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MULTISCALE MODELLING OF DAMAGE AND FRACTURE PROCESSES IN COMPOSITE MATERIALS

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PREFACE

Various types of composites are used in engineering practice. The most important are fibrous composites, laminates and materials with a more complicated geometry of reinforcement in the form of short fibres and particles of various properties, shapes and sizes.

The aim of course was to understand the basic principles of damage growth and fracture processes in ceramic, polymer and metal matrix composites. Nowadays, it is widely recognized that important macroscopic properties like the macroscopic stiffness and strength, are governed by processes that occur at one to several scales below the level of observation. Understanding how these processes influence the reduction of stiffness and strength is essential for the analysis of existing and the design of improved composite materials.

The study of how these various length scales can be linked together or taken into account simultaneously is particular attractive for composite materials, since they have a well-defined structure at the micro and meso-levels. Moreover, the microstructural and mesostructural levels are well-defined: the microstructural level can be associated with small particles or fibres, while the individual laminae can be indentified at the mesoscopic level. For this reason, advances in multiscale modelling and analysis made here, pertain directly to classes of materials which either have a range of relevant microstructural scales, such as metals, or do not have a very well-defined microstructure, e.g. cementitious composites.

In particular, the fracture mechanics and optimization techniques for the design of polymer composite laminates against the delamination type of failure was discussed. Computational modelling of laminated composites at different scales: microscopic mesoscopic and macroscopic with application of suitable plate/shell elements for thin composites was presented. The application of fracture and damage mechanics approaches to the description of the complete constitutive behaviour of high performance fibre-reinforced cementitious composites was discussed. With regard to ceramic matrix composites (CMC) the damage and fracture processes was described in three scales. The important problem of damage process of interfaces surrounding particles, grains or fibres in composites was analysed for different properties of the components of composites and in different scales.

The course brought together experts dealing with materials science, mechanics, experimental and computational techniques at the three mentioned scales. I acknowledge the commitment of Professors: H.Altenbach, R. de Borst, P.Ladeveze, B.Karihaloo and Z.Mroz in making the course possible in the nice atmosphere of the Palazzo del Torso in Udine. Lectures delivered by mentioned Professors presenting the latest achievements in the topic of the course and discussions with the course participants significantly enriched the scientific aim of this course. 58 participants PhD students, postdocs, senior researchers and engineers had good opportunity to *listen to interesting lectures and discuss their on going research problems with leading persons in the field of the course.*

I thank to the Rectors and staff of CISM for help and co-operation in the organization of the course and printing these lecture notes.

Tomasz Sadowski

CONTENTS

Modelling of anisotropic behavior in fiber and particle reinforced composites

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Abstract Fiber and particle reinforced composites are widely used in aircraft, spacecraft and automotive industries, but also in various branches of the traditional mechanical engineering. They substitute classical materials like steel, aluminium, etc. since their specific stiffness is significant higher. The optimal design of structures made of reinforced composites demands the mathematical description of the constitutive behavior of these materials characterized by anisotropic mechanical properties and inhomogeneities. This contribution is devoted to the phenomenological modelling of fiber and particle reinforced materials.

After a short introduction the modelling principles are briefly discussed. For a realistic material description the anisotropic elasticity is necessary. The generalized HOOKE's law is introduced and the symmetry relations of the stiffness and compliance tensors are discussed. For the analysis of the limit state of composite materials various failure and strength criteria are presented. Finally, a short introduction into modelling of polymer suspensions is given.

1 Introduction

Fiber and particle reinforced composites are used as structural materials in many application fields: aircraft and rocket industries, mechanical and civil engineering, sport goods and automotive industries, etc. The reason for this is a number of advantages in comparison with the traditional structural materials: high specific stiffness properties, small weight, etc. It must be noted that there are also disadvantages: for example, more complex design rules and failure analysis. The application fields, the advantages and disadvantages are discussed, for example by Altenbach et al. (2004); Altenbach & Becker (2003); Ashbee (1994); Chawla (1987); Ehrenstein (1992); Gay (2002); Gibson (1994); Hult & Rammerstorfer (1994); Jones (1975); Kim (1995) and Powell (1994).

The design of structures made of composites is connected with two main problems:

- the material behavior is usually anisotropic and
- the inhomogeneous distribution of all properties must be considered.

In the first case - anisotropic material behavior - one has to apply the anisotropic constitutive equations of continuum mechanics since the anisotropic behavior can be observed in the elastic, viscoelastic, plastic, etc. range. In addition, the classical failure and strength analysis based on the existence of an equivalent stress and a criterion, which allows to compare complex (multiaxial) stress states with some experimental data based on uniaxial tests must be extended. The problem is that in the case of anisotropic material behavior various failure modes are existing and a unique criterion for all cases cannot be established. The second item - the inhomogeneity of the material behavior - is more complicated. As is known from many practical applications for the general analysis of the stress or strain states one can use the overall properties assuming that the material is quasi-homogeneous and can be described with the help of effective ("smeared") properties. This approach works successfully in the case of structural elements made of composites if only the global mechanical characteristics (for example, the deflections of plates or the eigenfrequencies) are to be computed. In this case the comparison with the experimental data is satisfying. A quite different situation one obtains if the local behavior plays the main role. Now the averaged properties cannot be applied and the heterogeneity of the material must be considered.

Below the anisotropic analysis of composite materials and structures is discussed. The attention is paid to the elastic range and the limit state only. Both situations are mostly assumed in practical applications. In addition, two types of reinforcement are considered: the unidirectional continuous fiber and the short fiber (particle) reinforcements. They are assumed as a satisfying approximation in many practical cases. From the theoretical point of view the analysis of continuous fiber reinforced composites is much simpler - in the case of particle reinforcement the heterogeneity plays an important role.

After this brief introduction the basics of modelling the material behavior and anisotropic elasticity are presented. Some remarks concerning the principles of the global failure analysis are presented. Finally, some models of particle reinforced composites are discussed.

2 Materials behavior modelling

The modelling of the material behavior is a necessary first step for the engineering analysis of any structure. Since the geometry, the loading cases, etc. are often very complex the analysis must be performed computer-aided mostly. For this purpose one needs mathematical expressions describing the material behavior. In this section some problems in material behavior modelling will be discussed. For further reading one can recommend, for example, Altenbach & Skrzypek (1999); Haddad (2000a,b); Hergert et al. (2004); Lemaitre (2001); Lemaitre & Chaboche (1985) and Skrzypek & Ganczarski (2003).

2.1 Continuum mechanics background

The basic equations in *Continuum Mechanics* of deformable bodies can be divided into to groups, see Lai et al. (1993)

- the material independent equations and
- the material dependent equations.

The first group is following from the general balance equations, added by the statement of stresses and geometrical relations. As the main result one gets the equilibrium equations or the equations of motion. Since the material behavior can be reversible or irreversible from the energy and the entropy balance some statements of the physical admissibility of the deformation processes can be made.

The second group of equations allows the description of the individual response of any material on the applied stresses/forces or strains. The so-called constitutive equations (added, may be, by evolution equations) are related to some of the general balances (they describe the theoretical and the mathematical framework), but the concretion must be performed without any general physical rules. The theoretical framework for the concretion is presented by Haupt (2002); Krawietz (1886) or Palmov (1998).

In addition, the coefficients or parameters of the constitutive and the evolution equations must be identified by tests. There are different possibilities, discussed in Altenbach et al. (1995). Let us assume a macroscopic test, for example, the tension test. In this case one observes the stress-strain curve assuming that the stress and the strain is acting in the same direction. The problem is now how to describe mathematically this curve. At first, it is impossible to find a general analytical function for all stress and strain values. At second, it is clear that such a description is acceptable only for a very specific situation (for example, some parameters like the temperature or the moisture are fixed, the stresses lie in a small range, etc.). So we get from the tension test only a special law of the constitutive behavior.

It must be noted that this approach cannot be used for the modelling and simulation of the three-dimensional behavior, especially in the case of anisotropy since one needs experimental benefit from an infinite number of tests. In such a situation one has to perform a finite number of tests, that means one has to realize, for example, the tension test, the compression test, the shear test (torsion of a thin-walled cylindrical specimen), the two-dimensional tension test (biaxial tension test) and the hydrostatic compression test. In all these cases as a result one obtains stress-strain curves, but the curves can differ significantly. In addition, since the choice of tests is not unique the results depend on the kind of tests that are performed. Note that tests realizing homogeneous stress and strain states are preferred.

Limiting our further discussions to pure mechanical performances the mathematical description of the material behavior can be simplified since for the formulation of the constitutive and evolution equations one needs only a few variables. Let us introduce these variables.

