

Simons Symposia

Paolo Cascini
James M^cKernan
Jorge Vitório Pereira *Editors*

Foliation Theory in Algebraic Geometry

 Springer

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Preface

As part of the celebrations around the opening of the Simons Foundation offices in New York, the editors were invited to organise a conference on a topic of their choice. We chose birational geometry and foliation theory as there has been considerable activity in both areas in the last decade and there has also been increasing interaction between the two subjects. The conference “Foliation theory in algebraic geometry” took place September 3–7, 2013, at the recently opened Simons Foundation’s Gerald D. Fischbach Auditorium.

The conference attracted over seventy participants as well as locals from the New York area and was a great success. These are the proceedings of the conference. The talks included both survey talks on recent progress and original research and the articles are a reflection of these topics.

The articles in this proceedings should be of interest to people working in birational geometry and foliation theory and anyone wanting to learn about these subjects.

The editors would like to thank David Eisenbud for the initial invitation to organise a conference. They would also like to thank the Simons Foundation and Yuri Tschinkel for hosting the conference and to acknowledge the generous support of both the Simons Foundation and the NSF, under grant no. DMS 1339299. We would like to thank the speakers and all of the participants for making the conference a success. Finally they would like to recognise the invaluable support of Meghan Fazzi in the organisation of the conference.

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On Fano Foliations 2

Carolina Araujo and Stéphane Druel

Abstract In this paper we pursue the study of mildly singular del Pezzo foliations on complex projective manifolds started in [AD13].

Keywords Fano manifolds • Holomorphic foliations • Classification

Mathematical Subject Classification: 14M22, 37F75

1 Introduction

In recent years, techniques from higher dimensional algebraic geometry, specially from the minimal model program, have been successfully applied to the study of global properties of holomorphic foliations. This led, for instance, to the birational classification of foliations by curves on surfaces in [Bru04]. Motivated by these developments, we initiated in [AD13] a systematic study of *Fano foliations*. These are holomorphic foliations \mathcal{F} on complex projective manifolds with ample anti-canonical class $-K_{\mathcal{F}}$. One special property of Fano foliations is that their leaves are always covered by rational curves, even when these leaves are not algebraic (see, for instance, [AD13, Proposition 7.5]).

The index $\iota_{\mathcal{F}}$ of a Fano foliation \mathcal{F} on a complex projective manifold X is the largest integer dividing $-K_{\mathcal{F}}$ in $\text{Pic}(X)$. In analogy with Kobayashi-Ochiai's theorem on the index of Fano manifolds (Theorem 2.2), we proved in [ADK08, Theorem 1.1] that the index of a Fano foliation \mathcal{F} on a complex projective manifold is bounded above by its rank, $\iota_{\mathcal{F}} \leq r_{\mathcal{F}}$. Equality holds if and only if $X \cong \mathbb{P}^n$ and \mathcal{F} is induced by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^{n-r_{\mathcal{F}}}$. Our expectation is that Fano

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foliations with large index are the simplest ones. So we proceeded to investigate the next case, namely Fano foliation \mathcal{F} of rank r and index $\iota_{\mathcal{F}} = r - 1$. We call such foliations *del Pezzo foliations*, in analogy with the case of Fano manifolds. In contrast to the case when $\iota_{\mathcal{F}} = r_{\mathcal{F}}$, there are examples of del Pezzo foliations with non-algebraic leaves. For instance, let \mathcal{C} be a foliation by curves on \mathbb{P}^k induced by a global vector field. If we take this vector field to be general, then the leaves of \mathcal{C} are not algebraic. Now consider a linear projection $\psi : \mathbb{P}^n \dashrightarrow \mathbb{P}^k$, with $n > k$, and let \mathcal{F} be the foliation on \mathbb{P}^n obtained as pullback of \mathcal{C} via ψ . It is a del Pezzo foliation on \mathbb{P}^n , and its leaves are not algebraic (see Theorem 3.16(2) for the complete classification of del Pezzo foliations on \mathbb{P}^n). The first main result of [AD13] says that these are the only examples.

Theorem 1.1 ([AD13, Theorem 1.1]). *Let \mathcal{F} be a del Pezzo foliation on a complex projective manifold $X \not\cong \mathbb{P}^n$. Then \mathcal{F} is algebraically integrable, and its general leaves are rationally connected.*

One of the main ingredients in our study of Fano foliations is the notion of *log leaf* for an algebraically integrable foliation. Given an algebraically integrable foliation \mathcal{F} on a complex projective manifold X , denote by $\tilde{e} : \tilde{F} \rightarrow X$ the normalization of the closure of a general leaf of \mathcal{F} . There is a naturally defined effective Weil divisor $\tilde{\Delta}$ on \tilde{F} such that $K_{\tilde{F}} + \tilde{\Delta} = \tilde{e}^*K_{\mathcal{F}}$ (see Definition 3.6 for details). We call the pair $(\tilde{F}, \tilde{\Delta})$ a *general log leaf* of \mathcal{F} . In [AD13], we used the log leaf to define new notions of singularities for algebraically integrable foliations, following the theory of singularities of pairs from the minimal model program. Namely, we say that \mathcal{F} has *log canonical singularities along a general leaf* if $(\tilde{F}, \tilde{\Delta})$ is log canonical. By Theorem 1.1, these notions apply to del Pezzo foliations on projective manifolds $X \not\cong \mathbb{P}^n$. In [AD13], we established the following classification of del Pezzo foliations with mild singularities.

Theorem 1.2 ([AD13, 9.1 and Theorems 1.3, 9.2, 9.6]). *Let \mathcal{F} be a del Pezzo foliation of rank r on a complex projective manifold $X \not\cong \mathbb{P}^n$, and suppose that \mathcal{F} has log canonical singularities and is locally free along a general leaf. Then*

- either $\rho(X) = 1$;
- or $r \leq 3$ and X is a \mathbb{P}^m -bundle over \mathbb{P}^k .

