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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

MECHANICAL VIBRATION: WHERE DO WE STAND?

EDITED BY

ISAAC ELISHAKOFF

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The series presents lecture notes, monographs, edited works and proceedings in the field of Mechanics, Engineering, Computer Science and Applied Mathematics.
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EDITED BY

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PREFACE

"There is always a delightful sense of movement, vibration and life".

Theodore Robinson (1852-1896)

"I have never solved a major mechanical or interpretive problem at the keyboard. I have always solved it in my mind".

Jorge Bolet (1914-1990)

The idea of this book stems from the realization that scientists, not unlike laymen, should occasionally interrupt their regular work and reflect on the past, to see both the accomplishments and the drawbacks, so as to be able to plan for future research in the "proper" perspective. But an inquisitive reader may ask: Can one really document in any field, let alone mechanical vibrations (whose very name signifies change), "where do we stand"? Did not a Greek philosopher famously claim that one cannot enter a river twice? Another, on an even more sophisticated note, added that actually it is impossible to enter a river even once! For in the process of entering, both entrant and river change. Likewise, one can argue that it is nearly impossible to answer the question posed in the title of this volume.

But experience shows, despite the sage observations of the philosophers, that one does enter a river, lake, sea, or ocean. Likewise, scientists do stop (if not for a minute, for a conference) to reflect on the past, and if not in its detail, then at least in big strokes on various topics presented by the participants; questions by the listeners often change the research direction of the presenter.

The present writer was pleased to locate a short time ago, while searching for references devoted to the topic of statistical linearization, a paper written over 30 years ago under the title "Vibration: Where Do We Stand in 1975" (see the list of papers in the book Random Vibration — Status and Recent Developments: The Stephen Crandall Festschrift, edited by I. Elishakoff and R.H. Lyon, Elsevier, Amsterdam, 1986, page XVI), by a prominent contributor to mechanical vibrations.

The book in front of you inevitably reflects the personal contributions and preferences of the authors. It opens with several contributions of Professor Erasmo Viola of Bologna, Italy, and his co-authors on three-dimensional dynamic problems. In the following chapters Professor Daniel J. Inman of Blacksburg, Virginia, USA, deals with smart structures; Professor Ilya I. Blekhman of St. Petersburg, Russia, offers a new effective approach to the problem of nonlinear vibrations, whereas Professor Leslaw Socha of Warsaw, Poland, treats nonlinear stochastic problems. Dr. Sondipon Adhikari of Bristol, England, deals with problems of validation and verification, and stochastic eigenvalue problems. The contributions by Professor Ivo Caliò of Catania, Italy and the present writer deal with closed-form trigonometric solutions of inhomogeneous structures, preceded by a review of closed-form polynomial solutions of beam and plate problems by the present writer.

It is important to bring to the attention of the readers of this book another undertaking, namely, an extremely important volume titled Structural Dynamics @ 2000: Current Status and Future Directions, edited by D.J. Ewins and D.J. Inman and printed by Research Studies Press Ltd., in Baldock, Hertfordshire, England, 2001.

It is hoped that the readers will enjoy reading, and hopefully studying, the contributions in this volume and those in the above mentioned book. This may shape, albeit partially, their views on the study of mechanical vibrations and its future. We would like to express our sincere appreciation of the excellent atmosphere of the CISM, a true jewel of science and engineering, and its outstanding people — Secretary General Professor Bernhard Schrefler, Resident Rector Professor Giulio Maier, Ms. Elsa Venir, Ms. Carla Toros, Mr. Ezio Cum, and last but not least Ms. Monica del Pin, who had an extensive dedicated correspondence with us. Long before the modern trend of "globalization", CISM became a strong attractor of young and mature scientists and students from all over the world.

It will be rewarding to receive readers' comments by electronic mail elishako@fau.edu or by regular mail.

Isaac Elishakoff
July 2006

CONTENTS

Basic Equations of the Linearized Theory of Elasticity: a Brief Review

Erasmo Viola

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Abstract The basic relationships of the linearized theory of elasticity of a continuous system are reviewed in different notations. The governing equations are expressed in terms of the displacement field, together with the appropriate initial and boundary conditions. The equations of motion of a few structural members are deduced.

1 Introduction

In order to formulate the mathematical model of a continuous system S undergoing time–dependent deformation, the equations of motion for infinitesimal displacements will be summarized and discussed hereinafter.

The field equations for displacements and deformations refer to a linear elastic solid and the equilibrium under the action of externally applied loads is expressed in the undeformed state.

The coordinates of points of the continuous system in three–dimensional space are denoted by x, y, z or x_1, x_2, x_3, for the Cartesian coordinate systems $Oxyz$ and $Ox_1x_2x_3$ respectively (Fig. 1).

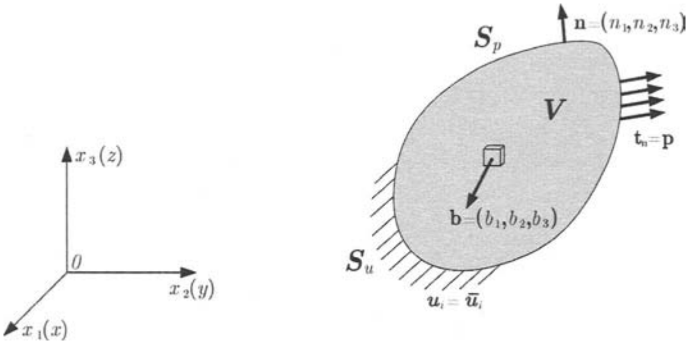

Figure 1. Applied body forces b_i and imposed surface tractions p_i and displacements $\bar{u}_i$ to the equilibrium position of the elastic system in the Cartesian coordinate space $Ox_1x_2x_3$ or $Oxyz$.

The displacement vector $\mathbf{u}$, which is defined at every point of the body, is also called the configuration variable and it can be expressed in the extended form

$$\mathbf{u}=\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}u\\v\\w\end{bmatrix}\quad\text{or}\quad\mathbf{u}(\mathbf{x},t)=\begin{bmatrix}u_1(x_1,x_2,x_3,t)\\u_2(x_1,x_2,x_3,t)\\u_3(x_1,x_2,x_3,t)\end{bmatrix}=\begin{bmatrix}u(x,y,z,t)\\v(x,y,z,t)\\w(x,y,z,t)\end{bmatrix}\tag{1}$$

or in the index notation

$$u_i=u_i(x_1,x_2.x_3.t)\quad\text{or}\quad u_i=u_i(\mathbf{x},t)\tag{2}$$

where the coordinate vector takes the form

$$\mathbf{x}^{\mathrm{T}}=[x_1\ x_2\ x_3]=[x\ y\ z]\tag{3}$$

The u_i $(i=1,2,3)$ components of the vector $\mathbf{u}$ refer to the Cartesian coordinate system $Ox_1x_2x_3$, whereas u, v, w denote the displacement components with respect to the coordinate system $Oxyz$.

From above, it should be noted that a vector quantity is denoted by a boldface letter.

The more explicit notation (x,y,z) will be used in the subsequent lectures when it is possible.

The volume occupied by the deformable body is denoted by V and μ indicates the volume density of the material.

The total surface of the body S can be split into two portions S_p and S_u such that $S=S_u\cup S_p$, where $\cup$ denotes the union symbol in set–theoretic notation. On the portion S_u of the surface, the displacements $\mathbf{u}$ are imposed $(i=1,2,3)$:

$$\mathbf{u}=\bar{\mathbf{u}}\quad\text{or}\quad u_i=\bar{u}_i\tag{4}$$

whereas, the surface traction vector $\mathbf{p}$

$$\mathbf{t_n}=\begin{bmatrix}t_{n1}\\t_{n2}\\t_{n3}\end{bmatrix}=\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix}=\mathbf{p}\quad\text{or}\quad t_{ni}=p_i\tag{5}$$

is imposed on the S_p surface, namely on the complementary part of the surface S_u.

The boundary conditions (4) are said to be of geometric or essential type, whereas the boundary conditions (5) are referred to as of forced or natural type.

Body forces $\mathbf{b}=(b_1,b_2,b_3)$ act on the unit volume of the body.

The vector of the applied forces $\mathbf{f}$ may be represented in the vector form

$$\mathbf{f}=\mathbf{b}-\mu\ddot{\mathbf{u}}\quad\text{or}\quad\begin{bmatrix}f_1\\f_2\\f_3\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}-\mu\frac{\partial^2}{\partial t^2}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\tag{6}$$

as the sum of two terms, namely the vector $\mathbf{b}$ of the body forces $b_i=b_i(x_1,x_2,x_3,t)$ and the vector $\mu\ddot{\mathbf{u}}$ of the inertia forces $\mu\ddot{u}_i$ $(i=1,2,3)$. A point over the u_i variable stands for derivation of the variable itself with respect to the time t.

In the following, three basic types of equations are mainly considered:

1. kinetic equations or motion equations;
2. kinematic equations of congruence relations;
3. constitutive equations or stress–strain relations.

Kinetic equations describe the static or dynamic equilibrium of a body under the action of applied loads. These equations are also called balance equations.

Kinematic relationships describe the deformation of the elastic body without considering the force causing the deformation itself. Kinematic relations are also called definition equations or congruence relationships.

The constitutive equations describe the constitutive behavior of the body and relate the variables of the kinetic description to the ones of the kinematic description.

These three types of equations can be combined to give the governing equations of the motion of the solid body. These governing equations will also be denoted as fundamental equations. Each basic set of equations will be represented in different notations. The index notation, the extended notation and the vector or matrix notation will be used.

In the above equations four vectors of variables are involved, namely the displacement vector, which describes the configuration of the continuum system; the force vector, which takes the forces applied to the system into account; the strain vector, which represents the strain tensor and the stress vector, which collects the components of the stress tensor.

The strain vector may be put into the extended form

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{x},t) = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{13} \\ 2\varepsilon_{23} \end{bmatrix} = \begin{bmatrix} \varepsilon_{11}(x_1,x_2,x_3,t) \\ \varepsilon_{22}(x_1,x_2,x_3,t) \\ \varepsilon_{33}(x_1,x_2,x_3,t) \\ 2\varepsilon_{12}(x_1,x_2,x_3,t) \\ 2\varepsilon_{13}(x_1,x_2,x_3,t) \\ 2\varepsilon_{23}(x_1,x_2,x_3,t) \end{bmatrix} = \begin{bmatrix} \varepsilon_x(x,y,z,t) \\ \varepsilon_y(x,y,z,t) \\ \varepsilon_z(x,y,z,t) \\ \gamma_{xy}(x,y,z,t) \\ \gamma_{xz}(x,y,z,t) \\ \gamma_{yz}(x,y,z,t) \end{bmatrix} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} \tag{7}$$

or in the index notation

$$\varepsilon_{ij} = \varepsilon_{ij}(x_1,x_2,x_3,t) \quad \text{or} \quad \varepsilon_{ij} = \varepsilon_{ij}(\mathbf{x},t) \tag{8}$$

In the same way, the stress vector can be expressed in the extended form

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x},t) = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} \sigma_{11}(x_1,x_2,x_3,t) \\ \sigma_{22}(x_1,x_2,x_3,t) \\ \sigma_{33}(x_1,x_2,x_3,t) \\ \sigma_{12}(x_1,x_2,x_3,t) \\ \sigma_{13}(x_1,x_2,x_3,t) \\ \sigma_{23}(x_1,x_2,x_3,t) \end{bmatrix} = \begin{bmatrix} \sigma_x(x,y,z,t) \\ \sigma_y(x,y,z,t) \\ \sigma_z(x,y,z,t) \\ \sigma_{xy}(x,y,z,t) \\ \sigma_{xz}(x,y,z,t) \\ \sigma_{yz}(x,y,z,t) \end{bmatrix} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{bmatrix} \tag{9}$$

or in the index notation

$$\sigma_{ij} = \sigma_{ij}(x_1,x_2,x_3,t) \quad \text{or} \quad \sigma_{ij} = \sigma_{ij}(\mathbf{x},t) \tag{10}$$

The state of strain and the state of stress within a body are determined when we know the values at each point of the six components of the vectors (7) and (9), respectively.

2 Dynamic equilibrium equations

In extended notation and in a rectangular Cartesian coordinate system, the equations of motion at time t take the form

$$\begin{aligned}
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + b_1 &= \mu \frac{\partial^2 u_1}{\partial t^2} \\
\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + b_2 &= \mu \frac{\partial^2 u_2}{\partial t^2} \\
\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + b_3 &= \mu \frac{\partial^2 u_3}{\partial t^2}
\end{aligned} \tag{11}$$

where μ is the density of the material, and the symmetry of the stress tensor is considered too:

$$\sigma_{12} = \sigma_{21} \quad \sigma_{13} = \sigma_{31} \quad \sigma_{23} = \sigma_{32} \tag{12}$$

Equations (11) are also called balance equations.

When no body moments and couples exist, the above system (11) of three equations contains 9 unknowns: 6 stress components σ_{11}, σ_{22}, σ_{33}, σ_{12}, σ_{13}, σ_{23}, and 3 displacements u_1, u_2 and u_3, since b_1, b_2 and b_3 are assumed as given.

The equations of dynamical equilibrium are not sufficient for the determination of the six stress components. Additional equations are needed such as the strain–displacement equations and the constitutive equations.

In index notation, the fundamental stress equations of motion (11) may be put into the form

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \mu \frac{\partial^2 u_i}{\partial t^2} \quad \text{or} \quad \sigma_{ij,j} + f_i = \mu \ddot{u}_i \tag{13}$$

On the right–hand side of Eq. (13), a comma followed by an index j denotes differentiation with respect to the variable x_j.

Equations of motion or dynamic equilibrium equations (11) can be written in the operatorial form

$$\mathbf{D}^* \boldsymbol{\sigma} = \mathbf{f} \quad \text{or} \quad \mathbf{D}^* \boldsymbol{\sigma} + \mathbf{b} = \mu \frac{\partial^2 \mathbf{u}}{\partial t^2} \tag{14}$$

where $\boldsymbol{\sigma}$ is the vector of the stress tensor and the balance operator $\mathbf{D}^*$, also known as the equilibrium operator, which assumes the aspect:

$$\mathbf{D}^* = - \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_1} & 0 & \frac{\partial}{\partial x_3} \\ 0 & 0 & \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} \end{bmatrix} \tag{15}$$

The equilibrium operator is denoted with the symbol $\mathbf{D}^*$ in order to underline the relationship of adjointness with the congruence operator $\mathbf{D}$, which will be introduced later.

Equations (11) state that at a point inside the body where an infinitesimal cube having unit edges can be supposed to be located, the sum of all the forces acting on the faces of the cube itself as well as at the center of its mass distribution is the vector zero. These equations can be derived using either Newton's second law of motion or the principle of conservation of linear momentum. Equations (11) must be satisfied at every interior point of the body.

2.1 Traction boundary conditions

Let us consider the normal n on a plane at a point B within a body. The components of the unit vector $\mathbf{n}$ of n are n_1, n_2, n_3. It is useful to express the relationship between the stress vector $\mathbf{t}_n$ at the specified point B, on the plane normal to n, and the stress vectors of three mutually perpendicular planes at the same point B, which are parallel to the coordinate planes. In matrix notation, one may write:

$$\begin{bmatrix} t_{n1} \\ t_{n2} \\ t_{n3} \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \quad \text{or} \quad \mathbf{t}_n = \boldsymbol{\sigma n} \tag{16}$$

in index notation, Eqs. (16) take the form:

$$t_{ni} = \sigma_{ij} n_j \tag{17}$$

for $i = 1, 2, 3$, $j = 1, 2, 3$. Eqs. (16) and (17) are called the Cauchy equations or the Cauchy stress formulas.

It is worth noting that it is customary in mechanics literature to abbreviate a summation of terms by repeating an index, which indicates summation over all the values of that index. This is the well known summation convention, whereby a repeated subscript implies a summation.