At first let us focus our attention on the strains. In Fig. 1 typical strains are shown. One can

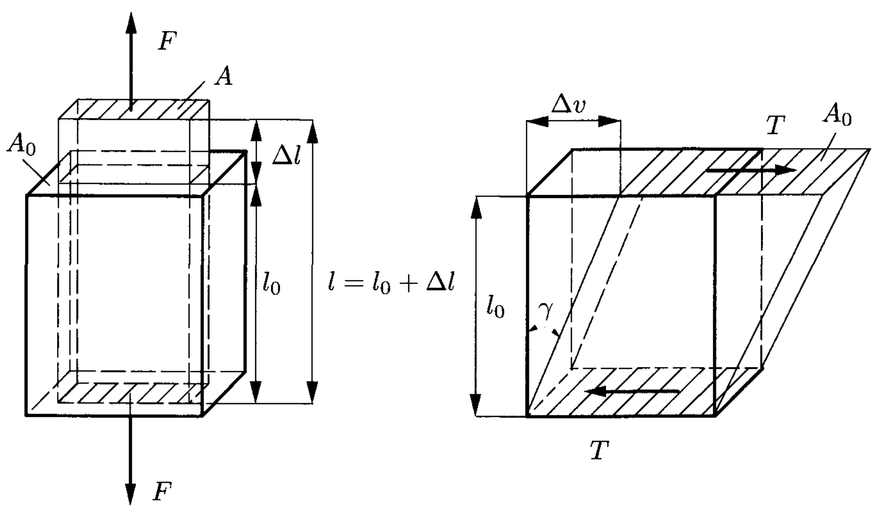

Figure 1. Possible strains: extensional (left) and shear (right) strains

consider that there are two types of strains:

- Extensional strains ε: The body changes only its volume but not its shape.
- Shear strains γ: The body changes only its shape but not its volume.

Concerning Fig. 1 in the one-dimensional case one can define the stresses and strains as follows. Assuming a uniform distribution of the forces F and T on the cross-section we introduce

$$\begin{aligned} \sigma &= \frac{F}{A_0} & & \text{normal stress} \quad \sigma, \\ \varepsilon &= \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0} & & \text{extensional strain} \quad \varepsilon, \\ \tau &= \frac{T}{A_0} & & \text{shear stress} \quad \tau, \\ \gamma &\approx \tan\gamma = \frac{\Delta v}{l_0} & & \text{shear strain} \quad \gamma \end{aligned}$$

In the general case of the classical material behavior the stress state is characterized by the stress tensor $\boldsymbol{\sigma}$. This is from the mathematical point of view a second rank tensor and assuming a orthonormal co-ordinate system (Cartesian co-ordinates x_i with the unit basic vectors $\boldsymbol{e}_i$ which have to fulfil the following conditions: $|\boldsymbol{e}_i| = 1, \boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{ij}$, δ_{ij} is the KRONECKER symbol, $i, j = 1, 2, 3$). The following representation is valid

$$\boldsymbol{\sigma} = \sigma_{ij}\boldsymbol{e}_i\boldsymbol{e}_j \tag{2.1}$$

Using $\boldsymbol{\sigma}$ we are applying the absolute or invariant notation, σ_{ij} are the coordinates in the index notation. The invariant notation used here is presented, for example, by Lurie (1990).

Let us discuss the meaning of the components of the stress tensor. The normal stresses are related to $i = j$ and the shear stresses to $i \neq j$. Note that $\sigma_{ij} = \sigma_{ji}$ and for this case the stresses are shown in Fig. 2 The three-dimensional state of strains is characterized by the strain tensor

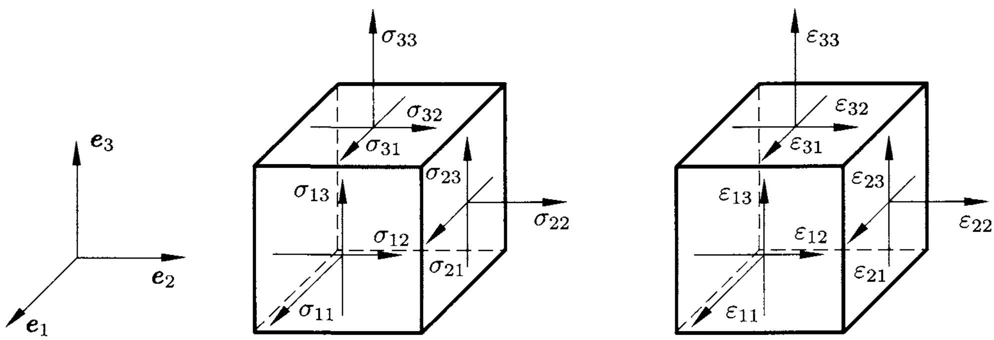

Figure 2. Stress and strain tensor components for Cartesian coordinates

ε_{ij} with the extensional strains in the case $i = j$ and the shear strains for $i \neq j$. Note that ε_{ij} with $i \neq j$ are the tensor shear coordinates, $2\varepsilon_{ij} = \gamma_{ij}, i \neq j$ the engineering shear strains. The coordinates of the strain tensor are also shown in Fig. 2.

Any second rank tensor can also be presented as a $[3 \times 3]$ matrix. For the stresses we obtain

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_2 & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_3 \end{bmatrix} \tag{2.2}$$

Since the symmetry of the stress tensor is assumed ($\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{T}}$ or $\sigma_{ij} = \sigma_{ji}$) the representation can be simplified as

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{12} & \sigma_2 & \tau_{23} \\ \tau_{13} & \tau_{23} & \sigma_3 \end{bmatrix} \tag{2.3}$$

In addition, the following vector representation is possible

$$\boldsymbol{\sigma} = [\ \sigma_1 \quad \sigma_2 \quad \sigma_3 \quad \sigma_4 \quad \sigma_5 \quad \sigma_6\ ]^{\mathrm{T}} \tag{2.4}$$

Between the components of the stress tensor (2.1) or the stress matrix (2.3) and the stress vector (2.4) the following relations exist

$$\sigma_{11} = \sigma_1, \quad \sigma_{22} = \sigma_2, \quad \sigma_{33} = \sigma_3, \quad \sigma_{23} = \sigma_4, \quad \sigma_{13} = \sigma_5, \quad \sigma_{12} = \sigma_6$$

Considering small deformations the following strain tensor can be introduced

$$\boldsymbol{\varepsilon} = [\boldsymbol{\nabla u}]^{\mathrm{sym}} = \frac{1}{2}[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^{\mathrm{T}}] \tag{2.5}$$

Here $\boldsymbol{u}$ denotes the displacement vector and $\boldsymbol{\nabla}$ is the Nabla operator. Assuming again Cartesian coordinates one can write

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \equiv \frac{1}{2}(u_{j,i} + u_{i,j}) \tag{2.6}$$

The strain tensor written down as a matrix

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & 2\varepsilon_{12} & 2\varepsilon_{13} \\ 2\varepsilon_{12} & \varepsilon_{22} & 2\varepsilon_{23} \\ 2\varepsilon_{13} & 2\varepsilon_{23} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{12} & \varepsilon_{22} & \gamma_{23} \\ \gamma_{13} & \gamma_{23} & \varepsilon_{33} \end{bmatrix} \tag{2.7}$$

or as a vector

$$\boldsymbol{\varepsilon} = [\ \varepsilon_1 \quad \varepsilon_2 \quad \varepsilon_3 \quad 2\varepsilon_4 = \gamma_4 \quad 2\varepsilon_5 = \gamma_5 \quad 2\varepsilon_6 = \gamma_6\ ]^{\mathrm{T}} \tag{2.8}$$

The components of the strain tensor are shown in Fig. 2.

It can be established that the symmetry of the stress tensor results in a symmetry of the strain tensor. This is not a general statement in *Continuum Mechanics*, but writing down the elastic energy, for example, one can see that only the symmetric part of the strain tensor plays a role in further discussions if the stress tensor is symmetrically. From the symmetry condition of the strain tensor follows that the strains can be represented by (2.8).

Remark: The starting point of discussion of the anisotropic behavior is connected with three principal assumptions:

- classical continuum assumption (no polar continua),

- small strains assumption, and
- elastic behavior assumption.

For many composite material applications these assumptions are valid since composites are mostly brittle, that means they behave linear elastically with the exception of the limit state characterized by the failure. From the *Material Science* it is well known that brittleness can be observed at small strains and after the elastic range the fracture starts immediately. The assumption of the stress tensor symmetry is under discussion, but using non-symmetric stress tensors the identification of the material properties is more complicated (Nowacki (1985)). So we decided for the main part of this contribution that the assumption of the stress tensor symmetry is valid.

2.2 Elastic behavior

The history of the theory of elasticity is presented in several monographs and textbooks (Todhunter & Pearson (1886), Todhunter & Pearson (1893), Love (1927), Timoshenko (1953), Hetnarski & Ignaczak (2004) among others). In parallel the theory of strength and failure was developed. Some important steps in the development of models for the elastic behavior were

- the establishment of HOOKE's law,
- the introduction of the YOUNG's modulus,
- the stress and strain concepts,
- the theory of linear elasticity,
- the discussion related to the number of material parameters,
- the anisotropic elasticity,
- isotropic failure and strength criteria,
- the anisotropic failure and strength,
- the application of continuous fiber reinforced composites, and
- particle reinforced composites.