In the latter case, one of the following holds.

- (1) $X \cong \mathbb{P}^1 \times \mathbb{P}^k$, and \mathcal{F} is the pullback via the second projection of a foliation $\mathcal{O}_{\mathbb{P}^k}(1)^{\oplus i} \subset T_{\mathbb{P}^k}$ for some $i \in \{1, 2\}$ ($r \in \{2, 3\}$).
- (2) There exist

- an exact sequence of vector bundles $0 \rightarrow \mathcal{K} \rightarrow \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$ on \mathbb{P}^k ; and
- a foliation by curves $\mathcal{C} \cong q^* \det(\mathcal{Q}) \subset T_{\mathbb{P}^k(\mathcal{K})}$, where $q : \mathbb{P}^k(\mathcal{K}) \rightarrow \mathbb{P}^k$ denotes the natural projection;

such that $X \cong \mathbb{P}_{\mathbb{P}^k}(\mathcal{E})$, and \mathcal{F} is the pullback of \mathcal{C} via the relative linear projection $\mathbb{P}_{\mathbb{P}^k}(\mathcal{E}) \dashrightarrow \mathbb{P}_{\mathbb{P}^k}(\mathcal{K})$. Moreover, one of the following holds.

- (a) $k = 1$, $\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, \mathcal{K} is an ample vector bundle such that $\mathcal{K} \not\cong \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus m}$ for any integer a , and $\mathcal{E} \cong \mathcal{Q} \oplus \mathcal{K}$ ($r = 2$).

- (b) $k = 1$, $\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(2)$, $\mathcal{K} \cong \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus m}$ for some integer $a \geq 1$, and $\mathcal{E} \cong \mathcal{Q} \oplus \mathcal{K}$ ($r = 2$).
- (c) $k = 1$, $\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $\mathcal{K} \cong \mathcal{O}_{\mathbb{P}^1}(a)^{\oplus(m-1)}$ for some integer $a \geq 1$, and $\mathcal{E} \cong \mathcal{Q} \oplus \mathcal{K}$ ($r = 3$).
- (d) $k \geq 2$, $\mathcal{Q} \cong \mathcal{O}_{\mathbb{P}^k}(1)$, and \mathcal{K} is V -equivariant for some $V \in H^0(\mathbb{P}^k, T_{\mathbb{P}^k} \otimes \mathcal{O}_{\mathbb{P}^k}(-1)) \setminus \{0\}$ ($r = 2$).

Conversely, given \mathcal{K} , \mathcal{E} , and \mathcal{Q} satisfying any of the conditions above, there exists a del Pezzo foliation of that type.

The goal of the present paper is to continue the classification of del Pezzo foliations on Fano manifolds $X \not\cong \mathbb{P}^n$ having log canonical singularities and being locally free along a general leaf. In view of Theorem 1.2, we need to understand del Pezzo foliations on Fano manifolds with Picard number 1. Our main result is the following.

Theorem 1.3. *Let \mathcal{F} be a del Pezzo foliation of rank $r \geq 3$ on an n -dimensional Fano manifold $X \not\cong \mathbb{P}^n$ with $\rho(X) = 1$, and suppose that \mathcal{F} has log canonical singularities and is locally free along a general leaf. Then $X \cong \mathbb{Q}^n$ and \mathcal{F} is induced by the restriction to \mathbb{Q}^n of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^{n-r}$.*

Remark 1.4. Codimension 1 del Pezzo foliations on Fano manifolds with Picard number 1 were classified in [LPT13, Proposition 3.7]. We extended this classification to mildly singular varieties, without restriction on the Picard number in [AD14, Theorem 1.3].

We also obtain a partial classification when $r = 2$ (Proposition 4.1).

In order to prove Theorem 1.3, we consider a general log leaf $(\tilde{F}, \tilde{\Delta})$ of \mathcal{F} . Under the assumptions of Theorem 1.3, $(\tilde{F}, \tilde{\Delta})$ is a *log del Pezzo pair*: it is a log canonical pair of dimension r satisfying $K_{\tilde{F}} + \tilde{\Delta} = (r - 1)L$, where L is an ample divisor on \tilde{F} . The first step in the proof of Theorem 1.3 consists in classifying all log del Pezzo pairs. This is done in Section 2.4, using Fujita’s theory of Δ -genus. Once we know the general log leaf $(\tilde{F}, \tilde{\Delta})$ of \mathcal{F} , we consider families of rational curves on X that restrict to special families of rational curves on \tilde{F} . The necessary results from the theory of rational curves are briefly reviewed in Section 2.1. The idea is to use these families of rational curves to bound the index of X from below. In order to obtain a good bound, we need to show that the dimension of these families of rational curves is big enough. Here enters a very special property of algebraically integrable Fano foliations having log canonical singularities along a general leaf: there is a common point contained in the closure of a general leaf [AD13, Proposition 5.3]. For our current purpose, we need the following strengthening of this result (see Definition 2.9 for the notion of log canonical center).

Proposition 1.5. *Let \mathcal{F} be an algebraically integrable Fano foliation on a complex projective manifold X having log canonical singularities along a general leaf. Then there is a closed irreducible subset $T \subset X$ satisfying the following property. For a general log leaf $(\tilde{F}, \tilde{\Delta})$, there exists a log canonical center S of $(\tilde{F}, \tilde{\Delta})$ whose image in X is T .*

When $r \geq 3$, this allows us to show that $\iota_X \geq n$, and then use Kobayashi-Ochiai's theorem (Theorem 2.2) to conclude that $X \cong Q^n$. The classification of del Pezzo foliations on Q^n is established in Proposition 3.18.

Proposition 1.5 still holds in the more general setting of \mathbb{Q} -Fano foliations on possibly singular projective varieties. Since this may be useful in other situations, we present the theory of foliations on normal projective varieties in Section 3, and prove a more general version of Proposition 1.5 (Proposition 3.14).

