579 LECTURE NOTES IN ECONOMICS AND MATHEMATICAL SYSTEMS

Dieter Sondermann

Introduction to Stochastic Calculus for Finance

A New Didactic Approach

Springer

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A New Didactic Approach

With 6 Figures

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To Freddy, Hans and Marek, who patiently helped me to a deeper understanding of stochastic calculus.

Preface

There are by now numerous excellent books available on stochastic calculus with specific applications to finance, such as Duffie (2001), Elliott-Kopp (1999), Karatzas-Shreve (1998), Lamberton-Lapeyre (1995), and Shiryaev (1999) on different levels of mathematical sophistication. What justifies another contribution to this subject? The motivation is mainly pedagogical. These notes start with an elementary approach to continuous time methods of Itô's calculus due to Föllmer. In an fundamental, but not well-known paper published in French in the Seminaire de Probabilité in 1981 (see Foellmer (1981)), Föllmer showed that one can develop Itô's calculus without probabilities as an exercise in real analysis. [1]

The notes are based on courses offered regularly to graduate students in economics and mathematics at the University of Bonn choosing "financial economics" as special topic. To students interested in finance the course opens a quick (but by no means "dirty") road to the tools required for advanced finance. One can start the course with what they know about real analysis (e.g. Taylor's Theorem) and basic probability theory as usually taught in undergraduate courses in economic departments and business schools. What is needed beyond (collected in Chap. 1) can be explained, if necessary, in a few introductory hours.

The content of these notes was also presented, sometimes in condensed form, to MA students at the IMPA in Rio, ETH Zürich, to practi-

[1] An English translation of Föllmer's paper is added to these notes in the Appendix. In Chap. 2 we use Föllmer's approach only for the relative simple case of processes with continuous paths. Föllmer also treats the more difficult case of jump-diffusion processes, a topic deliberately left out in these notes.

tioners in the finance industry, and to PhD students and professors of mathematics at the Weizmann institute. There was always a positive feedback. In particular, the pathwise Föllmer approach to stochastic calculus was appreciated also by mathematicians not so much familiar with stochastics, but interested in mathematical finance. Thus the course proved suitable for a broad range of participants with quite different background.

I am greatly indebted to many people who have contributed to this course. In particular I am indebted to Hans Föllmer for generously allowing me to use his lecture notes in stochastics. Most of Chapter 2 and part of Chapter 3 follows closely his lecture. Without his contribution these notes would not exist. Special thanks are due to my assistants, in particular to Rüdiger Frey, Antje Mahayni, Philipp Schönbucher, and Frank Thierbach. They have accompanied my courses in Bonn with great enthusiasm, leading the students with engagement through the demanding course material in tutorials and contributing many useful exercises. I also profited from their critical remarks and from comments made by Freddy Delbaen, Klaus Schürger, Michael Suchanecki, and an unknown referee. Finally, I am grateful to all those students who have helped in typesetting, in particular to Florian Schröder.

Bonn, June 2006 *Dieter Sondermann*

Contents

Introduction

The lecture notes are organized as follows: Chapter 1 gives a concise overview of the theory of Lebesgue and Stieltjes integration and convergence theorems used repeatedly in this course. For mathematic students, familiar e.g. with the content of Bauer (1996) or Bauer (2001), this chapter can be skipped or used as additional reference .

Chapter 2 follows closely Föllmer's approach to Itô's calculus, and is to a large extent based on lectures given by him in Bonn (see Foellmer (1991)). A motivation for this approach is given in Sect. 2.1. This section provides a good introduction to the course, since it starts with familiar concepts from real analysis.

In Chap. 3 the Girsanov transformation is treated in more detail, as usually contained in mathematical finance textbooks. Sect. 3.2 is taken from Revuz-Yor (1991) and is basic for the following applications to finance.