It is easy to see that both the theory of elasticity and the failure/strength theories were developed by the inductive way (the generalization was made step by step). Only during the last fifty years the deductive theory was formulated by Truesdell & Noll (1992) and others.

Let us now discuss the elasticity condition more in detail. The starting point is the introduction of two second rank tensors (the stress tensor $\boldsymbol{\sigma}$ and the strain tensor $\boldsymbol{\varepsilon}$) which are symmetrically and characterize the stress and the strain state. Now the question is how to formulate a constitutive equation for the elastic behavior.

The simplest case is the HOOKE's law

$$\sigma = E\varepsilon \tag{2.9}$$

containing only one material parameter. From the mathematical point of view the HOOKE's law is an algebraic linear equation of two scalar variables (the stress σ and the strain ε). The general form of a linear function of two variables is

$$\sigma = a\varepsilon + b$$

The coefficients can be estimated as follows: a is equal to E (the YOUNG's modulus) and b in many applications can be assumed to be 0 otherwise b characterizes the eigenstress. The result

$b = 0$ is identical to the statement that from the stress free state assumption follows no strains and vice versa.

The HOOKE's law is a special constitutive equation connecting the mechanical variables only. So, for example, isothermal conditions must be considered. The basic idea is coming from the original HOOKE's proposal that the loading state and the deformation state are proportional. At present this statement can be formulated for the normal stresses and strains as (2.9). For the shear stresses and shear strains the following relation is valid

$$\tau = G\gamma, \qquad G = \frac{\tau}{\gamma}, \qquad G \quad \text{shear modulus} \tag{2.10}$$

From the mathematical point of view (2.9) is, as was mentioned, a linear function of two variables. By this equation eigenstresses and eigenstrains cannot be described, and the nonlinear behavior cannot be presented. Since the stress and the strain states in the three-dimensional case are presented by the stress tensor and the strain tensor one has to built up a linear function between second rank tensors

$$\boldsymbol{\sigma} = {}^{(4)}\boldsymbol{E} \cdot\cdot\, \boldsymbol{\varepsilon}, \qquad \sigma_{ij} = E_{ijkl}\varepsilon_{kl}; k,l = 1,2,3 \tag{2.11}$$

The role of the proportionality factor plays the fourth rank HOOKEan tensor ${}^{(4)}\boldsymbol{E}$. Now the main problem is the analysis of the fourth rank tensor ${}^{(4)}\boldsymbol{E} = E_{ijkl}\boldsymbol{e}_i\boldsymbol{e}_j\boldsymbol{e}_k\boldsymbol{e}_l$ which must be related to the material properties of the linear-elastic anisotropic continuum. The experimental identification of the components of this tensor is non-trivial.

Considering the three-dimensional space $\mathbb{R}^3$ the number of the elasticity tensor components E_{ijkl} is $3^4 = 81$ with 3 as the dimension of the space and 4 as the rank of the tensor. With respect to the experimental effort one has to reduce this number. There are three main ideas for the reduction:

- to use general statements of the theory like the statements of symmetry for the stress and for the strain tensor or the statement of the elastic potential,
- to use symmetry considerations for the material behavior like the statement of monoclinic, orthotropic or transversally-isotropic material behavior and
- the statements of approximative stress or strain states (plane stress or plain strain conditions).

Let us focus our attention to the first and the second item. From the first item, see for example Altenbach & Altenbach (1994), $\sigma_{ij} = \sigma_{ji}$ results in $E_{ijkl} = E_{jikl}$ and $\varepsilon_{kl} = \varepsilon_{lk}$ in $E_{ijkl} = E_{ijlk}$. Using both assumptions the number of tensor components is reduced to 36. In addition, further reduction one gets from the existence of the elastic potential W. In this case one can write down

$$W = \frac{1}{2}\boldsymbol{\sigma} \cdot\cdot\, \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot\cdot\, {}^{(4)}\boldsymbol{E} \cdot\cdot\, \boldsymbol{\varepsilon}, \quad W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \tag{2.12}$$

Calculating the first and the second derivatives with respect to the strain tensor

$$\frac{\partial W}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{\sigma} = {}^{(4)}\boldsymbol{E} \cdot\cdot\, \boldsymbol{\varepsilon}, \quad \frac{\partial^2 W}{\partial \boldsymbol{\varepsilon}^2} = {}^{(4)}\boldsymbol{E}$$

or

$$\frac{\partial W}{\partial \varepsilon_{ij}} = \sigma_{ij} = E_{ijkl}\varepsilon_{kl}, \quad \frac{\partial^2 W}{\partial \varepsilon_{ij}\partial \varepsilon_{kl}} = E_{ijkl}$$

one can conclude that $E_{ijkl} = E_{klij}$. So finally, from the statements of the first item we obtain a reduction of the number of independent components E_{ijkl} from 81 to 21.

The discussed possibility of reducing the number of components allows to use a second representation of the elastic behavior. Considering the six stresses and the six strains as vectors (2.4) and (2.8) a linear functional relationship between these vectors can be formulated with the help of a $[6 \times 6]$ matrix (instead of the fourth rank elasticity tensor)

$$[\sigma_i] = [E_{ij}][\varepsilon_j]; \quad i, j = 1, 2, \ldots, 6$$

with the elasticity matrix E_{ij}. Assuming again the existence of an elastic potential one gets further reduction of the number of independent coordinates of the elasticity tensor. The elastic strain energy can be expressed by the strain energy density function

$$W(\varepsilon_i) = \frac{1}{2}\sigma_i \varepsilon_i = \frac{1}{2}E_{ij}\varepsilon_j \varepsilon_i$$

Let us calculate once more the first and the second derivatives of this function with respect to the strain vector

$$\frac{\partial W}{\partial \varepsilon_i} = \sigma_i, \quad \frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = E_{ij}, \quad \frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i} = E_{ji}, \quad \frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = \frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i}$$

From the last equation one can make the conclusion that the elasticity matrix must be symmetrically

$$E_{ij} = E_{ji}$$

and the number of the independent material coefficients is only 21.

The generalized relations in the contracted vector-matrix form in the case of the linear anisotropic elastic behavior can be written as follows

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & E_{55} & E_{56} \\ & & & & & E_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{2.13}$$

Let us summarize the basic formulae for transformation of the stress, the strain and the elasticity tensors in the relevant vectors or matrices. In Table 1 the transformation rules for the stress and the strain tensor coordinates are shown. Table 2 summarizes the transformation rules for the elasticity tensor.

In some cases it is more convenient to use the elasticity equation in the inverse form

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{2.14}$$

Table 1. Transformation of the tensor coordinates σ_{ij} and ε_{ij} to the vector coordinates σ_p and ε_p

σ_{ij}	σ_p	ε_{ij}	ε_p
σ_{11}	σ_1	ε_{11}	ε_1
σ_{22}	σ_2	ε_{22}	ε_2
σ_{33}	σ_3	ε_{33}	ε_3
$\sigma_{23} = \tau_{23}$	σ_4	$2\varepsilon_{23} = \gamma_{23}$	ε_4
$\sigma_{31} = \tau_{31}$	σ_5	$2\varepsilon_{31} = \gamma_{31}$	ε_5
$\sigma_{12} = \tau_{12}$	σ_6	$2\varepsilon_{12} = \gamma_{12}$	ε_6

Table 2. Transformation of the tensor coordinates E_{ijkl} to the matrix coordinates E_{pq}

E_{ijkl}				E_{pq}			
ij :	11,	22,	33	p :	1,	2,	3
	23,	31,	12		4,	5,	6
kl :	11,	22,	33	q :	1,	2,	3
	23,	31,	12		4,	5,	6

It is easy to show that

$$[E_{ij}][S_{jk}] = [\delta_{ik}] = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} \qquad i, j, k = 1, \ldots, 6$$

and

$$\boldsymbol{\sigma} = \boldsymbol{E}\boldsymbol{\varepsilon}, \sigma_i = E_{ij}\varepsilon_j, \quad \boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}, \varepsilon_i = S_{ij}\sigma_j; \quad i, j = 1, \ldots, 6$$

with $\boldsymbol{E} \equiv [E_{ij}]$ as the stiffness matrix and $\boldsymbol{S} \equiv [S_{ij}]$ as the compliance (flexibility) matrix.

2.3 Material science background

Further reduction of the number of independent components is possible if we take into account the material symmetry, see Nye (1992) among others. In dependence of the scale size each material has a special kind of symmetry. For example, metals have a crystalline microstructure. In this case each crystal has an individual orientation and the symmetry of each crystal can differ. In addition, they are oriented arbitrarily. On the macroscopic level the materials are polycrystalline materials with a huge number of crystals. Averaging the properties and the individual orientations over the volume one obtains an isotropic behavior on the macroscopic level in contrast to the anisotropic behavior on the microscopic level.