Notation and Conventions We always work over the field \mathbb{C} of complex numbers. Varieties are always assumed to be irreducible. We denote by $\text{Sing}(X)$ the singular locus of a variety X .

Given a sheaf \mathcal{F} of \mathcal{O}_X -modules on a variety X , we denote by \mathcal{F}^* the sheaf $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$. If r is the generic rank of \mathcal{F} , then we denote by $\det(\mathcal{F})$ the sheaf $(\wedge^r \mathcal{F})^{**}$. If \mathcal{G} is another sheaf of \mathcal{O}_X -modules on X , then we denote by $\mathcal{F}[\otimes]\mathcal{G}$ the sheaf $(\mathcal{F} \otimes \mathcal{G})^{**}$. For $m \in \mathbb{N}$, we denote by $\mathcal{F}^{[m]}$ the sheaf $(\mathcal{F}^{\otimes m})^{**}$.

If \mathcal{E} is a locally free sheaf of \mathcal{O}_X -modules on a variety X , we denote by $\mathbb{P}_X(\mathcal{E})$ the Grothendieck projectivization $\text{Proj}_X(\text{Sym}(\mathcal{E}))$, and by $\mathcal{O}_{\mathbb{P}}(1)$ its tautological line bundle.

If X is a normal variety, we denote by T_X the sheaf $(\Omega_X^1)^*$.

We denote by Q^n a (possibly singular) quadric hypersurface in \mathbb{P}^{n+1} . Given an integer $d \geq 0$, we denote by \mathbb{F}_d the surface $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-d))$. If moreover $d \geq 1$, we denote by $\mathbb{P}(1, 1, d)$ the cone in \mathbb{P}^{d+1} over the rational normal curve of degree d .

2 Preliminaries

2.1 Fano Manifolds and Rational Curves

Definition 2.1. A *Fano manifold* X is a complex projective manifold whose anti-canonical class $-K_X$ is ample. The index ι_X of X is the largest integer dividing $-K_X$ in $\text{Pic}(X)$.

Theorem 2.2 ([KO73]). *Let X be a Fano manifold of dimension $n \geq 2$ and index ι_X . Then $\iota_X \leq n + 1$, and equality holds if and only if $X \cong \mathbb{P}^n$. Moreover, $\iota_X = n$ if and only if $X \cong Q^n \subset \mathbb{P}^{n+1}$.*

Families of rational curves provide a useful tool in the study of Fano manifolds. Next we gather some results from the theory of rational curves. In what follows, rational curves are always assumed to be proper. A *family of rational curves* on a complex projective manifold X is a closed irreducible subset of $\text{RatCurves}^n(X)$. We refer to [Kol96] for details.

Definition 2.3. Let $\ell \subset X$ be a rational curve on a complex projective manifold, and consider its normalization $f: \mathbb{P}^1 \rightarrow X$. We say that ℓ is *free* if f^*T_X is globally generated.

2.4. Let X be a complex projective manifold, and $\ell \subset X$ a free rational curve. Let $x \in \ell$ be any point, and H_x an irreducible component of the scheme $\text{RatCurves}^n(X, x)$ containing a point corresponding to ℓ . Then

$$\dim(H_x) = -K_X \cdot \ell - 2.$$

Notation 2.5. Let X be a Fano manifold with $\rho(X) = 1$, and \mathcal{A} an ample line bundle on X such that $\text{Pic}(X) = \mathbb{Z}[\mathcal{A}]$. For any proper curve $C \subset X$, we refer to $\mathcal{A} \cdot C$ as the *degree* of C . Rational curves of degree 1 are called *lines*. Note that if $C \subset X$ is a proper curve of degree d , then $\iota_X = \frac{-K_X \cdot C}{d}$.

One can use free rational curves on Fano manifolds with Picard number 1 to bound their index. The following is an immediate consequence of paragraph 2.4 above.

Lemma 2.6. *Let X be a Fano manifold with $\rho(X) = 1$. Suppose that there is an m -dimensional family V of rational curves of degree d on X such that:*

- *all curves from V pass through some fixed point $x \in X$; and*
- *some curve from V is free.*

Then $\iota_X \geq \frac{m+2}{d}$.

Remark 2.7. Let V be a family of rational curves on a complex projective manifold X . To guarantee that some member of V is a free curve, it is enough to show that some curve from V passes through a *general* point of X . More precisely, let H be an irreducible component of $\text{RatCurves}^n(X)$ containing V . It comes with universal family morphisms

$$\begin{array}{ccc} U & \xrightarrow{e} & X, \\ \pi \downarrow & & \\ & & H \end{array}$$

where $\pi : U \rightarrow H$ is a \mathbb{P}^1 -bundle. Suppose that $e : U \rightarrow X$ is dominant. Then, by generic smoothness, there is a dense open subset $X^\circ \subset X$ over which $e : U \rightarrow X$ is smooth. On the other hand, by [Kol96, Proposition II.3.4], e is smooth at a point $u \in \pi^{-1}(t)$ if and only if the rational curve $\ell_t = e(\pi^{-1}(t))$ is free.

2.2 Singularities of Pairs

We refer to [KM98, section 2.3] and [Kol13, sections 2 and 4] for details.

Definition 2.8. Let X be a normal projective variety, and $\Delta = \sum a_i \Delta_i$ an effective \mathbb{Q} -divisor on X , i.e., Δ is a nonnegative \mathbb{Q} -linear combination of distinct prime Weil divisors Δ_i 's on X . Suppose that $K_X + \Delta$ is \mathbb{Q} -Cartier, i.e., some nonzero multiple of it is a Cartier divisor.