The core of this lecture is Chapter 4, which presents the fundamentals of "financial economics" in continuous time, such as the *market price of risk*, the *no-arbitrage principle*, the *fundamental pricing rule* and its invariance under *numeraire changes*. Special emphasis is laid on the economic interpretation of the so-called *"risk-neutral" arbitrage measure* and its relation to the "real world" measure considered in general equilibrium theory, a topic sometimes leading to confusion between economists and financial engineers.

Using the general Girsanov transformation, as developed in Sect. 3.2, the rather intricate problem of the *change of numeraire* can be treated in a rigorous manner, and the so-called "two-country" or "Siegel" paradox serves as an illustration. The section on Feynman-Kac relates the martingal approach used explicitly in these notes to the more classical approach based on partial differential equations.

In Chap. 5 the preceding methods are applied to term structure models. By looking at a term structure model in continuous time in the general form of Heath-Jarrow-Morton (1992) as an infinite collection of assets (the zerobonds of different maturities), the methods developed in Chap. 4 can be applied without modification to this situation. Readers who have gone through the original articles of HJM may appreciate the simplicity of this approach, which leads to the basic results of HJM in a straightforward way. The same applies to the now quite popular *Libor Market Model* treated in Sect. 5.5 .

Chapter 6 presents some more advanced topics of stochastic calculus such as *local times* and the *generalized Itô formula*. The basic question here is: Does one really need the apparatus of Itô's calculus in finance? A question which is tantamount to : are charts of financial assets in reality of unbounded variation? The answer is YES, as any practitioner experienced in "delta-hedging" can confirm. Chapter 6 provides the theoretical background for this phenomenon.

1

Preliminaries

Recommended literature : (Bauer 1996), (Bauer 2001)

We assume that the reader is familiar with the following basic concepts:

$(\Omega, \mathcal{F}, P)$ is a probability space, i.e.

$\mathcal{F}$ is a σ-algebra of subsets of the nonempty set Ω

P is a σ-additive measure on $(\Omega, \mathcal{F})$ with $P[\Omega] = 1$

X is a random variable on $(\Omega, \mathcal{F}, P)$ with values in $\overline{\mathbb{R}} := [-\infty, \infty]$, i.e. X is a map $X : \Omega \longrightarrow \overline{\mathbb{R}}$ with $[X \leq a] \in \mathcal{F}$ for all $a \in \mathbb{R}$

1.1 Brief Sketch of Lebesgue's Integral

The Lebesgue integral of a random variable X can be defined in three steps.

(a) For a discrete random variable of the form $X = \sum_{i=1}^{n} \alpha_i \, \mathbf{1}_{A_i}$, $\alpha_i \in \mathbb{R}$, $A_i \in \mathcal{F}$ the integral (resp. the expectation) of X is defined as

$$E[X] := \int_{\Omega} X(\omega) \, dP(\omega) := \sum_i \alpha_i \, P[A_i].$$

Note: In the following we will drop the argument ω in the integral and write shortly $\int_\Omega X\, dP$.

Let $\mathcal{E}$ denote the set of all discrete random variables.

(b) Consider the set of all random variables which are monotone limits of discrete random variables, i.e. define

$$\mathcal{E}^* := \Big\{ X \ : \ \exists \quad u_1 \leq \ldots \ , \ u_n \in \mathcal{E} \ , \ u_n \uparrow X \Big\}$$

Remark: X random variable with $X \geq 0 \Longrightarrow X \in \mathcal{E}^*$.

For $X \in \mathcal{E}^*$ define

$$\int_\Omega X\, dP := \lim_{n \longrightarrow \infty} \int_\Omega u_n\, dP.$$

(c) For an arbitrary random variable X consider the decomposition $X = X^+ - X^-$ with

$$X^+ := \sup(X, 0) \quad , \quad X^- := \sup(-X, 0).$$

According to (b), $X^+ \ , \ X^- \in \mathcal{E}^*$.