Another situation follows from technological treatment of materials. For example, anisotropic behavior can be established for initially isotropic materials after rolling processes. In the case of reinforced materials the situation is more complicated. The individual response of the matrix and the reinforcement can be isotropically, but the combination of both results in a macroscopic anisotropic behavior. The analysis of possible reductions of the number of the fourth rank material tensor components which are related to the independent material properties will be shown for special cases of the anisotropic behavior later.

In material science structural materials are classified as follows: metals, ceramics, and polymers. It is difficult to give an exact assessment of the advantages and disadvantages of these three basic material classes, because each category covers whole groups of materials within which the range of properties is often as broad as the differences between the three material classes. But at the simplistic level some obvious characteristic properties can be identified:

- Most metals are of medium to high density. They have good thermal stability and can be made corrosion-resistant by alloying. Metals have useful mechanical characteristics and it is moderately easy to shape and join. Metals became the preferred engineering material, they posed less problems to the designer than either ceramic or polymer materials.

- Ceramic materials have great thermal stability and are resistant to corrosion, abrasion, etc. They are very rigid but mostly brittle and can only be shaped with difficulty.
- Polymer materials (plastics) are of low density, have good chemical resistance but lack thermal stability. They have poor mechanical properties, but are easily fabricated and joined. Their resistance to environmental degradation, e.g. the photomechanical effects of sunlight, is moderate.

The main problem in modelling the material behavior is the necessity to describe the similar behavior (for example, the elastic behavior) using similar equations. This is the reason for the introduction of some basic terms.

A material is called *homogeneous* if its properties are the same at every point and therefore independent of the location. Homogeneity is associated with the scale of modelling and the so-called characteristic volume. In this sense the definition can be useful only for the average material behavior on a macroscopic level. On a microscopic level all materials are more or less inhomogeneous but depending on the scale, materials can be described approximately as homogeneous, otherwise as inhomogeneous. A material is inhomogeneous or heterogeneous if its properties depend on location. But in the average sense a material can be regarded as homogeneous, quasi-homogeneous or heterogeneous.

A material is *isotropic* if its properties are independent of the orientation, they do not vary with direction. Otherwise the material is anisotropic. A general anisotropic material has no planes or axes of material symmetry, but some special cases of material symmetries like orthotropy, transverse isotropy, etc., will be discussed later in detail.

Furthermore, a material can depend on several constituents or phases, single phase materials are called *monolithic*. The above three mentioned classes of conventional materials are on the macroscopic level more or less monolithic, homogeneous and isotropic.

3 Composites

3.1 Classification

The group of materials which can be defined as composite materials is extremely large. Its boundaries depend on definition. In the most general definition one can consider a composite as any material that is a combination (composition) of two or more materials (constituents) and have material properties derived from the individual constituents. These properties may have the combined characteristics of the constituents (for example, established by the weighted mixture rules) or they are substantially different. Sometimes the material properties of a composite material may exceed those of the constituents.

This general definition of composites includes natural materials like wood, traditional structural materials like concrete, as well as modern synthetic composites such as fiber or particle reinforced plastics which are now an important group of engineering materials where low weight in combination with high strength and stiffness are required in structural design.

In the more restrictive sense a structural composite consists of an assembly of two materials of different nature. In general, one material is discontinuous and is called the reinforcement, the other material is mostly less stiff and weaker, but continuously distributed. It is called the matrix. The properties of a composite material depend on

- the properties of the constituents,
- the geometry of the reinforcements,
- their distribution, orientation and concentration usually measured by the volume fraction or fiber volume ratio, and
- the nature and quality of the matrix-reinforcement interface.

In a less restrictive sense, a structural composite can consist of two or more phases on the macroscopic level. The mechanical performance and properties of composite materials are superior to those of their components or constituent materials taken separately. The concentration of the reinforcement phase is a determining parameter of the properties of the new material, their distribution determines the homogeneity or the heterogeneity on the macroscopic scale. The most important aspect of composite materials in which the reinforcement are fibers is the anisotropy caused by the fiber orientation. It is necessary to give special attention to this fundamental characteristic of fiber reinforced composites and the possibility to influence the anisotropy by material design for a desired quality.

The reinforcement constituent can be described as fibrous or particulate. The fibers are continuous or discontinuous. Continuous fibers are arranged usually uni- or bidirectional, but also irregular reinforcements by continuous fibers are possible. The arrangement and the orientation of continuous or short fibers determines the mechanical properties of composites and the behavior ranges between a general anisotropy to a quasi-isotropy. Particulate reinforcements have different shapes. They may be spherical, platelet or of any regular or irregular geometry. Their arrangement may be random or regular with preferred orientations. In the majority of practical applications particulate reinforced composites are considered to be randomly oriented and the mechanical properties are homogeneous and isotropic. The preferred orientation in the case of continuous fiber composites is unidirectional (UD) for each layer or lamina (UD-lamina). Examples of composite materials with different constituents and distributions of the reinforcements are shown in Fig. 3. Various classifications of composites are presented in the literature. One possibility is shown in Fig. 4.

Composite materials can also be classified by the nature of their constituents. According to the nature of the matrix material we classify organic, mineral or metallic matrix composites.

- Organic matrix composites are polymer resins or thermoplastics with fillers. The fibers can be mineral (glass, etc.), organic (Kevlar, etc.) or metallic (aluminium, etc.).
- Mineral matrix composites are ceramics with metallic fibers or with metallic or mineral particles.
- Metallic matrix composites are metals with mineral or metallic fibers.

The use of composites is connected with several functional requirements of fibers and matrices:

- fibers should have a high modulus of elasticity and a high ultimate strength,
- fibers should be stable and retain their strength during handling and fabrication,
- the variation of the mechanical characteristics of the individual fibers should be low, their diameters uniform and their arrangement in the matrix regular,
- matrices have to interface the fibers and protect their surfaces from damage,
- matrices have to transfer stress to the fibers by adhesion and/or friction, and
- matrices have to be chemically compatible with fibers over the whole working period.

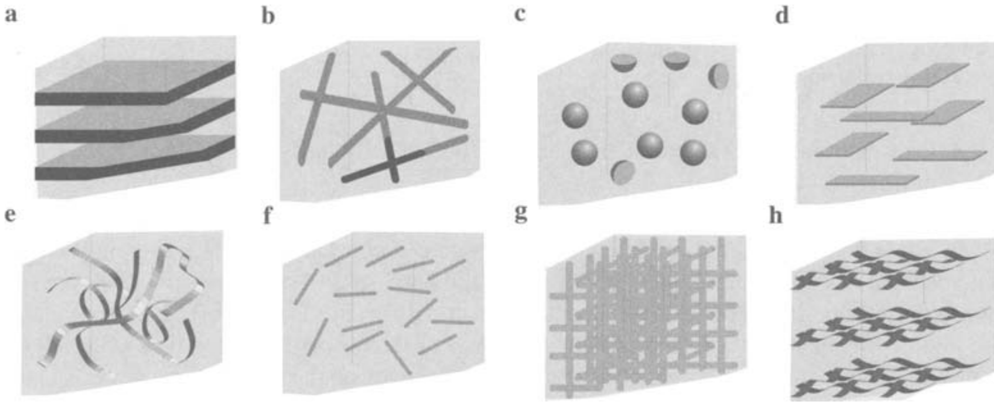

Figure 3. Classification of laminates. **a** Laminate with uni- or bidirectional layers, **b** irregular reinforcement with long fibers, **c** reinforcement with particles, **d** reinforcement with plate strapped particles, **e** random arrangement of continuous fibers, **f** irregular reinforcement with short fibers, **g** spatial reinforcement, **h** reinforcement with surface tissues as mats, woven fabrics, etc.

At present the main topics of composite material research and technology are

- investigation of all characteristics of the constituents and the composite material,
- material design and optimization for the given working conditions,
- development of analytical modelling and solution methods for determining material and structural behavior,

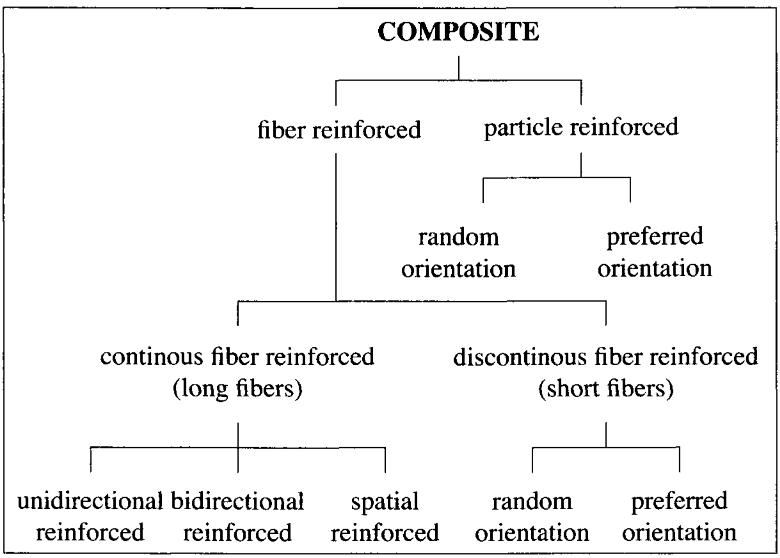

Figure 4. Classification of composites after Agarwal & Broutman (1990)