Let $f : \tilde{X} \rightarrow X$ be a log resolution of the pair (X, Δ) . There are uniquely defined rational numbers $a(E_i, X, \Delta)$'s such that

$$K_{\tilde{X}} + f_*^{-1}\Delta = f^*(K_X + \Delta) + \sum_{E_i} a(E_i, X, \Delta)E_i.$$

The $a(E_i, X, \Delta)$'s do not depend on the log resolution f , but only on the valuations associated to the E_i 's. The closed subvariety $f(E_i) \subset X$ is called the *center of E_i in X* . It also depends only on the valuation associated to E_i .

For a prime divisor D on X , we define $a(D, X, \Delta)$ to be the coefficient of D in $-\Delta$.

We say that the pair (X, Δ) is *log canonical* if, for some log resolution $f : \tilde{X} \rightarrow X$ of (X, Δ) , $a(E_i, X, \Delta) \geq -1$ for every f -exceptional prime divisor E_i . If this condition holds for some log resolution of (X, Δ) , then it holds for every log resolution of (X, Δ) .

Definition 2.9. Let (X, Δ) be a log canonical pair. We say that a closed irreducible subvariety $S \subset X$ is a *log canonical center* of (X, Δ) if there is a divisor E over X with $a(E, X, \Delta) = -1$ whose center in X is S .

2.3 Polarized Varieties and Fujita's Δ -Genus

Definition 2.10. A *polarized variety* is a pair (X, \mathcal{L}) consisting of a normal projective variety X , and an ample line bundle \mathcal{L} on X .

Definition 2.11 ([Fuj75]). The Δ -genus of an n -dimensional polarized variety (X, \mathcal{L}) is defined by the formula:

$$\Delta(X, \mathcal{L}) := n + c_1(\mathcal{L})^n - h^0(X, \mathcal{L}) \in \mathbb{Z}.$$

By [Fuj82, Corollary 2.12], $\Delta(X, \mathcal{L}) \geq 0$ for any polarized variety (X, \mathcal{L}) . Next we recall the classification of polarized varieties with Δ -genus zero from [Fuj82].

Theorem 2.12 ([Fuj82]). *Let X be a normal projective variety of dimension $n \geq 1$, and \mathcal{L} an ample line bundle on X . Suppose that $\Delta(X, \mathcal{L}) = 0$. Then one of the following holds.*

- (1) $(X, \mathcal{L}) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$.
- (2) $(X, \mathcal{L}) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1))$.
- (3) $(X, \mathcal{L}) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$, for some $d \geq 3$.
- (4) $(X, \mathcal{L}) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.
- (5) $(X, \mathcal{L}) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}), \mathcal{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})}(1))$, where \mathcal{E} is an ample vector bundle on \mathbb{P}^1 .
- (6) \mathcal{L} is very ample, and embeds X as a cone over a projective polarized variety of type (3–5) above.

2.4 Classification of Log Del Pezzo Pairs

Definition 2.13. Let X be a normal projective variety of dimension $n \geq 1$, and Δ an effective \mathbb{Q} -divisor on X . We say that (X, Δ) is a *log del Pezzo pair* if (X, Δ) is log canonical, and $-(K_X + \Delta) \equiv (n-1)c_1(\mathcal{L})$ for some ample line bundle \mathcal{L} on X .

Using Fujita's classification of polarized varieties with Δ -genus zero, we classify log del Pezzo pairs in Theorem 2.15 below.

Lemma 2.14. *Let (X, \mathcal{L}) be an n -dimensional polarized variety, with $n \geq 2$. Let (X, Δ) be a log del Pezzo pair such that $\Delta \neq 0$ and $-(K_X + \Delta) \equiv (n-1)c_1(\mathcal{L})$. Then $\Delta(X, \mathcal{L}) = 0$, and $\Delta \cdot c_1(\mathcal{L})^{n-1} = 2$.*

Proof. We follow the line of argumentation in the proof of [Fuj80, Lemma 1.10]. Since

$$c_1(\mathcal{L}^{\otimes t}) \equiv K_X + \Delta + c_1(\mathcal{L}^{\otimes n-1+t}),$$

we have that $h^i(X, \mathcal{L}^{\otimes t}) = 0$ for $i \geq 1$ and $t > 1 - n$ by [Fuj11, Theorem 8.1]. Therefore $\chi(X, \mathcal{O}_X) = 1$ and $\chi(X, \mathcal{L}^{\otimes t}) = 0$ for $2 - n \leq t \leq -1$. Hence, there are rational numbers a and b such that

$$\chi(X, \mathcal{L}^{\otimes t}) = (at^2 + bt + n(n-1)) \frac{\prod_{j=1}^{n-2} (t+j)}{n!}.$$

On the other hand, since X is normal, by Hirzebruch-Riemann-Roch,

$$\chi(X, \mathcal{L}^{\otimes t}) = \frac{c_1(\mathcal{L})^n}{n!} t^n - \frac{1}{2(n-1)!} K_X \cdot c_1(\mathcal{L})^{n-1} t^{n-1} + o(t^{n-1}).$$

Thus we have $a = c_1(\mathcal{L})^n$ and $b = \frac{n}{2} \Delta \cdot c_1(\mathcal{L})^{n-1} + (n-1)c_1(\mathcal{L})^n$. In particular,

$$h^0(X, \mathcal{L}) = \chi(X, \mathcal{L}) = n-1 + c_1(\mathcal{L})^n + \frac{1}{2} \Delta \cdot c_1(\mathcal{L})^{n-1}.$$

One then computes that

$$\Delta(X, \mathcal{L}) = 1 - \frac{1}{2} \Delta \cdot c_1(\mathcal{L})^{n-1}.$$

Since $\Delta \neq 0$ and $\Delta(X, \mathcal{L}) \geq 0$, we must have $\Delta(X, \mathcal{L}) = 0$ and $\Delta \cdot c_1(\mathcal{L})^{n-1} = 2$. \square