The stress components acting across planes parallel to the coordinate planes can also be connected with the external tractions $p_i = p_i(x_1, x_2, x_3, t)$ applied at any point on the bounding surface of a body. Taking account of Eq. (5), Eq. (16) leads to

$$\sigma_{ij} n_j = p_i \tag{18}$$

These relationships may be written in the matrix form

$$\mathbf{B}\boldsymbol{\sigma} = \mathbf{p} \tag{19}$$

where $\boldsymbol{\sigma}$ and $\mathbf{p}$ are the stress vector (9) and the vector of the surface external forces (5) and $\mathbf{B}$ is the matrix

$$\mathbf{B} = \begin{bmatrix} n_1 & 0 & 0 & 0 & n_3 & n_2 \\ 0 & n_2 & 0 & n_3 & 0 & n_1 \\ 0 & 0 & n_3 & n_2 & n_1 & 0 \end{bmatrix} \tag{20}$$

that collects the Cartesian components n_i $(i = 1,2,3)$ of the unit normal vector $\mathbf{n}$.

It should be noted that Eq. (19) must be specified at any point of the bounding surface S_p where external loads are applied.

3 Kinematic equations or congruence relations

The deformation of a deformable body in the neighborhood of a given point is characterized by the six components of the symmetrical strain tensor $\boldsymbol{\varepsilon}$, which can be written as a matrix:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \tag{21}$$

The strain components ε_{11}, ε_{22}, ε_{33} with equal subscripts are denoted by dilatations or normal strains, while the strain components ε_{12}, ε_{13}, ε_{23} with different subscripts are called shearing strains.

In short, ε_{ii} $(i = 1,2,3)$ denotes the change in length of a segment originally parallel to the x_i axis, while $\varepsilon_{ij} = \gamma_{ij}/2$ represents half of the change in angle between segments whose original directions were x_i and x_j.

The relationship between the strain components and the displacement components may be written in extended notation

$$\begin{aligned} &\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \qquad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \\ &\gamma_{12} = 2\varepsilon_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = u_{1,2} + u_{2,1} \\ &\gamma_{13} = 2\varepsilon_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = u_{1,3} + u_{3,1} \\ &\gamma_{23} = 2\varepsilon_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = u_{2,3} + u_{3,2} \end{aligned} \tag{22}$$

In Eq.(22), the derivative notation has been used. That is, differentiation with respect to a variable x_i $(i = 1,2,3)$ is indicated by a comma followed by the index i which represents the x_i variable.

Using index notation, the strain–displacement components take the form:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{23}$$

The matrix notation of the congruence equations assumes the aspect

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} \tag{24}$$

where

$$\mathbf{D} = \begin{bmatrix} \frac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} & 0 \\ \frac{\partial}{\partial x_3} & 0 & \frac{\partial}{\partial x_1} \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \end{bmatrix} \tag{25}$$

is called the congruence operator or the kinematic operator. Such an operator is also known as the definition operator, because the equations (22) in discussion are known as the definition equations too.

The displacement vector $\mathbf{u}$ and the vector of the strain components are shown by (1) and (7) respectively.

It is worth noting that the congruence operator is the adjoint of the balance operator.

The relationship of adjointness between balance and definition operators can be discovered with respect to the two bilinear functionals

$$< \mathbf{f}, \mathbf{u} >= \int_V f_i u_i \mathrm{d}V \quad , \quad < \boldsymbol{\sigma}, \boldsymbol{\varepsilon} >= \int_V \sigma_{ij} \varepsilon_{ij} \mathrm{d}V \tag{26}$$

3.1 Compatibility equations for strain

When a displacement field $u_i = u_i(x_1, x_2, x_3, t)$ is given, the computation of the strain components of the symmetric small–strain tensor $\boldsymbol{\varepsilon}$ is straightforward. In fact, using Eqs. (23) the strain components ε_{ij} can be expressed in terms of the gradients of the displacement vector $\mathbf{u}$. In addition, the components ω_{ij} of the rotation tensor $\boldsymbol{\omega}$ are defined as

$$\omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \tag{27}$$

The last tensor is an antisymmetric tensor, because $\omega_{ij} = -\omega_{ji}$.

However, when the strain components are given, the determination of the displacements is not always possible. The indetermination arises from the system (23) of the six congruence equations which involve only three unknowns u_1, u_2, u_3. Stated in other words, the six components of strain ε_{ij} cannot be given arbitrarily as functions of x_1, x_2, x_3 but must be subject to six differential equations of the type:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} = \varepsilon_{lj,ki} + \varepsilon_{ki,lj} \tag{28}$$

involving the three displacement components.

It should be noted that Eq. (28) provides the necessary and sufficient conditions for the existence of a single–valued displacement field $\mathbf{u} = \mathbf{u}(x_1, x_2, x_3, t)$, when the strains ε_{ij} are given and the region of integration is simply connected. Although Eq. (28) under consideration yields 81 equations, only six of them are different from each other and the remaining ones are either trivial or linear combinations of the above mentioned six independent equations. These 6 essential equations may be put into the operatorial form

$$\boldsymbol{R\varepsilon} = \mathbf{0} \tag{29}$$

where ε is the strain vector (7) and $\mathbf{R}$ is a 6×6 matrix of differential operators of the second order.

Eq. (29) is termed the operatorial form of the compatibility equations for the strain.

4 Constitutive equations or stress–strain relationships

In a linearly elastic isotropic material, the constitutive relations depend only on two elastic constants λ and G, called the Lame's constants:

$$G = \frac{E}{2(1+\nu)} \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{30}$$

where ν is the Poisson's ratio, E is the Young's modulus or normal modulus, and G is the shear modulus of the material.

The extended notation of the constitutive equations assumes the aspect:

$$\begin{array}{lll} \sigma_{11} = 2G\varepsilon_{11} + \lambda I_{1\varepsilon} & \sigma_{22} = 2G\varepsilon_{22} + \lambda I_{1\varepsilon} & \sigma_{33} = 2G\varepsilon_{33} + \lambda I_{1\varepsilon} \\ \sigma_{12} = G\gamma_{12} & \sigma_{13} = G\gamma_{13} & \sigma_{23} = G\gamma_{23} \end{array} \tag{31}$$

where

$$I_{1\varepsilon} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \tag{32}$$

is the cubic dilatation, namely the change in the unit volume of the elastic body.

The first term on the right hand side of the first three Eqs. (31) generates shear deformation, whereas the second one represents the principal stress due to volumetric dilatation.

The stress–strain relationship (31) can be expressed in index notation

$$\sigma_{ij} = 2G\varepsilon_{ij} + \delta_{ij}\lambda I_{1\varepsilon} \tag{33}$$

where δ_{ij} ($\delta_{ij} = 1$ when $i = j$ and $\delta_{ij} = 0$ when $i \neq j$) is the Kronecker delta.

In matrix notation, the relation between the stress components and the strain components takes the form

$$\boldsymbol{\sigma} = \boldsymbol{C\varepsilon} \tag{34}$$

where the stress vector $\boldsymbol{\sigma}$ and the strain vector $\boldsymbol{\varepsilon}$ were indicated above, and

$$\mathbf{C} = \begin{bmatrix} 2G+\lambda & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & 2G+\lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & 2G+\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & G & 0 & 0 \\ 0 & 0 & 0 & 0 & G & 0 \\ 0 & 0 & 0 & 0 & 0 & G \end{bmatrix} \tag{35}$$

is the constitutive operator, also called matrix of the material rigidity.

It is worth noting that in the case of isotropic material only two elastic constants must be considered. For the general anisotropic case, the elastic constants appearing in the constitutive equations are 21.

5 Fundamental equations

The three basic sets of equations, namely the balance, the kinematic and the constitutive equations may be combined to give the fundamental system of equations, also known as the governing system of equations. Firstly, the fundamental equations are deduced in matrix notation. So, if the strain displacement relations (24) are inserted into the constitutive equations (34), we have the relationships between stresses and displacements:

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon} = \mathbf{CDu} \tag{36}$$

When equations (36) are inserted into the equations of motion (14), the fundamental system of equations is derived:

$$\mathbf{D}^*\mathbf{CDu} = \mathbf{f} \tag{37}$$

The equations of motion in terms of displacements take all the three aspects of the problem of the elastic equilibrium into account.

By introducing the fundamental operator, also known as the elasticity operator,

$$\mathbf{L} = \mathbf{D}^*\mathbf{CD} \tag{38}$$

equation (37) can be written as

$$\mathbf{Lu} = \mathbf{f} \quad \text{or} \quad \mathbf{Lu} + \mathbf{b} = \mu\ddot{\mathbf{u}} \tag{39}$$

The fundamental system of equations (39) relates the configuration variable $\mathbf{u}$ to the source variable $\mathbf{b}$ of the phenomenon under investigation.

In extended notation, Eq. (39) takes the form:

$$\begin{aligned}
(\lambda+G)\frac{\partial}{\partial x_1}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+G\nabla^2 u_1+b_1&=\mu\frac{\partial^2 u_1}{\partial t^2}\\
(\lambda+G)\frac{\partial}{\partial x_2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+G\nabla^2 u_2+b_2&=\mu\frac{\partial^2 u_2}{\partial t^2}\\
(\lambda+G)\frac{\partial}{\partial x_3}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}+\frac{\partial u_3}{\partial x_3}\right)+G\nabla^2 u_3+b_3&=\mu\frac{\partial^2 u_3}{\partial t^2}
\end{aligned} \tag{40}$$

Using the index notation, Eqs. (40) assume the aspect

$$(\lambda+G)\,u_{i,ij}+Gu_{j,ii}+b_j=\mu\ddot{u}_j \tag{41}$$

Eqs. (40) may also be expressed in the vectorial form:

$$(\lambda+G)\,\nabla I_{1\varepsilon}+G\nabla^2\mathbf{u}+\mathbf{b}=\mu\ddot{\mathbf{u}} \tag{42}$$

where ∇ is the vector operator del (or nabla):

$$\nabla = \mathbf{i_1}\frac{\partial}{\partial x_1} + \mathbf{i_2}\frac{\partial}{\partial x_2} + \mathbf{i_3}\frac{\partial}{\partial x_3} \tag{43}$$

and ∇^2 is the scalar operator called the Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \tag{44}$$

and

$$I_{1\varepsilon} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \text{div}(\mathbf{u}) = \nabla \cdot \mathbf{u} \tag{45}$$

is the divergence of the displacement field, which denotes the cubical dilatation, that is the change in volume of an infinitesimal rectangular parallelepiped, with sides originally in the coordinate directions, divided by the original volume.

All the basic and fundamental equations, as well as the variables $\mathbf{u}$, $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$ and $\mathbf{b}$ which are involved, are reported in Tonti's diagram (see Appendix A).

The above three equilibrium equations in terms of displacements $u_i(x_1, x_2, x_3, t), i = 1, 2, 3$, are called Navier's equations for the forced vibration. Each form of these equations presents the body forces expressing themselves in different notation, namely as algebraic vector $\mathbf{b} = [b_1\ b_2\ b_3]^{\mathrm{T}}$ in (39), by means of the extended notation in (40), in index notation b_j in (41), and as a Cartesian vector $\mathbf{b} = b_1\mathbf{i_1} + b_2\mathbf{i_2} + b_3\mathbf{i_3} = b_i\mathbf{i_i}$ in (42).

It is worth noting that the complete formulation of the dynamic equilibrium problem expressed by the motion equations (40) requires that kinematic boundary conditions (4) on S_u and the forced boundary conditions (18) on S_p must be fixed. In addition, the initial conditions at the $t = 0$ time

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}_0(\mathbf{x}, 0) \quad , \quad \left.\frac{\partial \mathbf{u}(\mathbf{x}, t)}{\partial t}\right|_0 = \dot{\mathbf{u}}(\mathbf{x}, 0) = \dot{\mathbf{u}}_0 \tag{46}$$

must be imposed. In short, the considered mixed formulation of the boundary value problem is illustrated in Fig. 1, under the action of a given body force distribution $b_i = b_i(\mathbf{x}, t)$ inside the body, when a displacement field on the S_u surface and boundary forces on the S_p surface are specified.

When Eqs. (40) are solved in conjunction with appropriate initial (46) and boundary conditions, the displacements $u_i = u_i(\mathbf{x}, t)$ can be determined. Then, the application of the congruence relationships (22) and the constitutive relations (31) gives the strains ε_{ij} and the stresses, respectively.

5.1 Dilatational and distortional waves

In the absence of external loading, Navier's equations for the free vibration may be derived. Using the vectorial form (42), for example, we have

$$(\lambda + G)\,\nabla I_{1\varepsilon} + G\nabla^2\mathbf{u} = \mu\ddot{\mathbf{u}} \tag{47}$$

This equation cannot be integrated directly, so a form of solution must be assumed and checked for suitability by differentiation and substitution.

If we assume that the deformation produced by the waves is such that the volume expansion $I_{1\varepsilon}$ is zero, the vector form of the equations of motion becomes:

$$G\nabla^2\mathbf{u} = \mu\ddot{\mathbf{u}} \tag{48}$$

In extended notation, Eq. (48) assumes the aspect:

$$G\nabla^2 u_1 = \mu\frac{\partial^2 u_1}{\partial t^2}, \quad G\nabla^2 u_2 = \mu\frac{\partial^2 u_2}{\partial t^2}, \quad G\nabla^2 u_3 = \mu\frac{\partial^2 u_3}{\partial t^2} \tag{49}$$

These equations for waves are called equivolumic waves, shear waves, waves of distortion, rotational waves or secondary (S) waves. Eqs. (49) may also be put into the form

$$\nabla^2 u_i = \frac{1}{c_T^2}\frac{\partial^2 u_i}{\partial t^2} \tag{50}$$

where

$$c_T = \sqrt{\frac{G}{\mu}} \tag{51}$$

is the velocity of propagation of waves of distortion.

When the deformation produced by the waves is not accompanied by rotation, namely the deformation is irrotational, the components ω_1, ω_2, ω_3 of the rotation vector

$$\omega_1 = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) \quad \omega_2 = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) \quad \omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) \tag{52}$$

are zero. Therefore, the displacements $u_i = u_i(x_1, x_2, x_3, t)$ are derivable from a single function $g = g(x_1, x_2, x_3)$ as follows

$$u_1 = \frac{\partial g}{\partial x_1} \quad u_2 = \frac{\partial g}{\partial x_2} \quad u_3 = \frac{\partial g}{\partial x_3} \tag{53}$$

Then, the cubical dilatation and its derivatives take the form

$$I_{1\varepsilon} = \nabla^2 g \quad \text{and} \quad \frac{\partial I_{1\varepsilon}}{\partial x_i} = \nabla^2 u_i \tag{54}$$

Inserting Eqs. (54) into the motion equations (47) leads to the equation for longitudinal waves:

$$(\lambda + 2G)\,\nabla^2\mathbf{u} = \mu\ddot{\mathbf{u}} \tag{55}$$

In different notation we can write $(i = 1, 2, 3)$:

$$\nabla^2 u_i = \frac{1}{c_L^2}\frac{\partial^2 u_i}{\partial t^2} \quad \text{or} \quad \nabla^2\mathbf{u} = \frac{1}{c_L^2}\frac{\partial^2\mathbf{u}}{\partial t^2} \tag{56}$$

where

$$c_L = \sqrt{\frac{\lambda + 2G}{\mu}} \tag{57}$$

is the velocity of propagation of the longitudinal waves. A variety of terminology also exists for this wave–type. Longitudinal waves are also called dilatational waves, primary (P) waves ar irrotational waves.

The P and S wave designations have arisen in seismology, where they are also occasionally indicated as the "push" and "shake" waves.

6 Motion equations deduced for structural members

In the following the equations of motion of a few structural elements are derived from the fundamental equations. To this end, a more convenient form of Navier's equations of motion will be used:

$$(\lambda + G)\begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{bmatrix} I_{1\varepsilon} + G\nabla^2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \mu \frac{\partial^2}{\partial t^2} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{58}$$

The forced and free equations of motion are referred to as both one-dimensional and two-dimensional structural members.