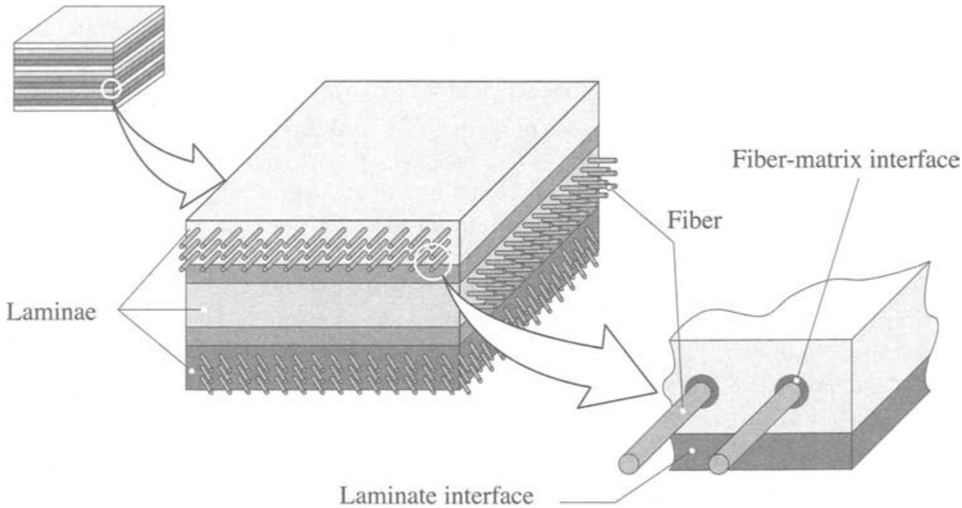

Figure 5. Hierarchical modelling of laminates

- experimental methods for material properties, stress and deformation states, failure, etc. characterization,
- modelling and analysis of creep and damage behavior of composites and their life prediction,
- development of new and efficient fabrication and recycling procedures.

The most significant mainspring in the composite research and application was weight saving in comparison to structures of conventional materials such as steel, alloys, etc. However, to have only material density, stiffness and strength in mind when thinking of composites is a very narrow view of the possibilities of such materials like fiber-reinforced plastics because they often may score over conventional materials like metals not only owing to their mechanical properties. Fiber reinforced plastics are extremely corrosion-resistant and have interesting electromagnetic properties. In consequence they are used for chemical plants and for structures which require non-magnetic materials. Further carbon fiber reinforced epoxy is used in medical applications because it is transparent to X-rays.

3.2 Modelling

Composite materials consist of two or more constituents and the modelling, analysis and design of structures built up of composites are different from conventional materials such as steel. There are three levels of modelling (Fig. 5):

- At the *micro-mechanical level* the average properties of a single reinforced layer have to be determined from the individual properties of the constituents, the fibers and the matrix. The average characteristics include the elastic moduli, the thermal and moisture expansion coefficients, etc. The micro-mechanics of a lamina does not consider the internal structure of the constituent elements, but the heterogeneity of the ply is regarded. The micro-mechanics is based on some simplifying approximations. These concern the fiber geometry and packing arrangement, so that the constituent characteristics together with the volume fractions of the constituents yield the average characteristics of the lamina.

- The calculated values of the average properties of a lamina provide the basis to predict the macrostructural properties. At the *macro-mechanical level*, only the averaged properties of a lamina are considered and the microstructure of the lamina is ignored. The properties along and perpendicular to the fiber direction, these are the principal directions of a lamina, are recognized and the so-called on-axis stress-strain relations for a unidirectional lamina can be developed. Loads may be applied not only on-axis but also off-axis and the relationships for stiffness and flexibility, for thermal and moisture expansion coefficients and the strength of an angle ply can be determined. Failure theories of a lamina are based on strength properties. This topic is called the macro-mechanics of a single layer or a lamina.
- A laminate is a stack of laminae. Each layer of fiber reinforcement can have different orientations and in principle each layer can be made of different materials. Knowing the macro-mechanics of a lamina, one develops the macro-mechanics of the laminate. Average stiffness, flexibility, strength, etc. can be determined for the whole laminate. The structure and orientation of the laminae in prescribed sequences to a laminate lead to significant advantages of composite materials when compared to a conventional monolithic material. In general, the mechanical response of laminates is anisotropic.

When the micro- and macro-mechanical analysis for laminae and laminates are carried out, the global behavior of laminated composite materials is known. The last step is the modelling on the structure level where the global behavior of a structure made of a composite material is to analyze.

By adapting the classical tools of structural analysis on anisotropic elastic structure elements the analysis of simple structures like beams or plates may be achieved by analytical methods, but for more general boundary conditions and/or loading and for complex structures, numerical methods are used. For laminated composites, assumptions are necessary to enable the mathematical modelling. These are an elastic behavior of fibers and matrices, a perfect bonding between fibers and matrices, a regular fiber arrangement in regular or repeating arrays, etc. Summarizing the different size scales of mechanical modelling of structure elements composed of fiber reinforced composites it must be noted that, independent of the different possibilities to formulate beam, plate or shell theories, three modelling levels must be considered:

- The microscopic level, where the average mechanical characteristics of a lamina have to be estimated from the known characteristics of the fibers and the matrix material taking into account the fiber volume fracture and the fiber packing arrangement. The micro-mechanical modelling leads to a correlation between constituent properties and average composite properties. In general, simple mixture rules are used in engineering applications. If possible, the average material characteristics of a lamina should be verified experimentally. On the micro-mechanical level a lamina is considered as a quasi-homogeneous orthotropic material.
- The macroscopic level, where the effective (average) material characteristics of a laminate have to be estimated from the average characteristics of a set of laminae taking into account their stacking sequence. The macro-mechanical modelling leads to a correlation between the known averaged laminae properties and effective laminate properties. On the macro-

mechanical level a laminate is considered generally as an equivalent single layer element with a quasi-homogeneous, anisotropic material behavior.

- The structural level, where the mechanical response of structural members like beams, plates, shells etc. have to be analyzed taking into account possibilities to formulate structural theories of different order.

4 Elastic composites as anisotropic solids

4.1 Basic assumptions

The classical theory of elastic solids is based on the following assumptions:

- The material behavior can be approximated as ideal linear elastic.
- All elastic properties are the same in tension and compression.
- All strains are small.
- The stress and the strain tensors are symmetric.
- The material behavior is homogeneous and isotropic.

All these assumptions are fulfilled in a satisfactory manner in the case of modelling and analysis of structure elements made of conventional monolithic materials like steel. The structural analysis of elements composed of composite materials is more complicated and based on the theory of anisotropic elasticity (see, for example, Ambarcumyan (1991), Berthelot (1999), Decolon (2002), Lekhnitskij (1981), Mālmeisters et al. (1977) and Rabinovich (1970)) since the elastic properties of composite materials now depend on the direction. In addition, the material is not homogeneous at all. The material is piecewise homogeneous and only after averaging it can be regarded as quasi-homogeneous.

For materials with isotropic and anisotropic behavior the governing equations are mostly the same. The equilibrium equations, the kinematic equations and the compatibility equations are identical because they do not depend on the behavior of the material. Let us summarize the material independent equations (see Altenbach & Altenbach (1994), Hahn (1985), Lai et al. (1993) among others). At first, we have the dynamic equilibrium equations

$$\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} + \boldsymbol{p} = \rho \ddot{\boldsymbol{u}}, \quad \sigma_{ij,i} + p_j = \rho \ddot{u}_j \tag{4.1}$$

with ρ as the density and $\boldsymbol{p}$ being the body force vector. In the index notation the spatial differentiation is written as $(\ldots)_{,i}$ (differentiation with respect to the coordinate x_i). At second, in the case of small strains the Eqs. (2.5) or (2.6) are valid. And last but not least the compatibility can be expressed as

$$\boldsymbol{\nabla} \times \boldsymbol{\varepsilon} \times \boldsymbol{\nabla} = \boldsymbol{0}, \quad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{il,jk} - \varepsilon_{jk,il} = 0 \tag{4.2}$$

All these equations are independent of the elastic properties of the material. Only the constitutive equations differ significantly for an isotropic and an anisotropic body.

Let us now consider that the material behavior can be anisotropically. Below the anisotropic elasticity in the most general form of the linear constitutive equations will be assumed. In addition, special cases of elastic symmetries are deduced (for example, the classical HOOKE's law for an isotropic body and the plane stress and plane strain cases). The final constitutive equations are applied in the analysis of the laminate stiffness and compliances.