Theorem 2.15. *Let (X, \mathcal{L}) be an n -dimensional polarized variety, with $n \geq 1$. Let (X, Δ) be a log del Pezzo pair such that Δ is integral and nonzero, and $-(K_X + \Delta) \equiv (n-1)c_1(\mathcal{L})$. Then one of the following holds.*

- (1) $(X, \mathcal{L}, \mathcal{O}_X(\Delta)) \cong (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}(2))$.
- (2) $(X, \mathcal{L}, \mathcal{O}_X(\Delta)) \cong (\mathbb{Q}^n, \mathcal{O}_{\mathbb{Q}^n}(1), \mathcal{O}_{\mathbb{Q}^n}(1))$.
- (3) $(X, \mathcal{L}, \mathcal{O}_X(\Delta)) \cong (\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d), \mathcal{O}_{\mathbb{P}^1}(2))$, for some integer $d \geq 3$.
- (4) $(X, \mathcal{L}, \mathcal{O}_X(\Delta)) \cong (\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2), \mathcal{O}_{\mathbb{P}^2}(1))$.
- (5) $(X, \mathcal{L}) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}), \mathcal{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})}(1))$ for an ample vector bundle \mathcal{E} on \mathbb{P}^1 . Moreover, one of the following holds.
 - (a) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \geq 2$, and $\Delta \sim_{\mathbb{Z}} \sigma + f$ where σ is the minimal section and f is a fiber of $\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}) \rightarrow \mathbb{P}^1$.
 - (b) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \geq 2$, and Δ is a minimal section.
 - (c) $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(a)$ for some $a \geq 1$, and $\Delta = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1))$.
- (6) \mathcal{L} is very ample, and embeds (X, Δ) as a cone over $((Z, \mathcal{M}), (\Delta_Z, \mathcal{M}|_{\Delta_Z}))$, where Z is smooth and $(Z, \mathcal{M}, \Delta_Z)$ satisfies one of the conditions (3–5) above.

Proof. By [CKP12, Theorem 0.1], we must have $-(K_X + \Delta) \sim_{\mathbb{Q}} (n-1)c_1(\mathcal{L})$.

If $n = 1$, then $-K_X \sim_{\mathbb{Q}} \Delta$ is ample, and hence $(X, \mathcal{L}, \mathcal{O}_X(\Delta))$ satisfies one of conditions (1–3) in the statement of Theorem 2.15.

Suppose from now on that $n \geq 2$. By Lemma 2.14, $\Delta(X, \mathcal{L}) = 0$, and so we can apply Theorem 2.12. Notice that if (X, \mathcal{L}) satisfies any of conditions (1–6) of Theorem 2.12, then $-(K_X + \Delta) \sim_{\mathbb{Z}} (n-1)c_1(\mathcal{L})$ since $X \setminus \text{Sing}(X)$ is simply connected.

If (X, \mathcal{L}) satisfies any of conditions (1–4) of Theorem 2.12, one checks easily that (X, \mathcal{L}, Δ) satisfies one of conditions (1–4) in the statement of Theorem 2.15.

Suppose that $(X, \mathcal{L}) \cong (\mathbb{P}_{\mathbb{P}^1}(\mathcal{E}), \mathcal{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})}(1))$ for an ample vector bundle \mathcal{E} on \mathbb{P}^1 , and write $\pi : X \rightarrow \mathbb{P}^1$ for the natural projection. Then

$$\Delta \in |\mathcal{O}_X(-K_X) \otimes \mathcal{L}^{\otimes 1-n}| = |\mathcal{L} \otimes \pi^*(\det(\mathcal{E}^*) \otimes \mathcal{O}_{\mathbb{P}^1}(2))|.$$

Write $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n)$, with $1 \leq a_1 \leq \cdots \leq a_n$. By the projection formula, $h^0(X, \mathcal{L} \otimes \pi^*(\det(\mathcal{E}^*) \otimes \mathcal{O}_{\mathbb{P}^1}(2))) = h^0(\mathbb{P}^1, \mathcal{E} \otimes \det(\mathcal{E}^*) \otimes \mathcal{O}_{\mathbb{P}^1}(2))$, hence we must have $a_1 + \cdots + a_{n-1} \leq 2$. This implies that $(n, a_1, \dots, a_{n-1}) \in \{(2, 1), (2, 2), (3, 1, 1)\}$. Thus either \mathcal{E} satisfies condition (5a-c) in the statement of Theorem 2.15, or $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$, $\Delta \in |\mathcal{O}_{\mathbb{P}_{\mathbb{P}^1}(\mathcal{E})}(1)|$, and hence X satisfies condition (2) with $n = 2$.

Finally, suppose that \mathcal{L} is very ample, and embeds X as a cone with vertex V over a smooth polarized variety (Z, \mathcal{M}) satisfying one of conditions (3–5) in the statement of Theorem 2.12. Set $m := \dim(Z)$ and $s := n - m = \dim(V) + 1$. Let $e : Y \rightarrow X$ be the blow-up of X along V , with exceptional divisor E . We have $Y \cong \mathbb{P}_Z(\mathcal{M} \oplus \mathcal{O}_Z^{\oplus s})$, with natural projection $\pi : Y \rightarrow Z$, and tautological line bundle $\mathcal{O}_Y(1) \cong e^*\mathcal{L}$. The exceptional divisor E corresponds to the projection $\mathcal{M} \oplus \mathcal{O}_Z^{\oplus s} \twoheadrightarrow \mathcal{O}_Z^{\oplus s}$.

Let Δ_Y be the strict transform of Δ in Y . We are done if we prove that $\Delta_Y = \pi^*\Delta_Z$ for some divisor Δ_Z on Z .