Physical systems such as bars in extension and vibrating strings are included in the category of one–dimensional systems.

The membrane element may be thought of as the two–dimensional analogue of the string.

6.1 Longitudinal vibration of rods

If we assume the one–dimensional displacement field and the vector of the body forces as follows:

$$\mathbf{u}^{\mathrm{T}} = [u_1\ u_2\ u_3] = [u_1(x,t)\ 0\ 0] \quad , \quad \mathbf{b}^{\mathrm{T}} = [b_1(x,t)\ 0\ 0] \tag{59}$$

the cubical dilatation $I_{1\varepsilon} = u_{1,1}$ and its derivative with respect to x_1, x_2, x_3 may be calculated. We have $I_{1\varepsilon,1} = u_{1,11}$, $I_{1\varepsilon,2} = I_{1\varepsilon,3} = 0$.

Using the above positions and results, the first equation of the system (58) of differential equations yields:

$$(\lambda + G)\frac{\partial^2 u_1}{\partial x_1^2} + G\frac{\partial^2 u_1}{\partial x_1^2} + b_1 = \mu\frac{\partial^2 u_1}{\partial t^2} \tag{60}$$

while the second and the third of the equations under consideration are identically verified. Eq. (60) denotes the forced motion equation for a one–dimensional continuous system.

Eq. (60) can be put into the form

$$\frac{\partial^2 u_1}{\partial x_1^2} = \frac{1}{c_L^2}\frac{\partial^2 u_1}{\partial t^2} \tag{61}$$

when the component b_1 of the body force is equal to zero. The velocity of propagation of longitudinal waves c_L can be expressed as

$$c_L = \sqrt{\frac{\lambda + 2G}{\mu}} = \sqrt{\frac{E}{\mu\,(1+\nu)}\left(\frac{1-\nu}{1-2\nu}\right)} \tag{62}$$

It is worth noting that if the Poisson's coefficient is set to zero, one obtains $c_L^2 = E/\mu$ that is the expression of the velocity of propagation of plane wave dilatation deduced by the application of approximate theories.

6.2 Transverse vibration of taut strings

The governing equation for a taut string can be obtained from the fundamental system of equations by assuming the displacement vector $\mathbf{u}$ and the force vector $\mathbf{b}$ as is indicated here:

$$\mathbf{u}^{\mathrm{T}} = [u_1\ u_2\ u_3] = [0\ u_2(x,t)\ 0] \quad , \quad \mathbf{b}^{\mathrm{T}} = [0\ b_2(x,t)\ 0] \tag{63}$$

The evaluation of $I_{1\varepsilon}$ and its derivative leads to $I_{1\varepsilon} = 0$, $I_{1\varepsilon,1} = I_{1\varepsilon,2} = I_{1\varepsilon,3} = 0$ and from the second equation of the system (58) the forced motion of the taut string under the external force $b_2 = b_2(x,t)$ is derived

$$G\frac{\partial^2 u_2}{\partial x_2^2} + b_2 = \mu\frac{\partial^2 u_2}{\partial t^2} \tag{64}$$

When $b_2 = 0$, the equation governing the harmonic motion is written as

$$\frac{\partial^2 u_2}{\partial x_2^2} = \frac{1}{c_T}\frac{\partial^2 u_2}{\partial t^2} \tag{65}$$

where

$$c_T = \sqrt{\frac{G}{\mu}} \tag{66}$$

is the shear wave velocity.

It is worthwhile to note that when an approximate theory is used for the waves in taut strings, the tangential elastic modulus G has to be replaced by the traction force T with which the string is initially taut.

6.3 Governing equation for membranes

In considering the vibration of elastic bodies involving two independent spatial variables, in what follows only the membrane element will be examined.

The equation governing the motion of the membrane can be derived from Eq. (58), by assuming

$$\mathbf{u}^{\mathrm{T}} = [u_1\ u_2\ u_3] = [0\ 0\ u_3(x_1,x_2,t)] \quad , \quad \mathbf{b}^{\mathrm{T}} = [0\ 0\ b_3(x_1,x_2,t)] \tag{67}$$

The assumption (67) leads to the following equation of the forced motion of the membrane:

$$G\left(\frac{\partial^2 u_3}{\partial x_1^2}+\frac{\partial^2 u_3}{\partial x_2^2}\right)+b_3=\mu\frac{\partial^2 u_3}{\partial t^2} \tag{68}$$

Setting $b_3 = 0$ in Eq. (68) the two dimensional form of the wave equation can be derived

$$\frac{\partial^2 u_3}{\partial x_1^2}+\frac{\partial^2 u_3}{\partial x_2^2}=\frac{1}{c_T^2}\frac{\partial^2 u_3}{\partial t^2} \tag{69}$$

7 Concluding remarks

The basic equations of the linearized theory of elasticity were presented and analyzed in a general framework. The equations for dilatational and distortional waves were derived in the three dimensional case and in some specific one and two dimensional structural elements.

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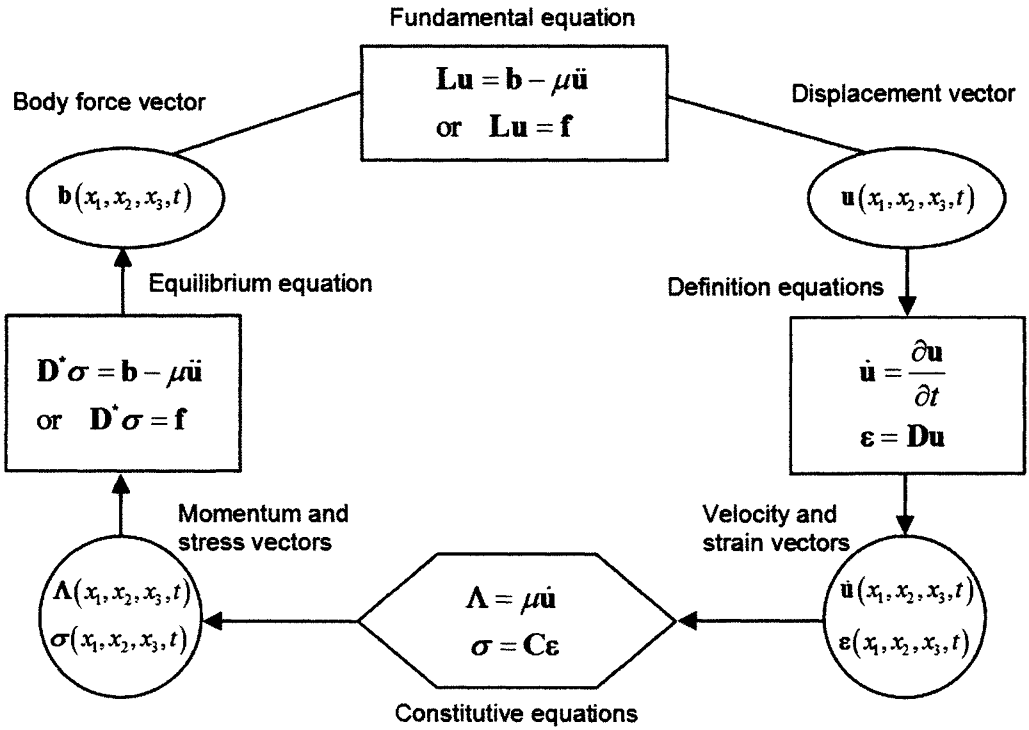
Appendix A Tonti's diagram



Dynamical Analysis of Spherical Structural Elements Using the First-order Shear Deformation Theory

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Abstract. This lecture deals with the dynamical behaviour of hemispherical domes and shell panels. The First-order Shear Deformation Theory (FSDT) is used to analyze the above moderately thick structural elements. The treatment is conducted within the theory of linear elasticity, when the material behaviour is assumed to be homogeneous and isotropic. The governing equations of motion, written in terms of internal resultants, are expressed as functions of five kinematic parameters, by using the constitutive and the congruence relationships. The boundary conditions considered are clamped (C), simply supported (S) and free (F) edge. Numerical solutions have been computed by means of the technique known as the Generalized Differential Quadrature (GDQ) Method. These results, which are based upon the FSDT, are compared with the ones obtained using commercial programs such as Ansys, Femap/Nastran, Straus, Pro/Engineer, which also elaborate a three-dimensional analysis.

1 Introduction

Structures of shell revolution type have been widespread in many fields of engineering, where they give rise to optimum conditions for dynamical behaviour, strength and stability. Pressure vessels, cooling towers, water tanks, dome-shaped structures, dams, turbine engine components and so forth, perform particular functions over different branches of structural engineering.

The purpose of this lecture is to study the dynamic behaviour of structures derived from shells of revolution. The equations given here incorporate the effects of transverse shear deformation and rotary inertia.

The geometric model refers to a moderately thick shell. The analysis will be performed by following two different investigations. In the first one, the solution is obtained by using the numerical technique termed GDQ method, which leads to a generalized eigenvalue problem. The main features of the numerical technique under discussion, as well as its historical development, are illustrated in section 3.

The solution is given in terms of generalized displacement components of the points lying on the middle surface of the shell. At the moment it can only be pointed out that by using the GDQ technique the numerical statement of the problem does not pass through any variational formulation, but deals directly with the governing equations of motion.

Numerical results will also be computed by using commercial programs, which also elaborate three-dimensional analyses.

It should be noted that there are various two-dimensional theories of thin shells. Any two-dimensional theory of shells is an approximation of the real three-dimensional problem. Starting from Love's theory about the thin shells, which dates back to 100 years ago, a lot of contributions on this topic have been made since then. The main purpose has been that of seeking better and better approximations for the exact three-dimensional elasticity solutions for shells.

In the last fifty years refined two-dimensional linear theories of thin shells have developed including important contributions by Sanders (1959), Flügge (1960), Niordson (1985). In these refined shell theories the deformation is based on the Kirchhoff-Love assumption. In other words, this theory assumes that normals to the shell middle-surface remain normal to it during deformations and unstretched in length.

It is worth noting that when the refined theories of thin shells are applied to thick shells, the errors could be quite large. With the increasing use of thick shells in various engineering applications, simple and accurate theories for thick shells have been developed. With respect to the thin shells, the thick shell theories take the transverse shear deformation and rotary inertia into account. The transverse shear deformation has been incorporated into shell theories by following the work of Reissner (1945) for the plate theory.

Several studies have been presented earlier for the vibration analysis of such revolution shells and the most popular numerical tool in carrying out these analyses is currently the finite element method. The generalized collocation method based on the ring element method has also been applied (Viola and Artioli (2004), Artioli, Gould and Viola (2004)). With regard to the latter method each static and kinematic variable is transformed into a theoretically infinite Fourier series of harmonic components, with respect to the circumferential co-ordinates.

In this paper, the governing equations of motion are a set of five bi-dimensional partial differential equations with variable coefficients. These fundamental equations are expressed in terms of kinematic parameters and can be obtained by combining the three basic sets of equations, namely balance, congruence and constitutive equations.

Referring to the formulation of the dynamic equilibrium in terms of harmonic amplitudes of mid-surface displacements and rotations, in this paper the system of second-order linear partial differential equations is solved, without resorting to the one-dimensional formulation of the dynamic equilibrium of the shell. Now, the discretization of the system leads to a standard linear eigenvalue problem, where two independent variables are involved.

In this way it is possible to compute the complete assessment of the modal shapes corresponding to natural frequencies of structures.

2 Basic Governing Equations

2.1 Shell Geometry and Kinematic Equations

The geometry of the shell considered hereafter is a surface of revolution with a circular curved meridian. The notation for the co-ordinates is shown in Figure 1. The total thickness of the shell is represented by h. The distance of each point from the shell mid-surface along the normal is ζ.

The co-ordinate along the meridional and circumferential directions are $\alpha_1 = \alpha_\varphi$ and $\alpha_2 = \alpha_\vartheta$, respectively. The distance of each point from the axis of revolution is $R_0(\varphi)$ and φ is the angle between the normal to the shell surface and the axis of revolution (Figure 2).

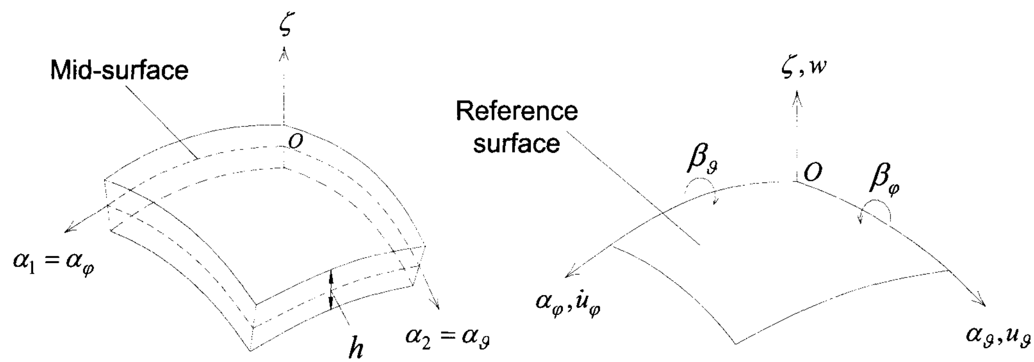

Figure 1. Co-ordinate system of the shell and reference surface.

The position of an arbitrary point within the shell material is known by the co-ordinates φ ($0 \le \varphi \le \pi$), ϑ ($0 \le \vartheta \le 2\pi$) upon the middle surface, and ζ directed along the outward normal and measured from the reference surface ($-h/2 \le \zeta \le h/2$).

R_φ and R_ϑ are, in the general case, the radii of curvature in the meridional and circumferential directions. For a spherical surface R_φ and R_ϑ are constant and equal to the radius of the shell R.

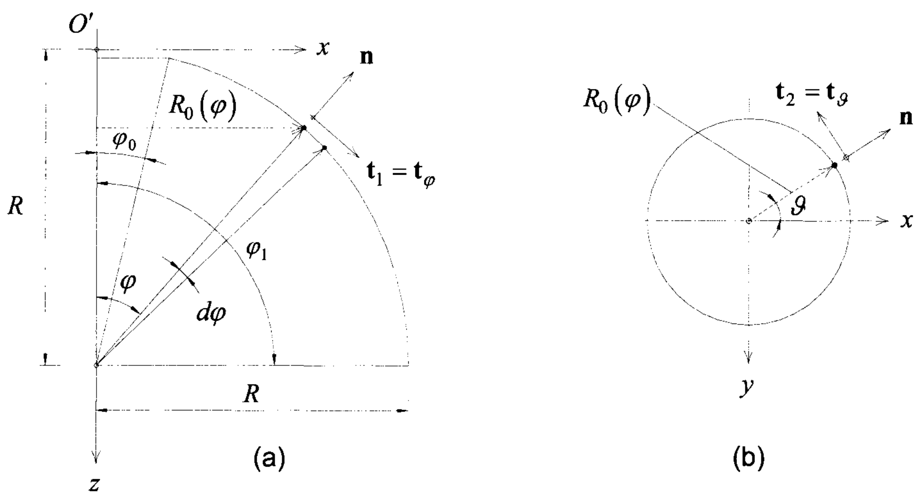

Figure 2. Geometry of hemispherical dome.