4.2 Elastic constitutive equations, transformation rules

The composite material engineering modelling neglects the real on the microscopic scale discontinuous structure and considers on a macroscopic or phenomenological scale the material models as continuous (quasi-homogeneous). Fiber and particle reinforced composites are highly heterogeneous materials as the consequence of the two constituents (fibers/particles and matrix). It must be defined a representative volume element of the material on a characteristic scale at which the properties of the material can be averaged and such a procedure results in a good approximation. If such an averaging is possible the composite material is macroscopic homogeneous, the designing structural elements composed of composite materials can be solved in an analogous manner as for conventional materials with the help of the average material properties (effective properties concept).

Let us present the main approaches in averaging material properties. We assume that a prismatic bar is composed of different materials as shown in Fig. 6. The starting point of the

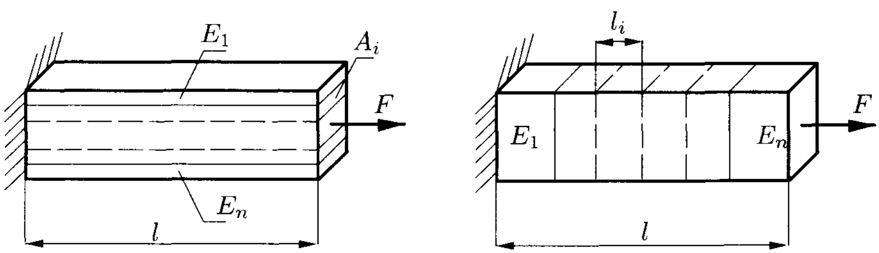

Figure 6. Prismatic bar composed of different materials

analysis of the mechanical behavior of such a bar is the stress definition $\sigma = F/A$ and the one-dimensional elastic law $\sigma = E\varepsilon$. From this follow $\sigma A = F = EA\varepsilon$ and finally $\varepsilon = (EA)^{-1}F$. EA is the tensile stiffness and $(EA)^{-1}$ the tensile flexibility or compliance. Now we assume that the different materials of the prismatic bar are arranged in parallel or series.

In the first case the arrangement is in parallel (VOIGT's model) that means

$$F = \sum_{i=1}^{n} F_i, \quad A = \sum_{i=1}^{n} A_i, \quad \varepsilon = \varepsilon_i$$

The F_i are the loading forces on A_i and the strains ε_i are equal for the total cross-section

$$F = EA\varepsilon \Rightarrow F_i = E_i A_i \varepsilon, \qquad \sum_{i=1}^{n} F_i = F = \sum_{i=1}^{n} E_i A_i \varepsilon$$

By coupling the equations for the stiffness $E_i A_i$ one observes the effective stiffness

$$EA = \sum_{i=1}^{n} E_i A_i, \qquad (EA)^{-1} = \frac{1}{\sum\limits_{i=1}^{n} E_i A_i}$$

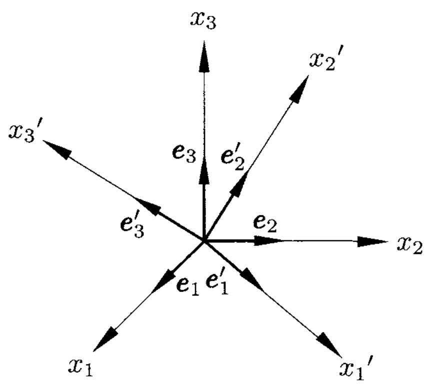

Figure 7. Rotation of the coordinate system. Reference system: e_1, e_2, e_3, rotated system: e_1', e_2', e_3'

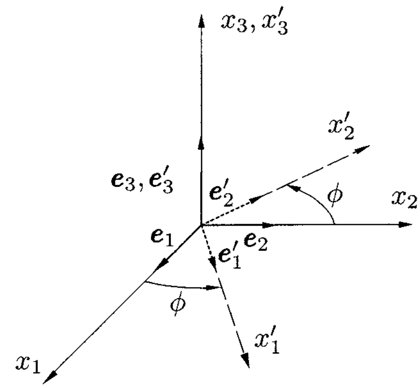

Figure 8. Rotation of the coordinate system around the direction e_3

The second case is the arrangement in series (REUSS' model). With $\triangle l = \sum_{i=1}^n \triangle l_i$ and $F = F_i$ one gets

$$\triangle l = l\varepsilon = l(EA)^{-1}F, \quad \triangle l_i = l_i\varepsilon_i = l_i(E_iA_i)^{-1}F$$

and

$$\sum_{i=1}^n \triangle l_i = \left[\sum_{i=1}^n l_i(E_iA_i)^{-1}\right] F$$

By coupling the equations for the stiffness E_iA_i one observes the effective stiffness as

$$EA = \frac{l}{\sum_{i=1}^n l_i(E_iA_i)^{-1}}, \quad (EA)^{-1} = \frac{\sum_{i=1}^n l_i(E_iA_i)^{-1}}{l}$$

The averaging in the VOIGT's or REUSS' sense can be applied as a first approximation for the properties of unidirectional reinforced layers. This is demonstrated, for example, by Altenbach et al. (2004). But it is well-known that the agreement with experimental data is partly not satisfying, see Hult & Rammerstorfer (1994) and Mālmeisters et al. (1977). So there are many proposals for improvements of the effective properties.

4.3 Transformation rules

Let us consider the rotation of the coordinate system as shown in Fig. 7. In this case the following transformation rules

$$e_i' = R_{ij}e_j, \quad e_i = R_{ji}e_j', \quad R_{ij} \equiv \cos(e_i', e_j), \quad R_{ji} \equiv \cos(e_i, e_j')$$

and

$$e' = Re, \quad e = R^{-1}e' = R^T e'$$

are valid. R is the transformation or rotation matrix. R is symmetric and unitary (Det $R = |R_{ij}| = 1, R^{-1} = R^T$).

Considering the special case of rotation ϕ around the direction e_3 (Fig. 8) the transformation matrix takes the form

$$\left[\overset{3}{R}_{ij}\right] = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \left[\overset{3}{R}_{ij}\right]^{-1} = \left[\overset{3}{R}_{ij}\right]^{T} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

Now the transformation rules are

$$\begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{bmatrix} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e'_1 \\ e'_2 \\ e'_3 \end{bmatrix}$$

with $c = \cos\phi$, $s = \sin\phi$.

After the introduction of the transformation rules for the coordinate axes one has to discuss the transformation rules for the tensors. Let us start with the second-rank tensors. For the stress tensor one gets

$$\sigma'_{ij} = R_{ik}R_{jl}\sigma_{kl}, \quad \sigma_{ij} = R_{ki}R_{lj}\sigma'_{kl} \tag{4.3}$$

The transformation rules for the contracted notation result in

$$\sigma'_p = T^{\sigma}_{pq}\sigma_q, \quad \sigma_p = \left(T^{\sigma}_{pq}\right)^{-1}\sigma'_q, \quad p, q = 1, \ldots, 6 \tag{4.4}$$

The transformation matrices T^{σ}_{pq} and $(T^{\sigma}_{pq})^{-1}$ follow by comparing (4.3) and (4.4). By analogy one gets for the strain tensor (contracted notation)

$$\varepsilon'_p = T^{\varepsilon}_{pq}\varepsilon_q, \quad \varepsilon_p = \left(T^{\varepsilon}_{pq}\right)^{-1}\varepsilon'_q, \quad p, q = 1, \ldots, 6$$

Summarizing all derivations the following equations can be established

$$\sigma' = T^{\sigma}\sigma, \quad \varepsilon' = T^{\varepsilon}\varepsilon, \quad \sigma = \left(T^{\sigma}\right)^{-1}\sigma', \quad \varepsilon = \left(T^{\varepsilon}\right)^{-1}\varepsilon' \tag{4.5}$$

Considering these equations the transformation relations for the elasticity matrix can be deduced. The starting point is the HOOKE's law

$$\sigma = E\varepsilon, \quad \sigma' = E'\varepsilon'$$

With Eqs (4.5) one can write down

$$\begin{aligned} (T^{\sigma})^{-1}\sigma' = \sigma = E\varepsilon = E(T^{\varepsilon})^{-1}\varepsilon' \quad &\Rightarrow \quad \sigma' = T^{\sigma}E(T^{\varepsilon})^{-1}\varepsilon' = E'\varepsilon', \\ T^{\sigma}\sigma = \sigma' = E'\varepsilon' = E'T^{\varepsilon}\varepsilon \quad &\Rightarrow \quad \sigma = (T^{\sigma})^{-1}E'T^{\varepsilon}\varepsilon = E\varepsilon, \end{aligned}$$

and the transformation relations for the stiffness matrix are

$$E' = T^{\sigma}E(T^{\sigma})^T, \quad E = (T^{\varepsilon})^T E' T^{\varepsilon}$$

or in index notation

$$E'_{ij} = T^{\sigma}_{ik} T^{\sigma}_{jl} E_{kl}, \quad E_{ij} = T^{\varepsilon}_{ik} T^{\varepsilon}_{jl} E'_{kl}$$

