf{v}_1^2\mathbf{v}_2}{\dot{\Phi}_c''\mathbf{v}_1^2} \tag{4.32}$$

5 Initial Imperfections

5.1 General Theory

If the structure under analysis is not perfect, in that it contains a displacement $\bar{\mathbf{u}}$ before the application of load, its potential energy functional $\Phi[\mathbf{u};\ \lambda]$ is modified as follows

$$\bar{\Phi} = \Phi\left[\mathbf{u};\lambda\right] + \Psi\left[\mathbf{u},\bar{\mathbf{u}};\lambda\right] \tag{5.1}$$

where $\Psi[\mathbf{u},\mathbf{0};\lambda] = 0$ for any $\mathbf{u}$. The equilibrium eq. (4.1) reads then

$$\Phi'[\mathbf{u};\lambda]\ \delta\mathbf{u} + \Psi'[\mathbf{u},\bar{\mathbf{u}};\lambda]\ \delta\mathbf{u} = 0 \tag{5.2}$$

Under the assumption that $\bar{\mathbf{u}}$ is small, let us take the series expansion of (5.2) in terms of $\bar{\mathbf{u}}$ from $\bar{\mathbf{u}} = \mathbf{0}$ by retaining only linear terms

$$\Phi'[\mathbf{u};\lambda]\ \delta\mathbf{u} + \tilde{\Psi}'[\mathbf{u},\mathbf{0};\lambda]\ \bar{\mathbf{u}}\delta\mathbf{u} = 0 \tag{5.3}$$

In eq. (5.3) the symbol ($\sim$) denotes differentiation with respect to $\bar{\mathbf{u}}$. Besides use has been made of the property that Ψ vanishes for $\bar{\mathbf{u}} = \mathbf{0}$.

Let

$$\mathbf{u}(\lambda) = \mathbf{u}_0(\lambda) + \bar{\mathbf{v}}(\lambda) \tag{5.4}$$

be an equilibrium path where $\mathbf{u}_0(\lambda)$ is the fundamental path of the perfect structure and $\bar{\mathbf{v}}(\lambda)$ an additional displacement measured from it. By replacing (5.4) into (5.3) and expanding in terms of $\bar{\mathbf{v}}$ we have

$$\{\,\Phi_0''\ \bar{\mathbf{v}} + \frac{1}{2}\Phi_0'''\ \bar{\mathbf{v}}^2 + \frac{1}{6}\Phi_0^{IV}\bar{\mathbf{v}}^3 + \ldots + \tilde{\Psi}_0'\bar{\mathbf{u}} + \tilde{\Psi}_0''\ \bar{\mathbf{u}}\,\bar{\mathbf{v}} + \ldots\}\ \delta\mathbf{u} = 0 \tag{5.5}$$

where $\tilde{\Psi}_0' = \tilde{\Psi}'[\mathbf{u}_0,\mathbf{0};\lambda],\ldots$ Further expansion of eq. (5.5) in terms of λ about the critical load $\lambda = \lambda_c$ gives

$$\begin{aligned}&[\Phi_c'' + (\lambda-\lambda_c)\dot{\Phi}_c'' + \frac{1}{2}(\lambda-\lambda_c)^2\ddot{\Phi}_c'' + \ldots]\ \bar{\mathbf{v}}\delta\mathbf{u}\\ &+ \frac{1}{2}[\Phi_c''' + (\lambda-\lambda_c)\dot{\Phi}_c''' + \ldots]\ \bar{\mathbf{v}}^2\delta\mathbf{u} + \frac{1}{6}[\Phi_c^{IV} + \ldots]\ \bar{\mathbf{v}}^3\delta\mathbf{u} + \ldots\\ &+ \left[\tilde{\Psi}_c' + (\lambda-\lambda_c)\dot{\tilde{\Psi}}_c' + \ldots\right]\bar{\mathbf{u}}\delta\mathbf{u} + \left[\tilde{\Psi}_c'' + \ldots\right]\bar{\mathbf{u}}\,\bar{\mathbf{v}}\delta\mathbf{u} + \ldots = 0\end{aligned} \tag{5.6}$$

being $\dot{\tilde{\Psi}}_c' = (d/d\lambda)\,\tilde{\Psi}_0'|_{\lambda=\lambda_c},\ldots$

It is now convenient to choose an initial imperfection in the form

$$\bar{\mathbf{u}} = \bar{\xi}\mathbf{u}^* = \alpha\xi^\gamma\mathbf{u}^* \tag{5.7}$$

where $\bar{\xi}$ is the imperfection amplitude and $\mathbf{u}$ gives the shape of the imperfection which is normalised according to $\Pi_2''\mathbf{u}^{*2} = 1$. Besides α is a scalar parameter and the exponent $\gamma > 0$ will be chosen to suit our convenience. Under suitable regularity condition we write

$$\begin{aligned}\lambda &= \lambda_c + \bar{\lambda}_1\xi + \frac{1}{2}\bar{\lambda}_2\xi^2 + \ldots\\ \bar{\mathbf{v}} &= \bar{\mathbf{v}}_1\xi + \frac{1}{2}\bar{\mathbf{v}}_2\xi^2 + \ldots\end{aligned} \tag{5.8}$$

Then replacing eqs. (5.8) into (5.6) and collecting terms with equal powers of ξ one gets