The parametric co-ordinates (φ,ϑ) define, respectively, the parallel circles and the meridional curves upon the middle surface of the shell (Figure 2). In developing a moderately thick shell theory we make certain assumptions. They are outlined below:

- The transverse normal is inextensible:

$$\varepsilon_n \approx 0$$

- Normals to the reference surface of the shell before deformation remain straight but not necessarily normal after deformation (a relaxed Kirchhoff-Love hypothesis).
- The transverse normal stress is negligible so that the plane assumption can be invoked:

$$\sigma_n = \sigma_n\left(\alpha_1,\alpha_2,\zeta,t\right) = 0$$

Consistent with the assumptions of a moderately thick shell theory, the displacement field assumed in this study is that of a *First-order Shear Deformation Theory* (FSDT) and can be put in the following form:

$$\begin{cases} U_\varphi\left(\alpha_\varphi,\alpha_\vartheta,\zeta,t\right) = u_\varphi\left(\alpha_\varphi,\alpha_\vartheta,t\right) + \zeta\beta_\varphi\left(\alpha_\varphi,\alpha_\vartheta,t\right) \\ U_\vartheta\left(\alpha_\varphi,\alpha_\vartheta,\zeta,t\right) = u_\vartheta\left(\alpha_\varphi,\alpha_\vartheta,t\right) + \zeta\beta_\vartheta\left(\alpha_\varphi,\alpha_\vartheta,t\right) \\ W\left(\alpha_\varphi,\alpha_\vartheta,\zeta,t\right) = w\left(\alpha_\varphi,\alpha_\vartheta,t\right) \end{cases} \tag{2.1}$$

where u_φ,u_ϑ,w are the displacement components of points lying on the middle surface ($\zeta = 0$) of the shell, along meridional, circumferential and normal directions, respectively. β_φ and β_ϑ are normals-to-mid-surface rotations, respectively.

The kinematics hypothesis expressed by equations (2.1) should be supplemented by the statement that the shell deflections are small and strains are infinitesimal, that is $w\left(\alpha_\varphi,\alpha_\vartheta,t\right) \ll h$.

It is worth noting that in-plane displacements U_φ and U_ϑ vary linearly through the thickness, while W remains independent of ζ. The relationships between strains and displacements along the shell reference (middle) surface $\zeta = 0$ are the following:

$$\begin{aligned} &\varepsilon_\varphi = \frac{1}{R}\left(\frac{\partial u_\varphi}{\partial\varphi} + w\right), \quad \varepsilon_\vartheta = \frac{1}{R_0}\left(\frac{\partial u_\vartheta}{\partial\vartheta} + u_\varphi\cos\varphi + w\sin\varphi\right), \quad \gamma_{\varphi\vartheta} = \frac{1}{R}\frac{\partial u_\vartheta}{\partial\varphi} + \frac{1}{R_0}\left(\frac{\partial u_\varphi}{\partial\vartheta} - u_\vartheta\cos\varphi\right) \\ &\kappa_\varphi = \frac{1}{R}\frac{\partial\beta_\varphi}{\partial\varphi}, \quad \kappa_\vartheta = \frac{1}{R_0}\left(\frac{\partial\beta_\vartheta}{\partial\vartheta} + \beta_\varphi\cos\varphi\right), \quad \kappa_{\varphi\vartheta} = \frac{1}{R}\frac{\partial\beta_\vartheta}{\partial\varphi} + \frac{1}{R_0}\left(\frac{\partial\beta_\varphi}{\partial\vartheta} - \beta_\vartheta\cos\varphi\right) \\ &\gamma_{\varphi n} = \frac{1}{R}\left(\frac{\partial w}{\partial\varphi} - u_\varphi\right) + \beta_\varphi, \quad \gamma_{\vartheta n} = \frac{1}{R_0}\left(\frac{\partial w}{\partial\vartheta} - u_\vartheta\sin\varphi\right) + \beta_\vartheta \end{aligned} \tag{2.2}$$

where $R_0(\varphi) = R\sin\varphi$ is the radius of a generic parallel of the spherical dome.

In the above, the first three strains $\varepsilon_\varphi,\varepsilon_\vartheta,\gamma_{\varphi\vartheta}$ are in-plane meridional, circumferential and shearing components, $\kappa_\varphi,\kappa_\vartheta,\kappa_{\varphi\vartheta}$ are the analogous curvature changes. The last two components are transverse shearing strains.

The matrix notation of the congruence equations assumes the aspect:

$$\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{u} \tag{2.3}$$

where

$$\mathbf{D}=\begin{bmatrix} \frac{1}{R}\frac{\partial}{\partial\varphi} & 0 & \frac{1}{R} & 0 & 0 \\ \frac{\cos\varphi}{R_0} & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} & \frac{\sin\varphi}{R_0} & 0 & 0 \\ \frac{1}{R_0}\frac{\partial}{\partial\vartheta} & \frac{1}{R}\frac{\partial}{\partial\varphi}-\frac{\cos\varphi}{R_0} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{R}\frac{\partial}{\partial\varphi} & 0 \\ 0 & 0 & 0 & \frac{\cos\varphi}{R_0} & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} \\ 0 & 0 & 0 & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} & \frac{1}{R}\frac{\partial}{\partial\varphi}-\frac{\cos\varphi}{R_0} \\ -\frac{1}{R} & 0 & \frac{1}{R}\frac{\partial}{\partial\varphi} & 1 & 0 \\ 0 & -\frac{\sin\varphi}{R_0} & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} & 0 & 1 \end{bmatrix} \tag{2.4}$$

is called the congruence operator or the kinematic operator and

$$\mathbf{u}(\alpha_\varphi,\alpha_\vartheta,t)=\begin{bmatrix} u_\varphi & u_\vartheta & w & \beta_\varphi & \beta_\vartheta \end{bmatrix}^T \tag{2.5}$$

$$\boldsymbol{\varepsilon}(\alpha_\varphi,\alpha_\vartheta,t)=\begin{bmatrix} \varepsilon_\varphi & \varepsilon_\vartheta & \gamma_{\varphi\vartheta} & \kappa_\varphi & \kappa_\vartheta & \kappa_{\varphi\vartheta} & \gamma_{\varphi n} & \gamma_{\vartheta n} \end{bmatrix}^T \tag{2.6}$$

denote the displacement vector and the generalized strain vector, respectively. The congruence operator is also known as the definition operator, because the equations (2.2) in discussion are known as the definition equations too.

2.2 Constitutive Equations

The shell material assumed in the following is a mono-laminar elastic isotropic one. Accordingly, the following constitutive equations relate internal stress resultants and internal couples with generalized strain components on the middle surface:

$$\begin{aligned} &N_\varphi = K(\varepsilon_\varphi+\nu\varepsilon_\vartheta),\quad M_\varphi = D(\kappa_\varphi+\nu\kappa_\vartheta),\quad Q_\varphi = K\frac{(1-\nu)}{2\chi}\gamma_{\varphi n} \\ &N_\vartheta = K(\varepsilon_\vartheta+\nu\varepsilon_\varphi),\quad M_\vartheta = D(\kappa_\vartheta+\nu\kappa_\varphi),\quad Q_\vartheta = K\frac{(1-\nu)}{2\chi}\gamma_{\vartheta n} \\ &N_{\varphi\vartheta} = N_{\vartheta\varphi} = K\frac{(1-\nu)}{2}\gamma_{\varphi\vartheta},\quad M_{\varphi\vartheta} = M_{\vartheta\varphi} = D\frac{(1-\nu)}{2}\kappa_{\varphi\vartheta} \end{aligned} \tag{2.7}$$

where $K = Eh/(1-\nu^2)$, $D = Eh^3/(12(1-\nu^2))$ are the membrane and bending rigidity, respectively. E is the Young modulus, ν is the Poisson ratio and χ is the shear factor which for isotropic materials is usually taken as $\chi = 6/5$. In equations (2.7), the first three components $N_\varphi, N_\vartheta, N_{\varphi\vartheta}$ are the in-plane meridional, circumferential and shearing force resultants, $M_\varphi, M_\vartheta, M_{\varphi\vartheta}$ are the analogous couples, while the last two Q_φ, Q_ϑ are the transverse shears.

In matrix notation, the relation between the generalized stress resultants per unit length and the generalized strain components takes the form:

$$\mathbf{S} = \mathbf{C}\boldsymbol{\varepsilon} \tag{2.8}$$

where

$$\mathbf{C}=\begin{bmatrix} K & \nu K & 0 & 0 & 0 & 0 & 0 & 0 \\ \nu K & K & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & K\frac{1-\nu}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & D & \nu D & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu D & D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D\frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K\frac{1-\nu}{2\chi} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & K\frac{1-\nu}{2\chi} \end{bmatrix} \tag{2.9}$$

is the constitutive operator, also called matrix of the material rigidity and

$$\mathbf{S}(\alpha_\varphi,\alpha_\vartheta,t)=\begin{bmatrix} N_\varphi & N_\vartheta & N_{\varphi\vartheta} & M_\varphi & M_\vartheta & M_{\varphi\vartheta} & Q_\varphi & Q_\vartheta \end{bmatrix}^T \tag{2.10}$$

is the vector of internal stress resultants also termed internal force vector.

2.3 Equations of Motion in terms of internal actions

Following the direct approach or the Hamilton's principle in dynamic version and remembering the Gauss-Codazzi relations for the shells of revolution $dR_0/d\varphi = R_\varphi \cos\varphi = R\cos\varphi$, five equations of dynamic equilibrium in terms of internal actions can be written for the shell element:

$$\begin{aligned}
&\frac{1}{R}\frac{\partial N_\varphi}{\partial \varphi}+\frac{1}{R_0}\frac{\partial N_{\varphi\vartheta}}{\partial \vartheta}+\frac{\cos\varphi}{R_0}\left(N_\varphi-N_\vartheta\right)+\frac{Q_\varphi}{R}+q_\varphi=I_0\ddot{u}_\varphi+I_1\ddot{\beta}_\varphi\\
&\frac{1}{R}\frac{\partial N_{\varphi\vartheta}}{\partial \varphi}+\frac{1}{R_0}\frac{\partial N_\vartheta}{\partial \vartheta}+2\frac{\cos\varphi}{R_0}N_{\varphi\vartheta}+\frac{\sin\varphi}{R_0}Q_\vartheta+q_\vartheta=I_0\ddot{u}_\vartheta+I_1\ddot{\beta}_\vartheta\\
&\frac{1}{R}\frac{\partial Q_\varphi}{\partial \varphi}+\frac{1}{R_0}\frac{\partial Q_\vartheta}{\partial \vartheta}+\frac{\cos\varphi}{R_0}Q_\varphi-\frac{N_\varphi}{R}-\frac{\sin\varphi}{R_0}N_\vartheta+q_n=I_0\ddot{w}\\
&\frac{1}{R}\frac{\partial M_\varphi}{\partial \varphi}+\frac{1}{R_0}\frac{\partial M_{\varphi\vartheta}}{\partial \vartheta}+\frac{\cos\varphi}{R_0}\left(M_\varphi-M_\vartheta\right)-Q_\varphi+m_\varphi=I_1\ddot{u}_\varphi+I_2\ddot{\beta}_\varphi\\
&\frac{1}{R}\frac{\partial M_{\varphi\vartheta}}{\partial \varphi}+\frac{1}{R_0}\frac{\partial M_\vartheta}{\partial \vartheta}+2\frac{\cos\varphi}{R_0}M_{\varphi\vartheta}-Q_\vartheta+m_\vartheta=I_1\ddot{u}_\vartheta+I_2\ddot{\beta}_\vartheta
\end{aligned} \tag{2.11}$$

where

$$I_0=\mu h\left(1+\frac{h^2}{12R^2}\right),\quad I_1=\frac{\mu h^3}{6R},\quad I_2=\mu h^3\left(\frac{1}{12}+\frac{h^2}{80R^2}\right) \tag{2.12}$$

are the mass inertias and μ is the mass density of the material. The first three equations (2.11) represent translational equilibriums along meridional, circumferential and normal directions, while the last two are rotational equilibrium equations about the φ and ϑ directions. Positive sign conventions for external loads per unit area as well as for stress resultants and couples are illustrated in Figure 3.

Equations of motion or dynamic equilibrium equations (2.11) can be written in the operatorial form:

$$\mathbf{D}^*\mathbf{S}=\mathbf{q}-\frac{\partial \mathbf{\Lambda}}{\partial t} \quad \text{or} \quad \mathbf{D}^*\mathbf{S}=\mathbf{f} \tag{2.13}$$

where

$$\mathbf{q}(\alpha_\varphi,\alpha_\vartheta,t)=\begin{bmatrix} q_\varphi & q_\vartheta & q_n & m_\varphi & m_\vartheta \end{bmatrix}^T \tag{2.14}$$

$$\Lambda(\alpha_\varphi,\alpha_\vartheta,t)=\mathbf{M}\dot{\mathbf{u}} \tag{2.15}$$

is the distributed external load and the momentum vectors, respectively, and

$$\mathbf{M}=\begin{bmatrix} I_0 & 0 & 0 & I_1 & 0 \\ 0 & I_0 & 0 & 0 & I_1 \\ 0 & 0 & I_0 & 0 & 0 \\ I_1 & 0 & 0 & I_2 & 0 \\ 0 & I_1 & 0 & 0 & I_2 \end{bmatrix} \tag{2.16}$$

is the mass matrix, while

$$\dot{\mathbf{u}}(\alpha_\varphi,\alpha_\vartheta,t)=\frac{\partial}{\partial t}\begin{bmatrix} u_\varphi & u_\vartheta & w & \beta_\varphi & \beta_\vartheta \end{bmatrix}^T \tag{2.17}$$

is the derivative of the displacement vector with respect to the variable t, that is the vector velocity.

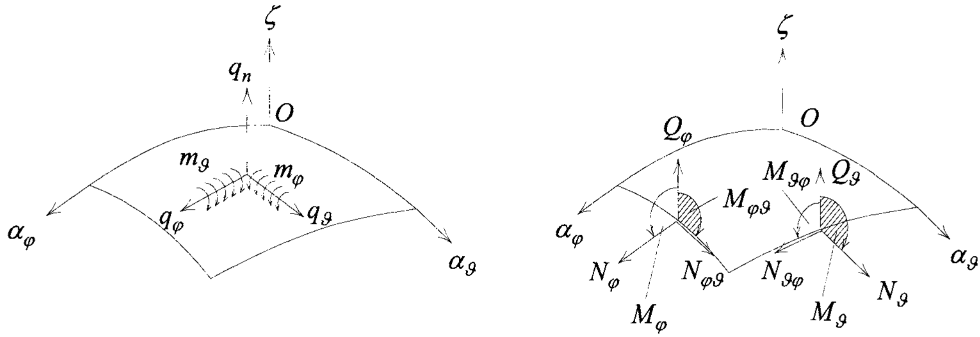

Figure 3. Distributed external loads and generalized stress resultants.

The balance operator, also known as the equilibrium operator, assumes the aspect:

$$\mathbf{D}^*=-\begin{bmatrix} \frac{\cos\varphi}{R_0}+\frac{1}{R}\frac{\partial}{\partial\varphi} & -\frac{\cos\varphi}{R_0} & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} & 0 & 0 & 0 & \frac{1}{R} & 0 \\ 0 & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} & 2\frac{\cos\varphi}{R_0}+\frac{1}{R}\frac{\partial}{\partial\varphi} & 0 & 0 & 0 & 0 & \frac{\sin\varphi}{R_0} \\ -\frac{1}{R} & -\frac{\sin\varphi}{R_0} & 0 & 0 & 0 & 0 & \frac{\cos\varphi}{R_0}+\frac{1}{R}\frac{\partial}{\partial\varphi} & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} \\ 0 & 0 & 0 & \frac{\cos\varphi}{R_0}+\frac{1}{R}\frac{\partial}{\partial\varphi} & -\frac{\cos\varphi}{R_0} & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} & -1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{R_0}\frac{\partial}{\partial\vartheta} & 2\frac{\cos\varphi}{R_0}+\frac{1}{R}\frac{\partial}{\partial\varphi} & 0 & -1 \end{bmatrix} \tag{2.18}$$

2.4 Fundamental Equations

The three basic sets of equations, namely the kinematic, the equilibrium and the constitutive equations may be combined to give the fundamental system of equations, also known as the governing system equations. Firstly, the fundamental equations are deducted in the matrix notation. So, if the strain-displacement relations (2.3) are inserted into the constitutive equations (2.8), we have the relationships between stress resultants and the generalized displacement components:

$$\mathbf{S} = \mathbf{C}\boldsymbol{\varepsilon} = \mathbf{C}\mathbf{D}\mathbf{u} \tag{2.19}$$

When the equations (2.19) are inserted into the equations of motion (2.13), the fundamental system of equations is derived:

$$\mathbf{D}^*\mathbf{C}\mathbf{D}\mathbf{u} = \mathbf{q} - \frac{\partial \boldsymbol{\Lambda}}{\partial t} \quad \text{or} \quad \mathbf{D}^*\mathbf{C}\mathbf{D}\mathbf{u} = \mathbf{f} \tag{2.20}$$

The equations of motion in terms of displacements take all the three aspects of the problem of the elastic equilibrium into account.