Write $\Delta_Y \sim_{\mathbb{Z}} \pi^* \Delta_Z + kE$ for some integral divisor Δ_Z on Z , and some integer $k \geq 0$. Let $\sigma : Z \rightarrow Y$ be the section of π corresponding to a general surjection $\mathcal{M} \oplus \mathcal{O}_Z^{\oplus s} \twoheadrightarrow \mathcal{M}$. Then $\sigma(Z) \cap E = \emptyset$, and $\mathcal{N}_{\sigma(Z)/Y} \cong \mathcal{M}^{\oplus s}$. Moreover, $(\sigma(Z), \Delta_{Y|_{\sigma(Z)}})$ is log canonical (see, for instance, [Kol97, Proposition 7.3.2]), and, by the adjunction formula, $-(K_{\sigma(Z)} + \Delta_{Y|_{\sigma(Z)}}) \sim_{\mathbb{Z}} (m-1)c_1(\mathcal{O}_Y(1))|_{\sigma(Z)}$.

We have $h^0(Y, \mathcal{O}_Y(kE + \pi^* \Delta_Z)) = h^0(Y, \mathcal{O}_Y(\Delta_Y)) \geq 1$. On the other hand,

$$\begin{aligned} h^0(Y, \mathcal{O}_Y(kE + \pi^* \Delta_Z)) &= h^0(Y, \mathcal{O}_Y(k) \otimes \pi^* \mathcal{M}^{\otimes -k} \otimes \mathcal{O}_Y(\pi^* \Delta_Z)) \\ &= h^0(Z, S^k(\mathcal{M} \oplus \mathcal{O}_Z^{\oplus s}) \otimes \mathcal{M}^{\otimes -k} \otimes \mathcal{O}_Z(\Delta_Z)) \\ &= h^0(Z, S^k(\mathcal{O}_Z \oplus \mathcal{M}_Z^{\otimes -1 \oplus s}) \otimes \mathcal{O}_Z(\Delta_Z)). \end{aligned}$$

We claim that $h^0(Z, \mathcal{M}_Z^{\otimes -1} \otimes \mathcal{O}_Z(\Delta_Z)) = 0$. Indeed, suppose that $h^0(Z, \mathcal{M}_Z^{\otimes -1} \otimes \mathcal{O}_Z(\Delta_Z)) \neq 0$. Then $-K_Z \sim_{\mathbb{Z}} \Delta_Z + (m-1)c_1(\mathcal{M})$ since $\Delta_{Y|_{\sigma(Z)}} \sim_{\mathbb{Z}} (\pi|_{\sigma(Z)})^* \Delta_Z$, and hence $-K_Z \geq mc_1(\mathcal{M})$. Under these conditions, [AD14, Theorem 2.5] implies that $(Z, \mathcal{M}, \mathcal{O}_Z(\Delta_Z))$ is isomorphic to either $(\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(1), \mathcal{O}_{\mathbb{P}^m}(2))$ or $(Q^m, \mathcal{O}_{Q^m}(1), \mathcal{O}_{Q^m}(1))$. This contradicts our current assumption that (Z, \mathcal{M}) satisfies one of conditions (3–5) in the statement of Theorem 2.12, and proves the claim.

Since $h^0(Z, \mathcal{M}_Z^{\otimes -1} \otimes \mathcal{O}_Z(\Delta_Z)) = 0$, we must have $h^0(Y, \mathcal{O}_Y(kE + \pi^* \Delta_Z)) = h^0(Z, \mathcal{O}_Z(\Delta_Z))$. Thus, replacing Δ_Z with a suitable member of its linear system if necessary, we may assume that $\Delta_Y = \pi^* \Delta_Z + kE$, and hence $k = 0$. Therefore (X, \mathcal{L}, Δ) satisfies condition (6) in the statement of Theorem 2.15. \square

In dimension 2, we have the following classification, without the assumption that (X, Δ) is log canonical.

Theorem 2.16 ([Nak07, Theorem 4.8]). *Let (X, Δ) be a pair with $\dim(X) = 2$ and $\Delta \neq 0$. Suppose that $-(K_X + \Delta)$ is Cartier and ample. Then one of the following holds.*

- (1) $X \cong \mathbb{P}^2$ and $\deg(\Delta) \in \{1, 2\}$.
- (2) $X \cong \mathbb{F}_d$ for some $d \geq 0$ and Δ is a minimal section.
- (3) $X \cong \mathbb{F}_d$ for some $d \geq 0$ and $\Delta \sim_{\mathbb{Z}} \sigma + f$, where σ is a minimal section and f a fiber of $\mathbb{F}_d \rightarrow \mathbb{P}^1$.
- (4) $X \cong \mathbb{P}(1, 1, d)$ for some $d \geq 2$ and $\Delta \sim_{\mathbb{Z}} 2\ell$ where ℓ is a ruling of the cone $\mathbb{P}(1, 1, d)$.

3 Foliations

3.1 Foliations and Pfaff Fields

Definition 3.1. Let X be normal variety. A *foliation* on X is a nonzero coherent subsheaf $\mathcal{F} \subsetneq T_X$ such that

- \mathcal{F} is closed under the Lie bracket, and
- \mathcal{F} is saturated in T_X (i.e., T_X/\mathcal{F} is torsion free).

The *rank* r of \mathcal{F} is the generic rank of \mathcal{F} . The *codimension* of \mathcal{F} is $q = \dim(X) - r$.

The *canonical class* $K_{\mathcal{F}}$ of \mathcal{F} is any Weil divisor on X such that $\mathcal{O}_X(-K_{\mathcal{F}}) \cong \det(\mathcal{F})$.

Definition 3.2. A foliation \mathcal{F} on a normal variety is said to be \mathbb{Q} -Gorenstein if its canonical class $K_{\mathcal{F}}$ is \mathbb{Q} -Cartier.

Definition 3.3. Let X be a variety, and r a positive integer. A *Pfaff field of rank r* on X is a nonzero map $\eta : \Omega_X^r \rightarrow \mathcal{L}$, where \mathcal{L} is a reflexive sheaf of rank 1 on X such that $\mathcal{L}^{[m]}$ is invertible for some integer $m \geq 1$.