If either $E[X^+] < \infty$ or $E[X^-] < \infty$, define

$$\int_\Omega X\, dP := \int_\Omega X^+\, dP - \int_\Omega X^-\, dP.$$

Properties of the Lebesgue Integral:

- *Linearity* : $\displaystyle\int_\Omega (\alpha\, X + \beta\, Y)\, dP = \alpha \int_\Omega X\, dP + \beta \int_\Omega Y\, dP$
- *Positivity* : $X \geq 0$ implies $\displaystyle\int X\, dP \geq 0$ and

$$\int X\, dP > 0 \Longleftrightarrow P[X > 0] > 0.$$

- *Monotone Convergence* (*Beppo Levi*).
  Let (X_n) be a monotone sequence of random variables (i.e. $X_n \leq X_{n+1}$) with $X_1 \geq C$. Then
  $$X := \lim_n X_n \in \mathcal{E}^*$$
  and
  $$\lim_{n \longrightarrow \infty} \int_\Omega X_n \, dP = \int_\Omega \lim_{n \longrightarrow \infty} X_n \, dP = \int_\Omega X \, dP.$$
- *Fatou's Lemma*
  (i) For any sequence (X_n) of random variables which are bounded from below one has
  $$\int_\Omega \lim_{n \longrightarrow \infty} \inf X_n \, dP \leq \lim_{n \longrightarrow \infty} \inf \int_\Omega X_n \, dP.$$
  (ii) For any sequence (X_n) of random variables bounded from above one has
  $$\int_\Omega \lim_{n \longrightarrow \infty} \sup X_n \, dP \geq \lim_{n \longrightarrow \infty} \sup \int_\Omega X_n \, dP.$$
- *Jensen's Inequality*
  Let X be an integrable random variable with values in $\mathbb{R}$ and $u : \overline{\mathbb{R}} \longrightarrow \bar{\mathbb{R}}$ a convex function.
  Then one has
  $$u(\mathrm{E}[X]) \leq \mathrm{E}[u(X)].$$
  Jensen's inequality is frequently applied, e.g. to $u(X) = |X|$, $u(X) = e^X$ or $u(X) = [X - a]^+$.

$\mathbf{L}^p$-Spaces $(1 \leq p < \infty)$

$\mathrm{L}^p(\Omega)$ denotes the set of all real-valued random variables X on $(\Omega, \mathcal{F}, P)$ with $\mathrm{E}[|X|^p] < \infty$ for some $1 \leq p < \infty$. For $X \in \mathrm{L}^p$, the L^p-norm is defined as
$$||X||_p := \Big(E[|X|^p]\Big)^{\frac{1}{p}}.$$

The L^p-norm has the following properties:

(a) *Hölder's Inequality*
Given $X \in L^p(\Omega)$ and $Y \in L^q(\Omega)$ with $\frac{1}{p} + \frac{1}{q} = 1$, one has

$$\int_\Omega |X| \cdot |Y| \, dP \leq \Big(\int_\Omega |X|^p \, dP \Big)^{\frac{1}{p}} \cdot \Big(\int_\Omega |Y|^q \, dP \Big)^{\frac{1}{q}} \, dP < \infty,$$

In particular, since $|X \cdot Y| \leq |X| \cdot |Y|$, implies $X \cdot Y \in L^1(\Omega)$.

(b) $L^p(\Omega)$ is a normed vector space. In particular, $X, Y \in L^p$ implies $X + Y \in L^p$ and one has

$$||X + Y||_p \leq ||X||_p + ||Y||_p. \quad \text{(triangle inequality)}$$

(c) $L^q \subset L^p$ for $p < q$.

Important special case: $p = 2$.

On L^2, the vector space of quadratically integrable random variables, there exists even a scalar product defined by

$$\langle X, Y \rangle := \int_\Omega X \cdot Y \, dP$$

Hence one has

$$||X||_2 = \sqrt{\langle X, X \rangle}$$

and Hölder's inequality takes the form

$$\langle X, Y \rangle = \int_\Omega X \cdot Y \, dP \leq ||X||_2 \cdot ||Y||_2.$$