By introducing the fundamental operator, also known as the elasticity operator,

$$\mathbf{L} = \mathbf{D}^*\mathbf{C}\mathbf{D}\mathbf{u} \tag{2.21}$$

equation (2.20) can be written as:

$$\mathbf{L}\mathbf{u} = \mathbf{q} - \frac{\partial \boldsymbol{\Lambda}}{\partial t} \quad \text{or} \quad \mathbf{L}\mathbf{u} = \mathbf{f} \tag{2.22}$$

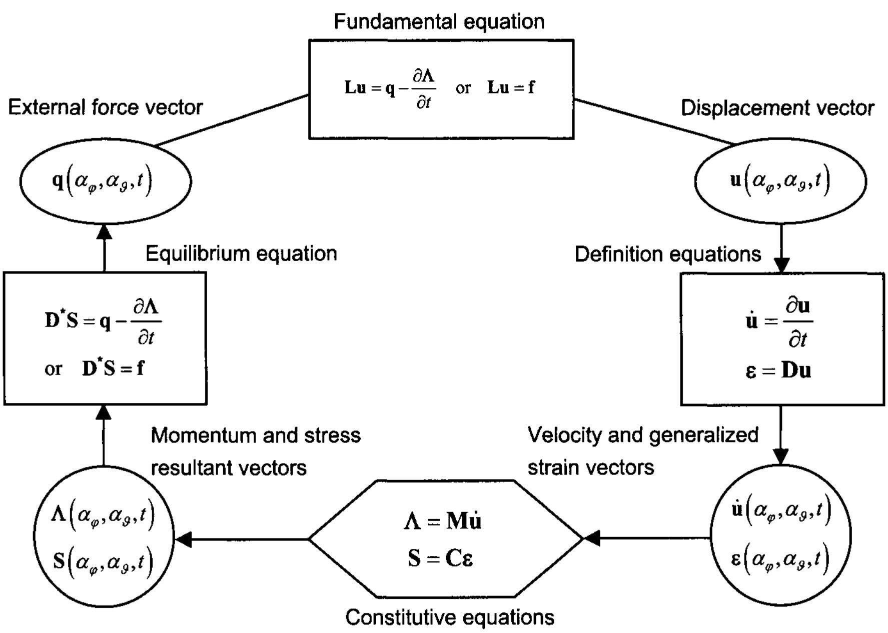

Figure 4. The scheme of the physical theories or the Tonti's diagram.

The fundamental system of equations (2.22) relates the configuration variable $\mathbf{u}$ to the source variable $\mathbf{q}$ of the phenomenon under investigation.

We can summarise all these aspects of each problem of elastic problem of equilibrium into the scheme of the physical theories or Tonti's diagram, which assumes the aspect reported in Figure 4.

Substituting the definition equations (2.2) into the constitutive equations (2.7) and the result of this substitution into the equilibrium equations (2.11), the complete equations of motion in terms of displacements can be written in the extended form as:

$$
\begin{aligned}
&\frac{K}{R^2}\frac{\partial^2 u_\varphi}{\partial \varphi^2}+\frac{(1-\nu)}{2}\frac{K}{R_0^2}\frac{\partial^2 u_\varphi}{\partial \vartheta^2}+\frac{(1+\nu)}{2}\frac{K}{RR_0}\frac{\partial^2 u_\vartheta}{\partial\varphi\partial\vartheta}+K\frac{\cos\varphi}{RR_0}\frac{\partial u_\varphi}{\partial\varphi}+\\
&+\frac{K}{R}\left[\frac{1}{R}\left(1+\frac{(1-\nu)}{2\chi}\right)+\frac{\nu\sin\varphi}{R_0}\right]\frac{\partial w}{\partial\varphi}-K\frac{(3-\nu)}{2}\frac{\cos\varphi}{R_0^2}\frac{\partial u_\vartheta}{\partial\vartheta}+\\
&-K\left[\frac{1}{R_0}\left(\frac{\nu\sin\varphi}{R}+\frac{\cos^2\varphi}{R_0}\right)+\frac{1}{R^2}\frac{(1-\nu)}{2\chi}\right]u_\varphi+K\frac{\cos\varphi}{R_0}\left[\frac{1}{R}-\frac{\sin\varphi}{R_0}\right]w+\\
&+\frac{(1-\nu)}{2\chi}\frac{K}{R}\beta_\varphi+q_\varphi=\mu h\left(1+\frac{h^2}{12R^2}\right)\ddot{u}_\varphi+\frac{\mu h^3}{6R}\ddot{\beta}_\varphi
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
&\frac{(1-\nu)}{2}\frac{K}{R^2}\frac{\partial^2 u_\vartheta}{\partial\varphi^2}+\frac{K}{R_0^2}\frac{\partial^2 u_\vartheta}{\partial\vartheta^2}+\frac{(1+\nu)}{2}\frac{K}{RR_0}\frac{\partial^2 u_\varphi}{\partial\varphi\partial\vartheta}+K\frac{(1-\nu)}{2}\frac{\cos\varphi}{RR_0}\frac{\partial u_\vartheta}{\partial\varphi}+\\
&+K\frac{(3-\nu)}{2}\frac{\cos\varphi}{R_0^2}\frac{\partial u_\varphi}{\partial\vartheta}+\frac{K}{R_0}\left[\frac{\nu}{R}+\frac{\sin\varphi}{R_0}\left(1+\frac{(1-\nu)}{2\chi}\right)\right]\frac{\partial w}{\partial\vartheta}+\\
&+\frac{(1-\nu)}{2}\frac{K}{R_0}\left[\frac{\sin\varphi}{R}-\frac{1}{R_0}\left(\cos^2\varphi+\frac{\sin^2\varphi}{\chi}\right)\right]u_\vartheta+\\
&+K\frac{(1-\nu)}{2\chi}\frac{\sin\varphi}{R_0}\beta_\vartheta+q_\vartheta=\mu h\left(1+\frac{h^2}{12R^2}\right)\ddot{u}_\vartheta+\frac{\mu h^3}{6R}\ddot{\beta}_\vartheta
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
&\frac{(1-\nu)}{2\chi}\frac{K}{R^2}\frac{\partial^2 w}{\partial\varphi^2}+\frac{(1-\nu)}{2\chi}\frac{K}{R_0^2}\frac{\partial^2 w}{\partial\vartheta^2}-\frac{K}{R}\left[\frac{1}{R}\left(1+\frac{(1-\nu)}{2\chi}\right)+\frac{\nu\sin\varphi}{R_0}\right]\frac{\partial u_\varphi}{\partial\varphi}+\\
&+K\frac{(1-\nu)}{2\chi}\frac{\cos\varphi}{RR_0}\frac{\partial w}{\partial\varphi}+\frac{(1-\nu)}{2\chi}\frac{K}{R}\frac{\partial\beta_\varphi}{\partial\varphi}-\frac{K}{R_0}\left[\frac{\nu}{R}+\frac{\sin\varphi}{R_0}\left(1+\frac{(1-\nu)}{2\chi}\right)\right]\frac{\partial u_\vartheta}{\partial\vartheta}+\\
&+\frac{(1-\nu)}{2\chi}\frac{K}{R_0}\frac{\partial\beta_\vartheta}{\partial\vartheta}-K\frac{\cos\varphi}{R_0}\left[\frac{\sin\varphi}{R_0}+\frac{1}{R}\left(\nu+\frac{(1-\nu)}{2\chi}\right)\right]u_\varphi+\\
&-K\left[\frac{1}{R}\left(\frac{1}{R}+\frac{2\nu\sin\varphi}{R_0}\right)+\frac{\sin^2\varphi}{R_0^2}\right]w+K\frac{(1-\nu)}{2\chi}\frac{\cos\varphi}{R_0}\beta_\varphi+q_n=\mu h\left(1+\frac{h^2}{12R^2}\right)\ddot{w}
\end{aligned}
\tag{2.25}
$$

$$\begin{gathered}\frac{D}{R^2}\frac{\partial^2\beta_\varphi}{\partial\varphi^2}+\frac{(1-\nu)}{2}\frac{D}{R_0^2}\frac{\partial^2\beta_\varphi}{\partial\vartheta^2}+\frac{(1+\nu)}{2}\frac{D}{RR_0}\frac{\partial^2\beta_\vartheta}{\partial\varphi\partial\vartheta}+\\-\frac{(1-\nu)}{2\chi}\frac{K}{R}\frac{\partial w}{\partial\varphi}+D\frac{\cos\varphi}{RR_0}\frac{\partial\beta_\varphi}{\partial\varphi}-D\frac{(3-\nu)}{2}\frac{\cos\varphi}{R_0^2}\frac{\partial\beta_\vartheta}{\partial\vartheta}+\\+\frac{(1-\nu)}{2\chi}\frac{K}{R}u_\varphi-\left[\frac{D}{R_0}\left(\frac{\nu\sin\varphi}{R}+\frac{\cos^2\varphi}{R_0}\right)+K\frac{(1-\nu)}{2\chi}\right]\beta_\varphi+\\+m_\varphi=\frac{\mu h^3}{6R}\ddot{u}_\varphi+\mu h^3\left(\frac{1}{12}+\frac{h^2}{80R^2}\right)\ddot{\beta}_\varphi\end{gathered}\tag{2.26}$$

$$\begin{gathered}\frac{(1-\nu)}{2}\frac{D}{R^2}\frac{\partial^2\beta_\vartheta}{\partial\varphi^2}+\frac{D}{R_0^2}\frac{\partial^2\beta_\vartheta}{\partial\vartheta^2}+\frac{(1+\nu)}{2}\frac{D}{RR_0}\frac{\partial^2\beta_\varphi}{\partial\varphi\partial\vartheta}+\\+D\frac{(1-\nu)}{2}\frac{\cos\varphi}{RR_0}\frac{\partial\beta_\vartheta}{\partial\varphi}-\frac{(1-\nu)}{2\chi}\frac{K}{R_0}\frac{\partial w}{\partial\vartheta}+D\frac{(3-\nu)}{2}\frac{\cos\varphi}{R_0^2}\frac{\partial\beta_\varphi}{\partial\vartheta}+\\+K\frac{(1-\nu)}{2\chi}\frac{\sin\varphi}{R_0}u_\vartheta+\left[\frac{(1-\nu)}{2}\frac{D}{R_0}\left(\frac{\sin\varphi}{R}-\frac{\cos^2\varphi}{R_0}\right)-K\frac{(1-\nu)}{2\chi}\right]\beta_\vartheta+\\+m_\vartheta=\frac{\mu h^3}{6R}\ddot{u}_\vartheta+\mu h^3\left(\frac{1}{12}+\frac{h^2}{80R^2}\right)\ddot{\beta}_\vartheta\end{gathered}\tag{2.27}$$

2.5 Boundary and Compatibility Conditions

In the following, three kinds of boundary conditions are considered, namely the fully clamped edge boundary conditions (C), the simply supported edge boundary conditions (S) and the free edge boundary conditions (F). The equations describing the boundary conditions can be written as follows:

Clamped edge boundary condition (C):

$$u_\varphi=u_\vartheta=w=\beta_\varphi=\beta_\vartheta=0 \quad \text{at } \varphi=\varphi_0 \text{ or } \varphi=\varphi_1,\ 0\le\vartheta\le\vartheta_0 \tag{2.28}$$

$$u_\varphi=u_\vartheta=w=\beta_\varphi=\beta_\vartheta=0 \quad \text{at } \vartheta=0 \text{ or } \vartheta=\vartheta_0,\ \varphi_0\le\varphi\le\varphi_1 \tag{2.29}$$

Simply supported edge boundary condition (S):

$$u_\varphi=u_\vartheta=w=\beta_\vartheta=0,\ M_\varphi=0 \quad \text{at } \varphi=\varphi_0 \text{ or } \varphi=\varphi_1,\ 0\le\vartheta\le\vartheta_0 \tag{2.30}$$

$$u_\varphi=u_\vartheta=w=\beta_\varphi=0,\ M_\vartheta=0 \quad \text{at } \vartheta=0 \text{ or } \vartheta=\vartheta_0,\ \varphi_0\le\varphi\le\varphi_1 \tag{2.31}$$

Free edge boundary condition (F):

$$N_\varphi=N_{\varphi\vartheta}=Q_\varphi=M_\varphi=M_{\varphi\vartheta}=0 \quad \text{at } \varphi=\varphi_0 \text{ or } \varphi=\varphi_1,\ 0\le\vartheta\le\vartheta_0 \tag{2.32}$$

$$N_\vartheta=N_{\varphi\vartheta}=Q_\vartheta=M_\vartheta=M_{\varphi\vartheta}=0 \quad \text{at } \vartheta=0 \text{ or } \vartheta=\vartheta_0,\ \varphi_0\le\varphi\le\varphi_1 \tag{2.33}$$

In addition to the external boundary conditions, the *kinematical* and *physical compatibility* should be satisfied at the common meridian with $\vartheta=0,2\pi$, if we want to consider a complete hemispherical dome of revolution. The kinematical compatibility conditions include the continuity of displacements. The physical compatibility conditions can only be the five continuous conditions for the generalized stress resultants. To consider a complete revolute hemispherical dome characterized by $\vartheta_0 = 2\pi$, it is necessary to implement the kinematical and physical compatibility conditions between the meridians with $\vartheta = 0$ and with $\vartheta_0 = 2\pi$:

Kinematical compatibility conditions:

$$\begin{aligned} &u_\varphi(\varphi,0,t) = u_\varphi(\varphi,2\pi,t), u_\vartheta(\varphi,0,t) = u_\vartheta(\varphi,2\pi,t), w(\varphi,0,t) = w(\varphi,2\pi,t), \\ &\beta_\varphi(\varphi,0,t) = \beta_\varphi(\varphi,2\pi,t), \beta_\vartheta(\varphi,0,t) = \beta_\vartheta(\varphi,2\pi,t) \end{aligned} \qquad \varphi_0 \le \varphi \le \varphi_1 \tag{2.34}$$

Physical compatibility conditions:

$$\begin{aligned} &N_\vartheta(\varphi,0,t) = N_\vartheta(\varphi,2\pi,t), N_{\varphi\vartheta}(\varphi,0,t) = N_{\varphi\vartheta}(\varphi,2\pi,t), Q_\vartheta(\varphi,0,t) = Q_\vartheta(\varphi,2\pi,t), \\ &M_\vartheta(\varphi,0,t) = M_\vartheta(\varphi,2\pi,t), M_{\varphi\vartheta}(\varphi,0,t) = M_{\varphi\vartheta}(\varphi,2\pi,t) \end{aligned} \qquad \varphi_0 \le \varphi \le \varphi_1 \tag{2.35}$$

3 Generalized Differential Quadrature Method

The GDQ method will be used to discretize the derivatives in the governing equations and the boundary conditions. The GDQ approach was developed by Shu and Richards (1992) to improve the Differential Quadrature technique for the computation of weighting coefficients, entering into the linear algebraic system of equations obtained from the discretization of the differential equation system, which can model the physical problem considered. The essence of the differential quadrature method is that the partial derivative of a smooth function with respect to a variable is approximated by a weighted sum of function values at all discrete points in that direction. Its weighting coefficients are not related to any special problem and only depend on the grid points and the derivative order. In this methodology, an arbitrary grid distribution can be chosen without any limitation.