Analogically the transformation relations for the compliance matrix can be formulated. The starting point is now

$$\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}' = \boldsymbol{S}'\boldsymbol{\sigma}'$$

and after some calculations

$$\begin{array}{ll} (\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma} = \boldsymbol{S}(\boldsymbol{T}^{\sigma})^{-1}\boldsymbol{\sigma}' & \Rightarrow \ \boldsymbol{\varepsilon}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{S}(\boldsymbol{T}^{\sigma})^{-1}\boldsymbol{\sigma}' = \boldsymbol{S}'\boldsymbol{\sigma}', \\ \boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}' = \boldsymbol{S}'\boldsymbol{\sigma}' = \boldsymbol{S}'\boldsymbol{T}^{\sigma}\boldsymbol{\sigma} & \Rightarrow \ \boldsymbol{\varepsilon} = (\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{S}'\boldsymbol{T}^{\sigma}\boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{\sigma}, \end{array}$$

one finally gets

$$\boldsymbol{S}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{S}(\boldsymbol{T}^{\varepsilon})^{T}, \quad \boldsymbol{S} = (\boldsymbol{T}^{\sigma})^{T}\boldsymbol{S}'\boldsymbol{T}^{\sigma},$$

or in index notation

$$S'_{ij} = T^{\varepsilon}_{ik} T^{\varepsilon}_{jl} S_{kl}, \quad S_{ij} = T^{\sigma}_{ik} T^{\sigma}_{jl} S'_{kl}$$

The complete estimation of the transformation rules is presented in Altenbach et al. (1996) and Altenbach et al. (2004).

For the special case of a rotation ϕ around the $\boldsymbol{e}_3$-direction (Fig. 8) the transformation matrices take the form

$$\left[\overset{3}{T}{}^{\sigma}_{pq}\right] = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \qquad \left[\overset{3}{T}{}^{\sigma}_{pq}\right]^{-1} = \left[\overset{3}{T}{}^{\varepsilon}_{pq}\right]^{T}$$

$$\left[\overset{3}{T}{}^{\varepsilon}_{pq}\right] = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \qquad \left[\overset{3}{T}{}^{\varepsilon}_{pq}\right]^{-1} = \left[\overset{3}{T}{}^{\sigma}_{pq}\right]^{T}$$

4.4 Symmetry Relations of Stiffness and Compliance Matrices

The most general case of the three-dimensional generalized HOOKE's law is connected with the stiffness and the compliance matrices containing 36 non-zero material parameters E_{ij} or S_{ij}, but due to the potential assumption only 21 are independent constants. In many cases the material show symmetries in their behavior. Important material symmetries are

- monoclinic material behavior,
- orthotropic material behavior,
- transversally isotropic material behavior, and
- isotropic material behavior.

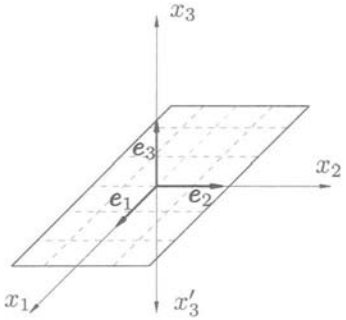

Figure 9. Example of monoclinic material behavior. 20 non-zero elements E_{ij} or S_{ij}, 13 independent elements

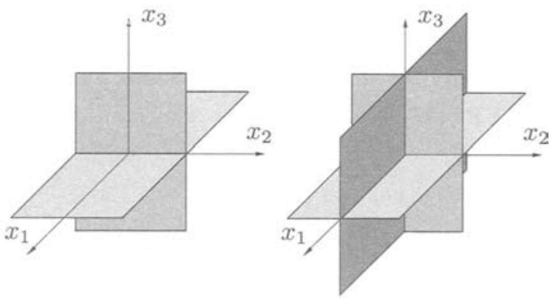

Figure 10. Example of orthotropic material behavior. 12 non-zero elements E_{ij} or S_{ij}, 9 independent elements

In all these cases the number of independent components of the stiffness or compliance matrices can be reduced.

Let us assume monoclinic (monotropic) material behavior. If we have one plane of symmetry (for example, Fig. 9) the elasticity matrix takes the form

$$[E_{ij}]^{\mathrm{MC}} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & E_{16} \\ E_{12} & E_{22} & E_{23} & 0 & 0 & E_{26} \\ E_{13} & E_{23} & E_{33} & 0 & 0 & E_{36} \\ 0 & 0 & 0 & E_{44} & E_{45} & 0 \\ 0 & 0 & 0 & E_{45} & E_{55} & 0 \\ E_{16} & E_{26} & E_{36} & 0 & 0 & E_{66} \end{bmatrix}$$

Assuming orthotropic material behavior (for example, Fig. 10) the elasticity matrix takes the following form

$$[E_{ij}]^{\mathrm{O}} = \begin{bmatrix} E_{11} & E_{12} & E_{13} & 0 & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{13} & E_{23} & E_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{66} \end{bmatrix}$$

The next example is the transversely isotropic material behavior. Now one obtains 12 non-zero elements and 5 independent elements

$$[E_{ij}]^{\mathrm{TI}} = \begin{bmatrix} E_{11} & E_{12} & E_{12} & 0 & 0 & 0 \\ E_{12} & E_{22} & E_{23} & 0 & 0 & 0 \\ E_{12} & E_{23} & E_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(E_{22} - E_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & E_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & E_{55} \end{bmatrix}$$

Table 3. Compliance matrix elements

Material model	Compliance matrix $[S_{ij}]$
Anisotropy: 21 independent material parameters	$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix}$
Monoclinic: 13 independent material parameters	Symmetry plane $x_3 = 0$: $S_{14} = S_{15} = S_{24} = S_{25} = S_{34} = S_{35} = S_{46} = S_{56} = 0$ Symmetry plane $x_2 = 0$: $S_{14} = S_{16} = S_{24} = S_{26} = S_{34} = S_{36} = S_{45} = S_{56} = 0$ Symmetry plane $x_1 = 0$: $S_{15} = S_{16} = S_{25} = S_{26} = S_{35} = S_{36} = S_{45} = S_{46} = 0$
Orthotropic: 9 independent material parameters	3 planes of symmetry $x_1 = 0, x_2 = 0, x_3 = 0$ $S_{14} = S_{15} = S_{16} = S_{24} = S_{25} = S_{26} = S_{34}$ $= S_{35} = S_{36} = S_{45} = S_{46} = S_{56} = 0$
Transversely isotropic: 5 independent material parameters	Plane of isotropy $x_3 = 0$: $S_{11} = S_{22}, S_{23} = S_{13}, S_{44} = S_{55}, S_{66} = 2(S_{11} - S_{12})$ Plane of isotropy $x_2 = 0$: $S_{11} = S_{33}, S_{12} = S_{23}, S_{44} = S_{66}, S_{55} = 2(S_{33} - S_{13})$ Plane of isotropy $x_1 = 0$: $S_{22} = S_{33}, S_{13} = S_{12}, S_{55} = S_{66}, S_{44} = 2(S_{22} - S_{23})$ all other S_{ij} like orthotropic
Isotropy: 2 independent material parameters	$S_{11} = S_{22} = S_{33}, S_{12} = S_{13} = S_{23},$ $S_{44} = S_{55} = S_{66} = 2(S_{11} - S_{12})$ all other $S_{ij} = 0$

The classical isotropic material behavior can be represented by

$$[E_{ij}]^{\mathrm{I}} = \begin{bmatrix} E_{11} & E_{12} & E_{12} & 0 & 0 & 0 \\ E_{12} & E_{11} & E_{12} & 0 & 0 & 0 \\ E_{12} & E_{12} & E_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_* & 0 & 0 \\ 0 & 0 & 0 & 0 & E_* & 0 \\ 0 & 0 & 0 & 0 & 0 & E_* \end{bmatrix}$$

with $E_* = \frac{1}{2}(E_{11} - E_{12})$. There are 12 non-zero elements, but only 2 independent parameters. The results for the tree-dimensional compliance matrices are shown in Table 3. The results for the three-dimensional stiffness matrices can be summarized as shown in Table 4.