The *singular locus* S of η is the closed subscheme of X whose ideal sheaf \mathcal{I}_S is the image of the induced map $\Omega_X^r[\otimes]\mathcal{L}^* \rightarrow \mathcal{O}_X$.

Notice that a \mathbb{Q} -Gorenstein foliation \mathcal{F} of rank r on normal variety X naturally gives rise to a Pfaff field of rank r on X :

$$\eta : \Omega_X^r = \wedge^r(\Omega_X^1) \rightarrow \wedge^r(T_X^*) \rightarrow \wedge^r(\mathcal{F}^*) \rightarrow \det(\mathcal{F}^*) \cong \det(\mathcal{F})^* \cong \mathcal{O}_X(K_{\mathcal{F}}).$$

Definition 3.4. Let \mathcal{F} be a \mathbb{Q} -Gorenstein foliation on a normal variety X . The *singular locus* of \mathcal{F} is defined to be the singular locus S of the associated Pfaff field. We say that \mathcal{F} is *regular at a point* $x \in X$ if $x \notin S$. We say that \mathcal{F} is *regular* if $S = \emptyset$.

Our definition of Pfaff field is more general than the one usually found in the literature, where \mathcal{L} is required to be invertible. This generalization is needed in order to treat \mathbb{Q} -Gorenstein foliations whose canonical classes are not Cartier.

3.5 (Foliations Defined by q -Forms). Let \mathcal{F} be a codimension q foliation on an n -dimensional normal variety X . The *normal sheaf* of \mathcal{F} is $N_{\mathcal{F}} := (T_X/\mathcal{F})^{**}$. The q -th wedge product of the inclusion $N_{\mathcal{F}}^* \hookrightarrow (\Omega_X^1)^{**}$ gives rise to a nonzero global section $\omega \in H^0(X, \Omega_X^q[\otimes]\det(N_{\mathcal{F}}))$ whose zero locus has codimension at least 2 in X . Moreover, ω is *locally decomposable* and *integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point of X , ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \dots, q\}$. The integrability condition for ω is equivalent to the condition that \mathcal{F} is closed under the Lie bracket.

Conversely, let \mathcal{L} be a reflexive sheaf of rank 1 on X , and $\omega \in H^0(X, \Omega_X^q[\otimes]\mathcal{L})$ a global section whose zero locus has codimension at least 2 in X . Suppose that ω is locally decomposable and integrable. Then the kernel of the morphism $T_X \rightarrow \Omega_X^{q-1}[\otimes]\mathcal{L}$ given by the contraction with ω defines a foliation of codimension q on X . These constructions are inverse of each other.

3.2 Algebraically Integrable Foliations

Let X be a normal projective variety, and \mathcal{F} a foliation on X . In this subsection we assume that \mathcal{F} is *algebraically integrable*. This means that \mathcal{F} is the relative tangent sheaf to a dominant rational map $\varphi : X \dashrightarrow Y$ with connected fibers. In this case, by a general leaf of \mathcal{F} we mean the fiber of φ over a general point of Y . We start by defining the notion of *log leaf* when \mathcal{F} is moreover \mathbb{Q} -Gorenstein. It plays a key role in our approach to \mathbb{Q} -Fano foliations.

Definition 3.6 (See [AD14, Definition 3.10] for details). Let X be a normal projective variety, \mathcal{F} a \mathbb{Q} -Gorenstein algebraically integrable foliation of rank r on X , and $\eta : \Omega_X^r \rightarrow \mathcal{O}_X(K_{\mathcal{F}})$ its corresponding Pfaff field. Let $F \subset X$ be the closure of a general leaf of \mathcal{F} , and $\tilde{e} : \tilde{F} \rightarrow X$ the normalization of F . Let $m \geq 1$ be the Cartier index of $K_{\mathcal{F}}$, i.e., the smallest positive integer m such that $mK_{\mathcal{F}}$ is Cartier. Then η induces a generically surjective map $\otimes^m \Omega_{\tilde{F}}^r \rightarrow \tilde{e}^* \mathcal{O}_X(mK_{\mathcal{F}})$. Hence there is a canonically defined effective Weil \mathbb{Q} -divisor $\tilde{\Delta}$ on \tilde{F} such that $mK_{\tilde{F}} + m\tilde{\Delta} \sim_{\mathbb{Z}} \tilde{e}^* mK_{\mathcal{F}}$.

We call the pair $(\tilde{F}, \tilde{\Delta})$ a *general log leaf* of \mathcal{F} .

The next lemma gives sufficient conditions under which the support of $\tilde{\Delta}$ is precisely the inverse image in \tilde{F} of the singular locus of \mathcal{F} . It is an immediate consequence of [AD13, Lemma 5.6].

Lemma 3.7. *Let \mathcal{F} be an algebraically integrable foliation on a complex projective manifold X . Suppose that \mathcal{F} is locally free along the closure of a general leaf F . Let $\tilde{e} : \tilde{F} \rightarrow X$ be its normalization, and $(\tilde{F}, \tilde{\Delta})$ the corresponding log leaf. Then $\text{Supp}(\tilde{\Delta}) = \tilde{e}^{-1}(F \cap \text{Sing}(\mathcal{F}))$.*

Definition 3.8. Let X be normal projective variety, \mathcal{F} a \mathbb{Q} -Gorenstein algebraically integrable foliation on X , and $(\tilde{F}, \tilde{\Delta})$ its general log leaf. We say that \mathcal{F} has *log canonical singularities along a general leaf* if $(\tilde{F}, \tilde{\Delta})$ is log canonical.

Remark 3.9. In [McQ08, Definition I.1.2], McQuillan introduced a notion of log canonicity for foliations, without requiring algebraic integrability. If a \mathbb{Q} -Gorenstein algebraically integrable foliation \mathcal{F} is log canonical in the sense of McQuillan, then \mathcal{F} has log canonical singularities along a general leaf (see [AD13, Proposition 3.11] and its proof).