The GDQ method is based on the analysis of a high-order polynomial approximation and the analysis of a linear vector space. For a general problem, it may not be possible to express the solution of the corresponding partial differential equation in a closed form. This solution function can be approximated by the two following types of function approximation: high-order polynomial approximation and Fourier series expansion (harmonic functions). It is well known that a smooth function in a domain can be accurately approximated by a high-order polynomial in accordance with the Weierstrass polynomial approximation theorem. In fact, from the Weierstrass theorem, if $f(x)$ is a real valued continuous function defined in the closed interval $[a,b]$, then there exists a sequence of polynomials $P_r(x)$ which converges to $f(x)$ uniformly as r goes to infinity. In practical applications, a truncated finite polynomial may be used. Thus, if $f(x)$ represents the solution of a partial differential equation, then it can be approximated by a polynomial of a degree less than or equal to $N-1$, for N large enough. The conventional form of this approximation is:

$$f(x) \cong P_N(x) = \sum_{j=1}^{N} d_j p_j(x) \tag{3.1}$$

where d_j is a constant. Then it is easy to show that the polynomial $P_N(x)$ constitutes an N-dimensional linear vector space V_N with respect to the operation of vector addition and scalar multiplication. Obviously, in the linear vector space V_N, $p_j(x)$ is a set of base vectors. It can be seen that, in the linear polynomial vector space, there exist several sets of base polynomials and each set of base polynomials can be expressed uniquely by another set of base polynomials in the space. Using vector space analysis, the method for computing the weighting coefficients can be generalized by a proper choice of base polynomials in a linear vector space. For generality, the Lagrange interpolation polynomials are chosen as the base polynomials. As a result, the weighting coefficients of the first order derivative are computed by a simple algebraic formulation without any restriction on the choice of the grid points, while the weighting coefficients of the second and higher order derivatives are given by a recurrence relationship.

When the Lagrange interpolated polynomials are assumed as a set of vector space base functions, the approximation of the function $f(x)$ can be written as:

$$f(x) \cong \sum_{j=1}^{N} p_j(x) f(x_j) \tag{3.2}$$

where N is the number of grid points in the whole domain, x_j, $j = 1,2,...,N$, are the co-ordinates of grid points in the variable domain and $f(x_j)$ are the function values at the grid points. $p_j(x)$ are the Lagrange interpolated polynomials, which can be defined by the following formula:

$$p_j(x) = \frac{\mathscr{L}(x)}{(x - x_j)\mathscr{L}^{(1)}(x_j)}, \quad j = 1,2,...,N \tag{3.3}$$

where:

$$\mathscr{L}(x) = \prod_{i=1}^{N}(x - x_i), \quad \mathscr{L}^{(1)}(x_j) = \prod_{i=1, i\neq j}^{N}(x_j - x_i) \tag{3.4}$$

Differentiating equation (3.2) with respect to x and evaluating the first derivative at a certain point of the function domain, it is possible to obtain:

$$f^{(1)}(x_i) \cong \sum_{j=1}^{N} p_j^{(1)}(x_i) f(x_j) = \sum_{j=1}^{N} \varsigma_{ij}^{(1)} f(x_j), \quad i = 1,2,...,N \tag{3.5}$$

where $\varsigma_{ij}^{(1)}$ are the GDQ weighting coefficients of the first order derivative and x_i denote the co-ordinates of the grid points. In particular, it is worth noting that the weighting coefficients of the first order derivative can be computed as:

$$p_j^{(1)}(x_i) = \varsigma_{ij}^{(1)} = \frac{\mathscr{L}^{(1)}(x_i)}{(x_i - x_j)\mathscr{L}^{(1)}(x_j)}, \quad i,j = 1,2,...,N, \quad i \neq j \tag{3.6}$$

From equation (3.6), $\varsigma_{ij}^{(1)}$ $(i \neq j)$ can be easily computed. However, the calculation of $\varsigma_{ii}^{(1)}$ is not easy to compute. According to the analysis of a linear vector space, one set of base functions can be expressed uniquely by a linear sum of another set of base functions. Thus, if one set of base polynomials satisfy a linear equation like (3.5), so does another set of base polynomials. As a consequence, the equation system for determining $\varsigma_{ij}^{(1)}$ and derived from the Lagrange interpolation polynomials should be equivalent to that derived from another set of base polynomials $p_j(x) = x^{j-1}$, $j = 1,2,...,N$. Thus, $\varsigma_{ij}^{(1)}$ satisfies the following equation, which is obtained by the base polynomials $p_j(x) = x^{j-1}$, when $j = 1$:

$$\sum_{j=1}^{N} \varsigma_{ij}^{(1)} = 0 \Rightarrow \varsigma_{ii}^{(1)} = -\sum_{j=1, j\neq i}^{N} \varsigma_{ij}^{(1)}, \quad i,j = 1,2,...,N \tag{3.7}$$

Equations (3.6) and (3.7) are two formulations to compute the weighting coefficients $\varsigma_{ij}^{(1)}$. It should be noted that, in the development of these two formulations, two sets of base polynomials were used in the linear polynomial vector space V_N. Finally, the n^{th} order derivative of function $f(x)$ with respect to x at grid points x_i, can be approximated by the GDQ approach:

$$\left.\frac{d^n f(x)}{dx^n}\right|_{x=x_i} = \sum_{j=1}^{N} \varsigma_{ij}^{(n)} f(x_j), \quad i=1,2,...,N \tag{3.8}$$

where $\varsigma_{ij}^{(n)}$ are the weighting coefficients of the n^{th} order derivative. Similar to the first order derivative and according to the polynomial approximation and the analysis of a linear vector space, it is possible to determine a recurrence relationship to compute the second and higher order derivatives. Thus, the weighting coefficients can be generated by the following recurrent formulation:

$$\varsigma_{ij}^{(n)} = n\left(\varsigma_{ii}^{(n-1)}\varsigma_{ij}^{(1)} - \frac{\varsigma_{ij}^{(n-1)}}{x_i - x_j}\right), \quad i \neq j,\ n=2,3,...,N-1,\ i,j=1,2,...,N \tag{3.9}$$

$$\sum_{j=1}^{N} \varsigma_{ij}^{(n)} = 0 \Rightarrow \varsigma_{ii}^{(n)} = -\sum_{j=1, j\neq i}^{N} \varsigma_{ij}^{(n)}, \quad n=2,3,...,N-1,\ i,j=1,2,...,N \tag{3.10}$$

It is obvious from the above equations that the weighting coefficients of the second and higher order derivatives can be determined from those of the first order derivative. Furthermore, it is interesting to note that, the preceding coefficients $\varsigma_{ij}^{(n)}$ are dependent on the derivative order *n*, on the grid point distribution $x_j, j=1,2,...,N$, and on the specific point x_i, where the derivative is computed. There is no need to obtain the weighting coefficients from a set of algebraic equations which could be ill-conditioned when the number of grid points is large. Furthermore, this set of expressions for the determination of the weighting coefficients is so compact and simple that it is very easy to implement them in formulating and programming, because of the recurrence feature.

Another important point for successful application of the GDQ method is how to distribute the grid points. In fact, the accuracy of this method is usually sensitive to the grid point distribution. The optimal grid point distribution depends on the order of derivatives in the boundary condition and the number of grid points used. The grid point distribution also plays an essential role in determining the convergence speed and stability of the GDQ method. It is demonstrated that non-uniform grid distribution usually yields better results than equally spaced distribution. Quan and Chang (1989) compared numerically the performances of the often-used non-uniform meshes and concluded that the grid points originating from the Chebyshev polynomials of the first kind are optimum in all the cases examined there. The zeros of orthogonal polynomials are the rational basis for the grid points. Shu and Richards (1992) have used other choice which has given better results than the zeros of Chebyshev and Legendre polynomials. Bert and Malik (1996) indicated that the preferred type of grid points changes with problems of interest and recommended the use of Chebyshev-Gauss-Lobatto grid for the structural mechanics computation. With Lagrange interpolating polynomials, the Chebyshev-Gauss-Lobatto sampling point rule proves efficient for numerical reasons [Shu, Chen, Xue and Du (2001)] so that for such a collocation the approximation error of the dependent variables decreases as the number of nodes increases. For the numerical computations presented in this paper, the co-ordinates of grid points (φ_i, ϑ_j) are chosen as:

Chebyshev-Gauss-Lobatto sampling points (C-G-L)

$$\varphi_i = \frac{1-\cos\left(\frac{i-1}{N-1}\pi\right)}{2}(\varphi_1-\varphi_0)+\varphi_0, \quad i=1,2,...,N, \quad \text{for } \varphi\in[\varphi_0,\varphi_1] \ (\text{with } \varphi_0>0 \text{ and } \varphi_1\le 90°)$$
$$\vartheta_j = \frac{1-\cos\left(\frac{j-1}{M-1}\pi\right)}{2}\vartheta_0, \quad j=1,2,...,M, \quad \text{for } \vartheta\in[0,\vartheta_0] \ (\text{with } \vartheta_0\le 2\pi) \tag{3.11}$$

where N, M are the total number of sampling points used to discretize a domain in φ and ϑ directions, respectively, of the hemispherical shell.

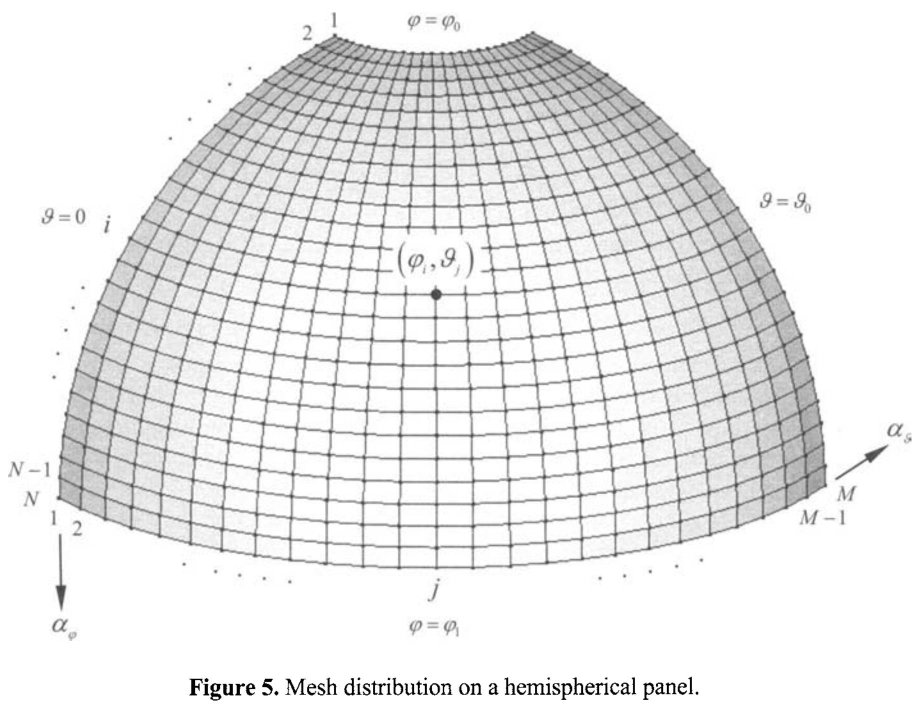

Figure 5. Mesh distribution on a hemispherical panel.

4 Numerical Implementation

A novel approach in numerically solving the governing equations (2.23), (2.24), (2.25), (2.26), and (2.27) is represented by the Generalized Differential Quadrature (GDQ) method. This method, for the problem studied herein, demonstrates its numerical accuracy and extreme coding simplicity.

In the following, only the free vibration of hemispherical dome or panel will be studied. So, setting $\mathbf{q}(\alpha_\varphi,\alpha_\vartheta,t)=\mathbf{0}$ and using the method of variable separation, it is possible to seek solutions that are harmonic in time and whose frequency is ω; then, the displacements and the rotations can be written as follows:

$$\begin{aligned}
u_\varphi(\alpha_\varphi,\alpha_\vartheta,t) &= U^\varphi(\alpha_\varphi,\alpha_\vartheta)e^{i\omega t}\\
u_\vartheta(\alpha_\varphi,\alpha_\vartheta,t) &= U^\vartheta(\alpha_\varphi,\alpha_\vartheta)e^{i\omega t}\\
w(\alpha_\varphi,\alpha_\vartheta,t) &= W^\varsigma(\alpha_\varphi,\alpha_\vartheta)e^{i\omega t}\\
\beta_\varphi(\alpha_\varphi,\alpha_\vartheta,t) &= B^\varphi(\alpha_\varphi,\alpha_\vartheta)e^{i\omega t}\\
\beta_\vartheta(\alpha_\varphi,\alpha_\vartheta,t) &= B^\vartheta(\alpha_\varphi,\alpha_\vartheta)e^{i\omega t}
\end{aligned} \tag{4.1}$$

where the vibration spatial amplitude values ($U^\varphi(\alpha_\varphi,\alpha_\vartheta)$, $U^\vartheta(\alpha_\varphi,\alpha_\vartheta)$, $W^\varsigma(\alpha_\varphi,\alpha_\vartheta)$, $B^\varphi(\alpha_\varphi,\alpha_\vartheta)$, $B^\vartheta(\alpha_\varphi,\alpha_\vartheta)$) fulfil the fundamental differential system.
The basic steps in the GDQ solution of the free vibration problem of hemispherical dome type structures are as in the following:

- Discretization of independent variables $\varphi \in]0,90°]$, $\vartheta \in [0,\vartheta_0]$ (with $\vartheta_0 \le 2\pi$).
- The spatial derivatives are approximated according to GDQ rule.
- The differential governing systems (2.23), (2.24), (2.25), (2.26), and (2.27) are transformed into linear eigenvalue problems for the natural frequencies. The boundary conditions are imposed in the sampling points corresponding to the boundary. All these relations are imposed pointwise.
- The solution of the previously stated discrete system in terms of natural frequencies and mode shape components is worked out. For each mode, local values of dependent variables are used to obtain the complete assessment of the deformed configuration.