The structural analysis in engineering is mostly based on the so-called engineering constants. Considering orthotropic material behavior with material parameters E_i, G_{ij} and ν_{ij} one can write

Table 4. Stiffness matrix elements

Material model	Elasticity matrix $[E_{ij}]$
Anisotropy: 21 independent material parameters	$\begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ & E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ & & E_{33} & E_{34} & E_{35} & E_{36} \\ & & & E_{44} & E_{45} & E_{46} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & E_{55} & E_{56} \\ & & & & & E_{66} \end{bmatrix}$
Monoclinic: 13 independent material parameters	Symmetry plane $x_3 = 0$: $E_{14} = E_{15} = E_{24} = E_{25} = E_{34} = E_{35} = E_{46} = E_{56} = 0$ Symmetry plane $x_2 = 0$: $E_{14} = E_{16} = E_{24} = E_{26} = E_{34} = E_{36} = E_{45} = E_{56} = 0$ Symmetry plane $x_1 = 0$: $E_{15} = E_{16} = E_{25} = E_{26} = E_{35} = E_{36} = E_{45} = E_{46} = 0$
Orthotropic: 9 independent material parameters	3 planes of symmetry $x_1 = 0, x_2 = 0, x_3 = 0$ $E_{14} = E_{15} = E_{16} = E_{24} = E_{25} = E_{26} = E_{34}$ $= E_{35} = E_{36} = E_{45} = E_{46} = E_{56} = 0$
Transversely isotropic: 5 independent material parameters	Plane of isotropy $x_3 = 0$: $E_{11} = E_{22}, E_{23} = E_{13}, E_{44} = E_{55}, E_{66} = \frac{1}{2}(E_{11} - E_{12})$ Plane of isotropy $x_2 = 0$: $E_{11} = E_{33}, E_{12} = E_{23}, E_{44} = E_{66}, E_{55} = \frac{1}{2}(E_{33} - E_{13})$ Plane of isotropy $x_1 = 0$: $E_{22} = E_{33}, E_{12} = E_{13}, E_{55} = E_{66}, E_{44} = \frac{1}{2}(E_{22} - E_{23})$ all other E_{ij} like orthotropic
Isotropy: 2 independent material parameters	$E_{11} = E_{22} = E_{33}, E_{12} = E_{13} = E_{23},$ $E_{44} = E_{55} = E_{66} = \frac{1}{2}(E_{11} - E_{12})$ all other $E_{ij} = 0$

down

$$\begin{aligned} \sigma_1 &= E_{11}\varepsilon_1 + E_{12}\varepsilon_2 + E_{13}\varepsilon_3, & \sigma_4 &= E_{44}\varepsilon_4, \\ \sigma_2 &= E_{12}\varepsilon_1 + E_{22}\varepsilon_2 + E_{23}\varepsilon_3, & \sigma_5 &= E_{55}\varepsilon_5, \\ \sigma_3 &= E_{13}\varepsilon_1 + E_{23}\varepsilon_2 + E_{33}\varepsilon_3, & \sigma_6 &= E_{66}\varepsilon_6 \end{aligned}$$

The inverted generalized HOOKE's law takes the form

$$\begin{aligned} \varepsilon_1 &= S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3, & \varepsilon_4 &= S_{44}\sigma_4, \\ \varepsilon_2 &= S_{12}\sigma_1 + S_{22}\sigma_2 + S_{23}\sigma_3, & \varepsilon_5 &= S_{55}\sigma_5, \\ \varepsilon_3 &= S_{13}\sigma_1 + S_{23}\sigma_2 + S_{33}\sigma_3, & \varepsilon_6 &= S_{66}\sigma_6 \end{aligned}$$

Let us now identify the constants.

At first, we perform the tension test. The uniaxial tension in x_i-direction, $\sigma_1 \neq 0$, $\sigma_i = 0$, $i = 2, \dots, 6$ can be presented by

$$\varepsilon_1 = S_{11}\sigma_1, \quad \varepsilon_2 = S_{12}\sigma_1, \quad \varepsilon_3 = S_{13}\sigma_1, \quad \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0,$$

Physical tensile tests yield the elastic constants E_1, ν_{12}, ν_{13}

$$E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}}, \quad \nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} = -S_{12}E_1, \quad \nu_{13} = -\frac{\varepsilon_3}{\varepsilon_1} = -S_{13}E_1$$

or

$$S_{11} = \frac{1}{E_1}, \quad S_{12} = -\frac{\nu_{12}}{E_1}, \quad S_{13} = -\frac{\nu_{13}}{E_1}$$

Analogous relations resulting from uniaxial tension in x_2- and x_3-directions and all S_{ij} are related to the nine measured engineering constants (3 YOUNG's moduli and 6 POISSON's ratios) by uniaxial tension tests in three directions x_1, x_2 and x_3. From the symmetry of the compliance matrix one can conclude

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}, \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}$$

or

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \quad \frac{\nu_{ij}}{\nu_{ji}} = \frac{E_i}{E_j}, \quad i,j = 1,2,3 \quad (i \neq j)$$

Remember that the first and the second subscript in POISSON's ratios denote stress and strain directions, respectively.

At second, one can perform the shear test

$$\varepsilon_4 = S_{44}\sigma_4, \quad \varepsilon_5 = S_{55}\sigma_5, \quad \varepsilon_6 = S_{66}\sigma_6$$

The compliances can be estimated as

$$S_{44} = \frac{1}{G_{23}} = \frac{1}{E_4}, \quad S_{55} = \frac{1}{G_{13}} = \frac{1}{E_5}, \quad S_{66} = \frac{1}{G_{12}} = \frac{1}{E_6}$$

Finally one gets

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ & & \frac{1}{E_3} & 0 & 0 & 0 \\ & & & \frac{1}{E_4} & 0 & 0 \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \frac{1}{E_5} & 0 \\ & & & & & \frac{1}{E_6} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

Let us now estimate the components of the elasticity matrix. The following trivial relations between stiffness and compliance matrices can be obtained

$$E_{44} = \frac{1}{S_{44}} = G_{23}, \quad E_{55} = \frac{1}{S_{55}} = G_{13}, \quad E_{66} = \frac{1}{S_{66}} = G_{12}$$

In addition, a symmetric [3x3]-matrix must be inverted

$$E_{ij} = S_{ij}^{-1} = \frac{(-1)^{i+j}U_{ij}}{\text{Det}[S_{ij}]}, \quad \text{Det}[S_{ij}] = \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{vmatrix}$$

The U_{ij} are submatrices of $\boldsymbol{S}$ to the element S_{ij}

$$E_{11} = \frac{S_{22}S_{33} - S_{23}^2}{\text{Det}[S_{ij}]}, \qquad E_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{\text{Det}[S_{ij}]}, \quad E_{22} = \frac{S_{33}S_{11} - S_{13}^2}{\text{Det}[S_{ij}]},$$
$$E_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{\text{Det}[S_{ij}]}, \quad E_{33} = \frac{S_{11}S_{22} - S_{12}^2}{\text{Det}[S_{ij}]}, \qquad E_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{\text{Det}[S_{ij}]}$$

Finally, the stiffness matrix can be expressed by engineering constants as follows

$$E_{11} = \frac{(1 - \nu_{23}\nu_{32})E_1}{\Delta}, \quad E_{12} = \frac{(\nu_{12} + \nu_{13}\nu_{32})E_2}{\Delta}, \quad E_{13} = \frac{(\nu_{13} + \nu_{12}\nu_{23})E_3}{\Delta},$$
$$E_{22} = \frac{(1 - \nu_{31}\nu_{13})E_2}{\Delta}, \quad E_{23} = \frac{(\nu_{23} + \nu_{21}\nu_{13})E_3}{\Delta}, \quad E_{33} = \frac{(1 - \nu_{21}\nu_{12})E_3}{\Delta}$$

with $\Delta = 1 - \nu_{21}\nu_{12} - \nu_{32}\nu_{23} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{13}\nu_{32}$.

Taking into account $E_i/\Delta \equiv \overline{E}_i, 1/S_i \equiv E_i$ one gets

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} (1 - \nu_{23}\nu_{32})\overline{E}_1 & (\nu_{12} + \nu_{13}\nu_{32})\overline{E}_2 & (\nu_{13} + \nu_{12}\nu_{23})\overline{E}_3 \\ & (1 - \nu_{31}\nu_{13})\overline{E}_2 & (\nu_{23} + \nu_{21}\nu_{13})\overline{E}_3 \\ \text{SYM} & & (1 - \nu_{21}\nu_{12})\overline{E}_3 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix},$$

$$\begin{bmatrix} \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_4 & 0 & 0 \\ & E_5 & 0 \\ \text{SYM} & & E_6 \end{bmatrix} \begin{bmatrix} \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

The most general case of monoclinic material behavior with the plane of elastic symmetry $(x_1 - x_2)$ results in

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & \frac{\eta_{61}}{E_6} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & \frac{\eta_{62}}{E_6} \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & \frac{\eta_{63}}{E_6} \\ 0 & 0 & 0 & \frac{1}{E_4} & \frac{\mu_{54}}{E_5} & 0 \\ 0 & 0 & 0 & \frac{\mu_{45}}{E_4} & \frac{1}{E_5} & 0 \\ \frac{\eta_{16}}{E_1} & \frac{\eta_{26}}{E_2} & \frac{\eta_{36}}{E_3} & 0 & 0 & \frac{1}{E_6} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

with the following reciprocal relations

$$\frac{\eta_{61}}{E_6} = \frac{\eta_{16}}{E_1}, \quad \frac{\eta_{62}}{E_6} = \frac{\eta_{26}}{E_2}, \quad \frac{\eta_{63}}{E_6} = \frac{\eta_{36}}{E_3}, \quad \frac{\mu_{54}}{E_5} = \frac{\mu_{45}}{E_4}$$

The μ_{ij} are the shear-shear stress coupling parameters, the η_{ij} the normal-shear stress coupling parameters (Lai et al. (1993)).