4.1 Discretization of Motion Equations

The simple numerical operations illustrated here, applying the GDQ procedure, enable one to write the equations of motion in discrete form, transforming each space derivative into a weighted sum of node values of dependent variables. Each of the approximated equations is valid in a single sampling point. The governing equations can be discretized as:

$$\begin{aligned}
&\frac{K}{R^2}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(2)}U_{kj}^{\varphi}+\frac{(1-\nu)}{2}\frac{K}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(2)}U_{mj}^{\varphi}+\frac{(1+\nu)}{2}\frac{K}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}U_{km}^{\vartheta}\right)+\\
&+K\frac{\cos\varphi_i}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{kj}^{\varphi}+\frac{K}{R}\left[\frac{1}{R}\left(1+\frac{(1-\nu)}{2\chi}\right)+\frac{\nu\sin\varphi_i}{R_{0i}}\right]\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}W_{kj}^{\varsigma}+\\
&-K\frac{(3-\nu)}{2}\frac{\cos\varphi_i}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}U_{im}^{\vartheta}-K\left[\frac{1}{R_{0i}}\left(\frac{\nu\sin\varphi_i}{R}+\frac{\cos^2\varphi_i}{R_{0i}}\right)+\frac{1}{R^2}\frac{(1-\nu)}{2\chi}\right]U_{ij}^{\varphi}+\\
&+K\frac{\cos\varphi_i}{R_{0i}}\left[\frac{1}{R}-\frac{\sin\varphi_i}{R_{0i}}\right]W_{ij}^{\varsigma}+\frac{(1-\nu)}{2\chi}\frac{K}{R}B_{ij}^{\varphi}=-\omega^2\mu h\left(1+\frac{h^2}{12R^2}\right)U_{ij}^{\varphi}-\omega^2\frac{\mu h^3}{6R}B_{ij}^{\varphi}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\frac{(1-\nu)}{2}\frac{K}{R^2}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(2)}U_{kj}^{\vartheta}+\frac{K}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(2)}U_{im}^{\vartheta}+\frac{(1+\nu)}{2}\frac{K}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}U_{km}^{\varphi}\right)+\\&+K\frac{(1-\nu)}{2}\frac{\cos\varphi_i}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{kj}^{\vartheta}+K\frac{(3-\nu)}{2}\frac{\cos\varphi_i}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}U_{im}^{\varphi}+\\&+\frac{K}{R_{0i}}\left[\frac{\nu}{R}+\frac{\sin\varphi_i}{R_{0i}}\left(1+\frac{(1-\nu)}{2\chi}\right)\right]\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}W_{im}^{\zeta}+K\frac{(1-\nu)}{2\chi}\frac{\sin\varphi_i}{R_{0i}}B_{ij}^{\vartheta}+\\&+\frac{(1-\nu)}{2}\frac{K}{R_{0i}}\left[\frac{\sin\varphi_i}{R}-\frac{1}{R_{0i}}\left(\cos^2\varphi_i+\frac{\sin^2\varphi_i}{\chi}\right)\right]U_{ij}^{\vartheta}=-\omega^2\mu h\left(1+\frac{h^2}{12R^2}\right)U_{ij}^{\vartheta}-\omega^2\frac{\mu h^3}{6R}B_{ij}^{\vartheta}\end{aligned}\tag{4.3}$$

$$\begin{aligned}&\frac{(1-\nu)}{2\chi}\frac{K}{R^2}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(2)}W_{kj}^{\zeta}+\frac{(1-\nu)}{2\chi}\frac{K}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(2)}W_{im}^{\zeta}-\frac{K}{R}\left[\frac{1}{R}\left(1+\frac{(1-\nu)}{2\chi}\right)+\frac{\nu\sin\varphi_i}{R_{0i}}\right]\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{kj}^{\varphi}+\\&+K\frac{(1-\nu)}{2\chi}\frac{\cos\varphi_i}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}W_{kj}^{\zeta}+\frac{(1-\nu)}{2\chi}\frac{K}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}B_{kj}^{\varphi}-\frac{K}{R_{0i}}\left[\frac{\nu}{R}+\frac{\sin\varphi_i}{R_{0i}}\left(1+\frac{(1-\nu)}{2\chi}\right)\right]\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}U_{im}^{\vartheta}+\\&+\frac{(1-\nu)}{2\chi}\frac{K}{R_{0i}}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}B_{im}^{\vartheta}-K\frac{\cos\varphi_i}{R_{0i}}\left[\frac{\sin\varphi_i}{R_{0i}}+\frac{1}{R}\left(\nu+\frac{(1-\nu)}{2\chi}\right)\right]U_{ij}^{\varphi}+\\&-K\left[\frac{1}{R}\left(\frac{1}{R}+\frac{2\nu\sin\varphi_i}{R_{0i}}\right)+\frac{\sin^2\varphi_i}{R_{0i}^2}\right]W_{ij}^{\zeta}+K\frac{(1-\nu)}{2\chi}\frac{\cos\varphi_i}{R_{0i}}B_{ij}^{\varphi}=-\omega^2\mu h\left(1+\frac{h^2}{12R^2}\right)W_{ij}^{\zeta}\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\frac{D}{R^2}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(2)}B_{kj}^{\varphi}+\frac{(1-\nu)}{2}\frac{D}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(2)}B_{im}^{\varphi}+\frac{(1+\nu)}{2}\frac{D}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}B_{km}^{\vartheta}\right)+\\&-\frac{(1-\nu)}{2\chi}\frac{K}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}W_{kj}^{\zeta}+D\frac{\cos\varphi_i}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}B_{kj}^{\varphi}-D\frac{(3-\nu)}{2}\frac{\cos\varphi_i}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}B_{im}^{\vartheta}+\\&+\frac{(1-\nu)}{2\chi}\frac{K}{R}U_{ij}^{\varphi}-\left[\frac{D}{R_{0i}}\left(\frac{\nu\sin\varphi_i}{R}+\frac{\cos^2\varphi_i}{R_{0i}}\right)+K\frac{(1-\nu)}{2\chi}\right]B_{ij}^{\varphi}=\\&\qquad=-\omega^2\frac{\mu h^3}{6R}U_{ij}^{\varphi}-\omega^2\mu h^3\left(\frac{1}{12}+\frac{h^2}{80R^2}\right)B_{ij}^{\varphi}\end{aligned}\tag{4.5}$$

$$\begin{aligned}
&\frac{(1-\nu)}{2}\frac{D}{R^2}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(2)}B_{kj}^{\vartheta}+\frac{D}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(2)}B_{im}^{\vartheta}+\frac{(1+\nu)}{2}\frac{D}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}B_{km}^{\varphi}\right)+\\
&+D\frac{(1-\nu)}{2}\frac{\cos\varphi_i}{RR_{0i}}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}B_{kj}^{\vartheta}-\frac{(1-\nu)}{2\chi}\frac{K}{R_{0i}}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}W_{im}^{\varsigma}+D\frac{(3-\nu)}{2}\frac{\cos\varphi_i}{R_{0i}^2}\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}B_{im}^{\varphi}+\\
&+K\frac{(1-\nu)}{2\chi}\frac{\sin\varphi_i}{R_{0i}}U_{ij}^{\vartheta}+\left[\frac{(1-\nu)}{2}\frac{D}{R_{0i}}\left(\frac{\sin\varphi_i}{R}-\frac{\cos^2\varphi_i}{R_{0i}}\right)-K\frac{(1-\nu)}{2\chi}\right]B_{ij}^{\vartheta}=\\
&\qquad=-\omega^2\frac{\mu h^3}{6R}U_{ij}^{\vartheta}-\omega^2\mu h^3\left(\frac{1}{12}+\frac{h^2}{80R^2}\right)B_{ij}^{\vartheta}
\end{aligned}\tag{4.6}$$

where $i=2,3,...,N-1$, $j=2,3,...,M-1$ and $\varsigma_{ik}^{\varphi(1)}$, $\varsigma_{jm}^{\vartheta(1)}$, $\varsigma_{ik}^{\varphi(2)}$ and $\varsigma_{jm}^{\vartheta(2)}$ are the weighting coefficients of the first and second derivatives in φ and ϑ directions, respectively. On the other hand, N, M are the total number of grid points in φ and ϑ directions.

4.2 Implementation of Boundary and Compatibility Conditions

Applying the GDQ methodology, the discretized forms of the boundary conditions are given as follows:

Clamped edge boundary condition (C):

$$\begin{aligned}
&U_{aj}^{\varphi}=U_{aj}^{\vartheta}=W_{aj}^{\varsigma}=B_{aj}^{\varphi}=B_{aj}^{\vartheta}=0 \qquad \text{for } a=1,N \text{ and } j=1,2,...,M\\
&U_{ib}^{\varphi}=U_{ib}^{\vartheta}=W_{ib}^{\varsigma}=B_{ib}^{\varphi}=B_{ib}^{\vartheta}=0 \qquad \text{for } b=1,M \text{ and } i=1,2,...,N
\end{aligned}\tag{4.7}$$

Simply supported edge boundary condition (S):

$$\begin{aligned}
&\begin{cases}
U_{aj}^{\varphi}=U_{aj}^{\vartheta}=W_{aj}^{\varsigma}=B_{aj}^{\vartheta}=0\\
\dfrac{1}{R}\displaystyle\sum_{k=1}^{N}\varsigma_{ak}^{\varphi(1)}B_{kj}^{\varphi}+\frac{\nu}{R_{0a}}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}B_{am}^{\vartheta}+B_{aj}^{\varphi}\cos\varphi_a\right)=0
\end{cases} \qquad \text{for } a=1,N \text{ and } j=1,2,...,M\\
&\begin{cases}
U_{ib}^{\varphi}=U_{ib}^{\vartheta}=W_{ib}^{\varsigma}=B_{ib}^{\varphi}=0\\
\dfrac{1}{R_{0i}}\left(\displaystyle\sum_{m=1}^{M}\varsigma_{bm}^{\vartheta(1)}B_{im}^{\vartheta}+B_{ib}^{\varphi}\cos\varphi_i\right)+\frac{\nu}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}B_{kb}^{\varphi}=0
\end{cases} \qquad \text{for } b=1,M \text{ and } i=1,2,...,N
\end{aligned}\tag{4.8}$$

Free edge boundary condition (F):

$$\begin{cases}\frac{1}{R}\left(\sum_{k=1}^{N}\varsigma_{ak}^{\varphi(1)}U_{kj}^{\varphi}+W_{aj}^{\zeta}\right)+\frac{\nu}{R_{0a}}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}U_{am}^{\vartheta}+U_{aj}^{\varphi}\cos\varphi_a+W_{aj}^{\zeta}\sin\varphi_a\right)=0\\ \frac{1}{R}\sum_{k=1}^{N}\varsigma_{ak}^{\varphi(1)}U_{kj}^{\vartheta}+\frac{1}{R_{0a}}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}U_{am}^{\varphi}-U_{aj}^{\vartheta}\cos\varphi_a\right)=0\\ \frac{1}{R}\left(\sum_{k=1}^{N}\varsigma_{ak}^{\varphi(1)}W_{kj}^{\zeta}-U_{aj}^{\varphi}\right)+B_{aj}^{\varphi}=0 \qquad \text{for } a=1,N \text{ and } j=1,2,...,M\\ \frac{1}{R}\sum_{k=1}^{N}\varsigma_{ak}^{\varphi(1)}B_{kj}^{\varphi}+\frac{\nu}{R_{0a}}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}B_{am}^{\vartheta}+B_{aj}^{\varphi}\cos\varphi_a\right)=0\\ \frac{1}{R}\sum_{k=1}^{N}\varsigma_{ak}^{\varphi(1)}B_{kj}^{\vartheta}+\frac{1}{R_{0a}}\left(\sum_{m=1}^{M}\varsigma_{jm}^{\vartheta(1)}B_{am}^{\varphi}-B_{aj}^{\vartheta}\cos\varphi_a\right)=0\end{cases} \tag{4.9}$$

$$\begin{cases}\frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{bm}^{\vartheta(1)}U_{im}^{\vartheta}+U_{ib}^{\varphi}\cos\varphi_i+W_{ib}^{\zeta}\sin\varphi_i\right)+\frac{\nu}{R}\left(\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{kb}^{\varphi}+W_{ib}^{\zeta}\right)=0\\ \frac{1}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{kb}^{\vartheta}+\frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{bm}^{\vartheta(1)}U_{im}^{\varphi}-U_{ib}^{\vartheta}\cos\varphi_i\right)=0\\ \frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{bm}^{\vartheta(1)}W_{im}^{\zeta}-U_{ib}^{\vartheta}\sin\varphi_i\right)+B_{ib}^{\vartheta}=0 \qquad \text{for } b=1,M \text{ and } i=1,2,...,N\\ \frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{bm}^{\vartheta(1)}B_{im}^{\vartheta}+B_{ib}^{\varphi}\cos\varphi_i\right)+\frac{\nu}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}B_{kb}^{\varphi}=0\\ \frac{1}{R}\sum_{m=1}^{M}\varsigma_{bm}^{\varphi(1)}B_{im}^{\vartheta}+\frac{1}{R_{0i}}\left(\sum_{k=1}^{N}\varsigma_{ik}^{\vartheta(1)}B_{kb}^{\varphi}-B_{ib}^{\vartheta}\cos\varphi_i\right)=0\end{cases} \tag{4.10}$$

Kinematical and physical compatibility conditions:

$$U_{i1}^{\varphi}=U_{iM}^{\varphi},U_{i1}^{\vartheta}=U_{iM}^{\vartheta},W_{i1}^{\zeta}=W_{iM}^{\zeta},B_{i1}^{\varphi}=B_{iM}^{\varphi},B_{i1}^{\vartheta}=B_{iM}^{\vartheta}$$

$$\begin{cases}\frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{1m}^{\vartheta(1)}U_{im}^{\vartheta}+U_{i1}^{\varphi}\cos\varphi_i+W_{i1}^{\zeta}\sin\varphi_i\right)+\frac{\nu}{R}\left(\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{k1}^{\varphi}+W_{i1}^{\zeta}\right)=\\ \qquad =\frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{bm}^{\vartheta(1)}U_{im}^{\vartheta}+U_{iM}^{\varphi}\cos\varphi_i+W_{iM}^{\zeta}\sin\varphi_i\right)+\frac{\nu}{R}\left(\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{kM}^{\varphi}+W_{iM}^{\zeta}\right)\\ \frac{1}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{k1}^{\vartheta}+\frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{1m}^{\vartheta(1)}U_{im}^{\varphi}-U_{i1}^{\vartheta}\cos\varphi_i\right)=\frac{1}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}U_{kM}^{\vartheta}+\frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{Mm}^{\vartheta(1)}U_{im}^{\varphi}-U_{iM}^{\vartheta}\cos\varphi_i\right)\\ \frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{1m}^{\vartheta(1)}W_{im}^{\zeta}-U_{i1}^{\vartheta}\sin\varphi_i\right)+B_{i1}^{\vartheta}=\frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{Mm}^{\vartheta(1)}W_{im}^{\zeta}-U_{iM}^{\vartheta}\sin\varphi_i\right)+B_{iM}^{\vartheta}\\ \frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{1m}^{\vartheta(1)}B_{im}^{\vartheta}+B_{i1}^{\varphi}\cos\varphi_i\right)+\frac{\nu}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}B_{k1}^{\varphi}=\frac{1}{R_{0i}}\left(\sum_{m=1}^{M}\varsigma_{Mm}^{\vartheta(1)}B_{im}^{\vartheta}+B_{iM}^{\varphi}\cos\varphi_i\right)+\frac{\nu}{R}\sum_{k=1}^{N}\varsigma_{ik}^{\varphi(1)}B_{kM}^{\varphi}\\ \frac{1}{R}\sum_{m=1}^{M}\varsigma_{1m}^{\varphi(1)}B_{im}^{\vartheta}+\frac{1}{R_{0i}}\left(\sum_{k=1}^{N}\varsigma_{ik}^{\vartheta(1)}B_{k1}^{\varphi}-B_{i1}^{\vartheta}\cos\varphi_i\right)=\frac{1}{R}\sum_{m=1}^{M}\varsigma_{Mm}^{\varphi(1)}B_{im}^{\vartheta}+\frac{1}{R_{0i}}\left(\sum_{k=1}^{N}\varsigma_{ik}^{\vartheta(1)}B_{kM}^{\varphi}-B_{iM}^{\vartheta}\cos\varphi_i\right)\end{cases} \tag{4.11}$$

for $i=2,...,N-1$

4.3 Solution Procedure

Applying the differential quadrature procedure, the whole system of differential equations has been discretized and the global assembling leads to the following set of linear algebraic equations:

$$\begin{bmatrix} \mathbf{K}_{bb} & \mathbf{K}_{bd} \\ \mathbf{K}_{db} & \mathbf{K}_{dd} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_b \\ \boldsymbol{\delta}_d \end{bmatrix} = \omega^2 \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{dd} \end{bmatrix} \begin{bmatrix} \boldsymbol{\delta}_b \\ \boldsymbol{\delta}_d \end{bmatrix} \tag{4.12}$$

In the above matrices and vectors, the partitioning is set forth by subscripts b and d, referring to the system degrees of freedom and standing for boundary and domain, respectively. In order to make the computation more efficient, kinematic condensation of non-domain degrees of freedom is performed:

$$\left(\mathbf{K}_{dd} - \mathbf{K}_{db}\left(\mathbf{K}_{bb}\right)^{-1}\mathbf{K}_{bd}\right)\boldsymbol{\delta}_d = \omega^2\mathbf{M}_{dd}\boldsymbol{\delta}_d \tag{4.13}$$

The natural frequencies of the structure considered can be determined by making the following determinant vanish:

$$\left|\left(\mathbf{K}_{dd} - \mathbf{K}_{db}\left(\mathbf{K}_{bb}\right)^{-1}\mathbf{K}_{bd}\right) - \omega^2\mathbf{M}_{dd}\right| = 0 \tag{4.14}$$

5 Applications and Results

Based on the above derivations, in the present paragraph some results and considerations about the free vibration problem of a hemispherical panel and a hemispherical dome are presented. The analysis has been carried out by means of numerical procedures illustrated above. The mechanical characteristics for the considered structures are listed in Table 1. In order to verify the accuracy of the present method, some comparisons have also been performed. The first ten natural frequencies of a hemispherical panel and a hemispherical dome are reported in Tables 2, 3, 4, 5, 6 and 7. The detail regarding the geometry of the considered structures are indicated below:

- *Hemispherical panel*: $R = 1\text{m}$, $h = 0.1\text{m}$, $\varphi_0 = 15°$, $\varphi_1 = 90°$, $\vartheta_0 = 120°$ (Tables 2, 3 and 4).
- *Hemispherical dome*: $R = 1\text{m}$, $h = 0.1\text{m}$, $\varphi_0 = 6°$, $\varphi_1 = 90°$, $\vartheta_0 = 360°$ (Tables 5, 6 and 7).

The geometrical boundary conditions for the hemispherical panel are identified by the following convention. For example, the symbolism C-S-C-F indicates that the edges $\varphi = \varphi_1$, $\vartheta = 0$, $\varphi = \varphi_0$, $\vartheta = \vartheta_0$ are clamped, simply supported, clamped and free, respectively. In particular, we have considered the hemispherical panels characterized by C-F-F-F and S-F-F-F boundary conditions (Tables 2, 3 and 4). For the hemispherical dome, for example, the symbolism C-F indicates that the edges $\varphi = \varphi_1$ and $\varphi = \varphi_0$ are clamped and free, respectively. In this case, the missing boundary conditions are the kinematical and physical compatibility conditions that are applied at the same meridian for $\vartheta = 0$ and $\vartheta = 2\pi$. In this work the hemispherical dome that we have examined is characterized by C-F and S-F boundary conditions (Tables 5, 6 and 7).

One of the aims of this paper is to compare results from the present analysis with those obtained with finite element techniques and based on the same shell theory or 3D element theory. In Tables 2, 4, 5 and 7, we have compared the 2D shell theory results obtained by the GDQ Method with the FEM results obtained by some commercial programs using the same 2D shell theory. On the other hand, the FEM solutions using 3D element theory obtained with the same commercial programs are reported in Tables 3 and 6.

For the GDQM results reported in Tables 2, 4, 5 and 7, we have considered the grid distribution (3.11) with $N=21$ and $M=21$. For the commercial programs, we have used shell elements with 4 and 8 nodes in Tables 2, 4, 5 and 7. On the other hand, brick-elements with 20-nodes were used for the 3D element theory in Tables 3 and 6.

It is noteworthy that the results from the present methodology are very close to those obtained by the commercial programs. As can be seen, the numerical results show excellent agreement. Furthermore, it is significant that the computational effort in terms of time and number of grid points is smaller for the GDQ method results than for the finite element method.

In Figure 6, we have reported the first mode shapes for the hemispherical panel characterized by C-F-F-F boundary conditions, while in Figure 7 the mode shapes for the hemispherical dome characterized by C-F boundary conditions are illustrated. In particular, for the hemispherical dome there are some symmetrical mode shapes due to the symmetry of the problem considered in 3D space. In these cases, we have summarized the symmetrical mode shapes in one figure.

6 Conclusions

A Generalized Differential Quadrature Method application to free vibration analysis of spherical shells has been presented to illustrate the versatility and the accuracy of this methodology. The adopted shell theory is a first order shear deformation theory. The dynamic equilibrium equations are discretized and solved with the present method giving a standard linear eigenvalue problem. The vibration results are obtained without the modal expansion methodology. The complete 2D differential system, governing the structural problem, has been solved. Due to the theoretical framework, no approximation (δ-point technique) is needed in modelling the boundary edge conditions.

The GDQ method provides a very simple algebraic formula for determining the weighting coefficients required by the differential quadrature approximation, without restricting in any way the choice of mesh grids. Examples presented show that the generalized differential quadrature method can produce accurate results, utilizing only a small number of sampling points. Furthermore, discretizing and programming procedures are quite easy. Fast convergence and very good stability can be obtained. Furthermore, the computational effort in terms of time and number of grid points is smaller for the GDQ method results than for the finite element method.

Table 1. Physical parameters used in the analysis of free vibrations of the structures considered.

Parameter	Value
Density of mass μ	$7800\ \mathrm{kg/m^3}$
Young's modulus E	$2.1 \cdot 10^{11}$ Pa
Poisson coefficient ν	0.3
Shear factor χ	6/5

Table 2. Shell theory for the hemispherical panel C-F-F-F.

Frequencies [Hz]	GDQ Method	Ansys 8	Femap\Nastran 8.3	Straus 7	Pro\Engineer WildFire 2
f_1	195.35	195.84	196.20	195.11	195.36
f_2	196.01	196.54	196.70	195.85	196.20
f_3	433.95	436.00	435.69	433.48	433.91
f_4	526.27	530.34	531.48	528.90	526.71
f_5	766.43	772.52	771.54	767.01	765.57
f_6	854.57	860.01	860.15	856.80	856.47
f_7	935.20	947.12	943.32	936.96	935.20
f_8	1062.02	1075.60	1065.55	1068.02	1060.61
f_9	1069.86	1084.10	1080.86	1079.32	1069.56
f_{10}	1122.32	1137.80	1137.89	1130.51	1122.62

Table 3. 3D element theory for the hemispherical panel C-F-F-F.

Frequencies [Hz]	Ansys 8	Femap\Nastran 8.3	Straus 7	Pro\Engineer WildFire 2
f_1	195.80	196.52	196.56	196.37
f_2	196.56	197.32	197.40	197.00
f_3	435.02	435.56	435.69	435.77
f_4	527.79	530.55	530.67	529.59
f_5	767.37	766.86	766.93	768.48
f_6	856.50	857.32	857.38	857.35
f_7	937.52	937.12	937.68	940.26
f_8	1060.70	1052.11	1052.27	1063.47
f_9	1069.90	1068.47	1068.61	1072.50
f_{10}	1125.60	1130.15	1130.64	1130.61

Table 4. Shell theory for the hemispherical panel S-F-F-F.

Frequencies [Hz]	f_1	f_2	f_3	f_4	f_5
GDQ Method	159.29	173.05	392.09	453.32	731.42
Ansys 8	158.94	171.90	393.68	454.04	738.00
Femap\Nastran 8.3	159.61	172.40	393.77	456.50	736.96
Straus 7	158.02	171.34	391.20	453.79	733.09
Pro\Engineer WF 2	158.63	171.59	392.07	452.04	730.62

Table 5. Shell theory for the hemispherical dome C-F.

Frequencies [Hz]	GDQ Method	Ansys 8	Femap\Nastran 8.3	Straus 7	Pro\Engineer WildFire 2
f_1	508.93	509.19	509.79	513.14	509.09
f_2	508.93	509.19	509.79	513.14	509.09
f_3	704.71	705.22	705.71	712.14	703.95
f_4	764.88	766.09	767.91	771.44	763.94
f_5	764.88	766.09	767.91	771.44	763.95
f_6	872.77	875.50	876.63	888.37	871.74
f_7	872.77	875.53	876.64	888.37	871.76
f_8	932.50	936.54	931.95	942.37	931.43
f_9	932.50	936.55	931.95	942.37	931.50
f_{10}	1018.30	1024.40	1021.78	1026.28	1016.95

Table 6. 3D element theory for the hemispherical dome C-F.

Frequencies [Hz]	Ansys 8	Femap\Nastran 8.3	Straus 7	Pro\Engineer WildFire 2
f_1	510.09	510.01	509.95	509.32
f_2	510.29	510.05	509.99	509.48
f_3	706.00	705.76	705.46	704.88
f_4	764.35	765.28	764.91	764.23
f_5	764.38	765.39	765.02	764.39
f_6	873.50	873.61	873.12	872.19
f_7	874.10	874.00	873.47	872.74
f_8	932.20	932.11	931.23	931.46
f_9	932.27	932.16	931.30	931.94
f_{10}	1021.10	1020.36	1018.72	1019.64

Table 7. Shell theory for the hemispherical dome S-F.

Frequencies [Hz]	f_1	f_2	f_3	f_4	f_5
GDQ Method	478.49	478.49	666.82	750.93	750.93
Ansys 8	478.71	478.71	667.35	751.65	751.66
Femap\Nastran 8.3	479.59	479.59	668.09	753.39	753.39
Straus 7	479.80	479.80	669.81	755.08	755.08
Pro\Engineer WF 2	478.57	478.57	666.14	749.78	749.78

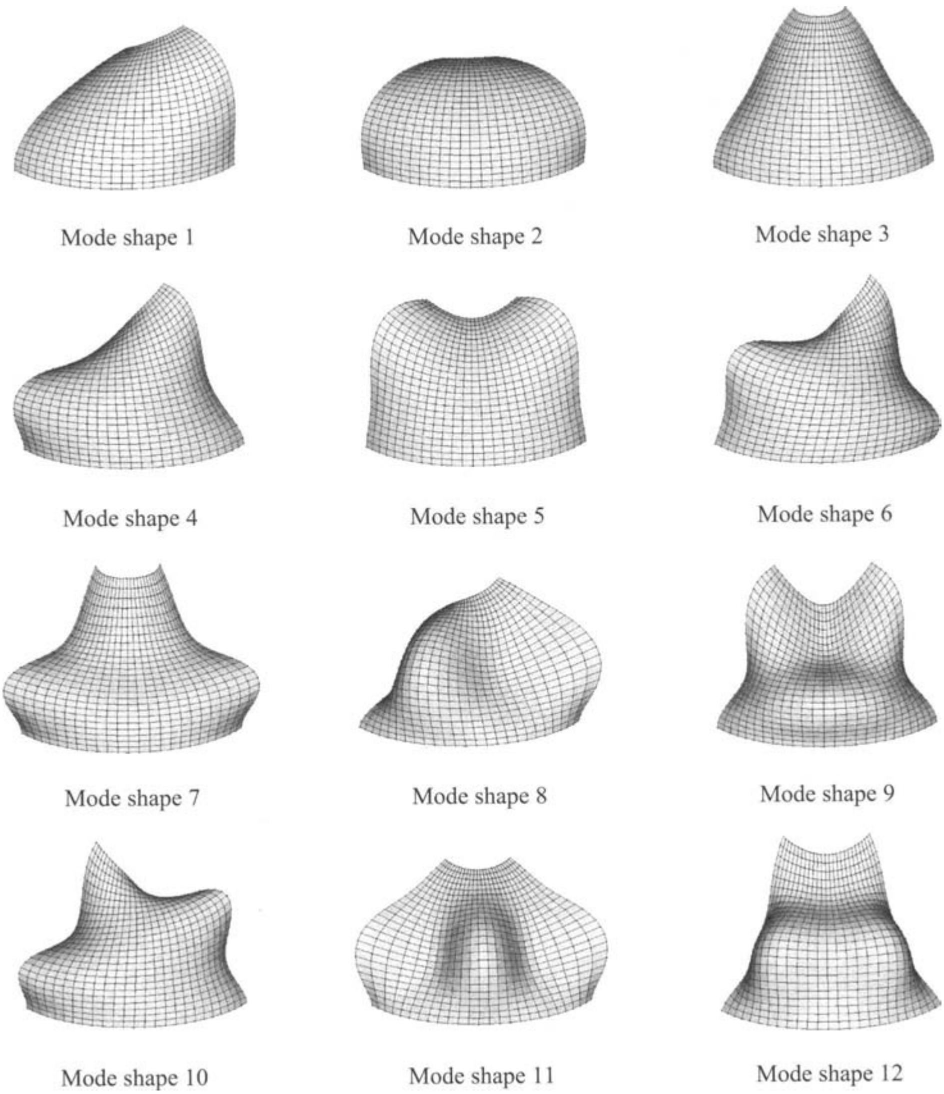

Figure 6. Mode shapes for the hemispherical panel C-F-F-F.

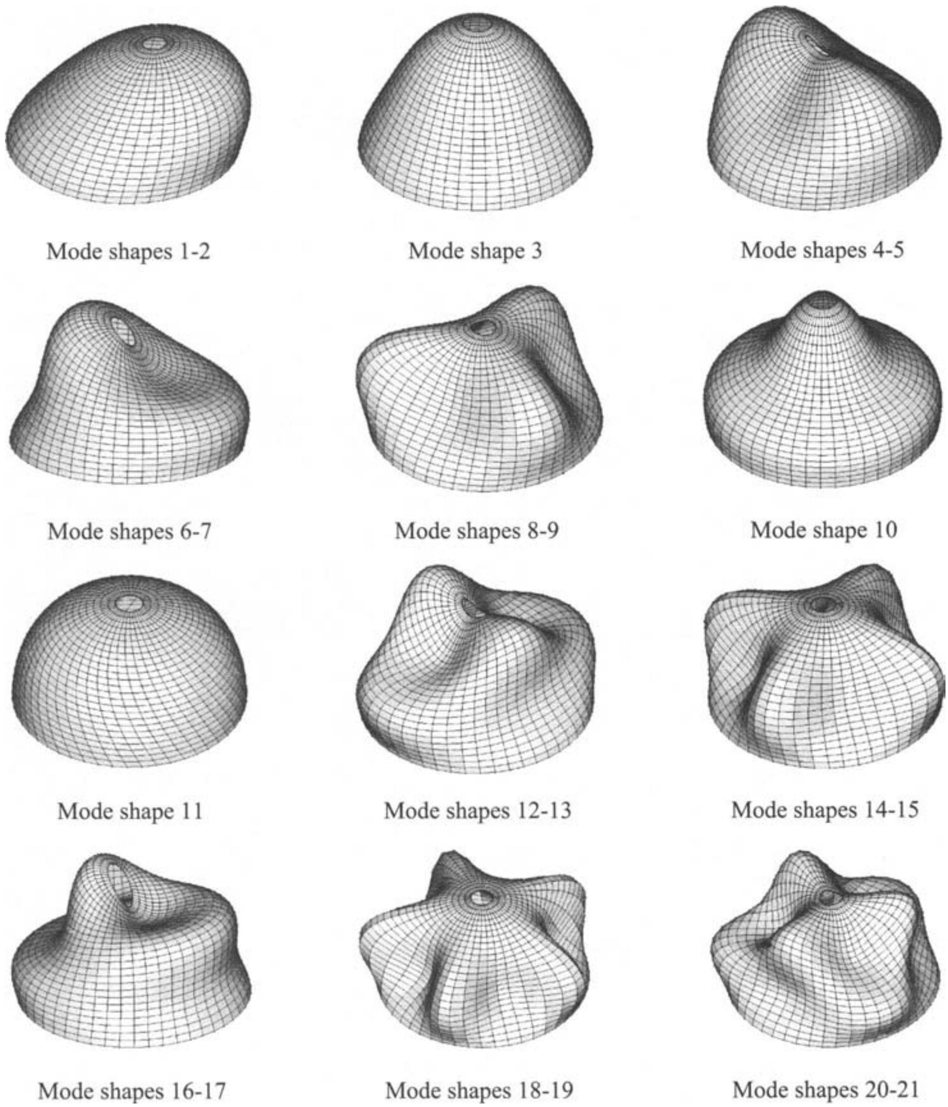

Figure 7. Mode shapes for the hemispherical dome C-F.

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