3.10 (The Family of Log Leaves). Let X be normal projective variety, and \mathcal{F} an algebraically integrable foliation on X . We describe the *family of leaves* of \mathcal{F} (see [AD13, Lemma 3.2 and Remark 3.8] for details). There is a unique irreducible closed subvariety W of $\text{Chow}(X)$ whose general point parameterizes the closure of a general leaf of \mathcal{F} (viewed as a reduced and irreducible cycle in X). It comes with a universal cycle $U \subset W \times X$ and morphisms:

$$\begin{array}{ccc} U & \xrightarrow{e} & X, \\ \pi \downarrow & & \\ W & & \end{array}$$

where $e : U \rightarrow X$ is birational and, for a general point $w \in W$, $e(\pi^{-1}(w)) \subset X$ is the closure of a leaf of \mathcal{F} .

The variety W is called the *family of leaves* of \mathcal{F} .

Suppose moreover that \mathcal{F} is \mathbb{Q} -Gorenstein, denote by $m \geq 1$ the Cartier index of $K_{\mathcal{F}}$, by r the rank of \mathcal{F} , and by $\eta : \Omega_X^r \rightarrow \mathcal{O}_X(K_{\mathcal{F}})$ the corresponding Pfaff field. Given a morphism $V \rightarrow W$ from a normal variety, let U_V be the normalization of $U \times_V W$, with induced morphisms:

$$\begin{array}{ccc} U_V & \xrightarrow{e_V} & X. \\ \pi_V \downarrow & & \\ V & & \end{array}$$

Then η induces a generically surjective map $\otimes^m \Omega_{U_V/V}^r \rightarrow e_V^* \mathcal{O}_X(mK_{\mathcal{F}})$. Thus there is a canonically defined effective Weil \mathbb{Q} -divisor Δ_V on U_V such that $\det(\Omega_{U_V/V}^1)^{\otimes m} [\otimes] \mathcal{O}_{U_V}(m\Delta_V) \cong e_V^* \mathcal{O}_X(mK_{\mathcal{F}})$. Suppose that $v \in V$ is mapped to a general point of W , set $U_v := (\pi_V)^{-1}(v)$, and $\Delta_v := (\Delta_V)|_{U_v}$. Then (U_v, Δ_v) coincides with the general log leaf $(\tilde{F}, \tilde{\Delta})$ defined above.

3.3 \mathbb{Q} -Fano Foliations

Definition 3.11. Let X be a normal projective variety, and \mathcal{F} a \mathbb{Q} -Gorenstein foliation on X . We say that \mathcal{F} is a *\mathbb{Q} -Fano foliation* if $-K_{\mathcal{F}}$ is ample. In this case, the *index* of \mathcal{F} is the largest positive rational number $\iota_{\mathcal{F}}$ such that $-K_{\mathcal{F}} \sim_{\mathbb{Q}} \iota_{\mathcal{F}} H$ for a Cartier divisor H on X .

If \mathcal{F} is a \mathbb{Q} -Fano foliation of rank r on a normal projective variety X , then, by [Hör13, Corollary 1.2], $\iota_{\mathcal{F}} \leq r$. Moreover, equality holds if and only if X is a generalized normal cone over a normal projective variety Z , and \mathcal{F} is induced by the natural rational map $X \dashrightarrow Z$ (see also [ADK08, Theorem 1.1], and [AD14, Theorem 4.11]).

Definition 3.12. A \mathbb{Q} -Fano foliation \mathcal{F} of rank $r \geq 2$ is called a *del Pezzo foliation* if $\iota_{\mathcal{F}} = r - 1$.

In [AD13, Proposition 5.3], we proved that algebraically integrable Fano foliations having log canonical singularities along a general leaf have a very special

property: there is a common point contained in the closure of a general leaf. We strengthen this result in Proposition 3.14 below. It will be a consequence of the following theorem.

Theorem 3.13 ([ADK08, Theorem 3.1]). *Let X be a normal projective variety, $f : X \rightarrow C$ a surjective morphism onto a smooth curve, and Δ an effective Weil \mathbb{Q} -divisor on X such that (X, Δ) is log canonical over the generic point of C . Then $-(K_{X/C} + \Delta)$ is not ample.*

Proposition 3.14. *Let \mathcal{F} be an algebraically integrable \mathbb{Q} -Fano foliation on a normal projective variety X , having log canonical singularities along a general leaf. Then there is a closed irreducible subset $T \subset X$ satisfying the following property. For a general log leaf $(\tilde{F}, \tilde{\Delta})$ of \mathcal{F} , there exists a log canonical center S of $(\tilde{F}, \tilde{\Delta})$ whose image in X is T .*

Proof. Let W be the normalization of the family of leaves of \mathcal{F} , U the normalization of the universal cycle over W , with universal family morphisms $\pi : U \rightarrow W$ and $e : U \rightarrow X$. As explained in 3.10, there is a canonically defined effective \mathbb{Q} -Weil divisor Δ on U such that $\det(\Omega_{U/W}^1)^{\otimes m}[\otimes]\mathcal{O}_U(m\Delta) \cong e^*\mathcal{O}_X(mK_{\mathcal{F}})$, where $m \geq 1$ denotes the Cartier index of $K_{\mathcal{F}}$. Moreover, there is a smooth dense open subset $W_0 \subset W$ with the following properties. For any $w \in W_0$, denote by U_w the fiber of π over w , and set $\Delta_w := \Delta|_{U_w}$. Then

- U_w is integral and normal, and
- (U_w, Δ_w) has log canonical singularities.

To prove the proposition, suppose to the contrary that, for any two general points $w, w' \in W_0$, and any log canonical centers S_w and $S_{w'}$ of (U_w, Δ_w) and $(U_{w'}, \Delta_{w'})$ respectively, we have $e(S_w) \neq e(S_{w'})$.

Let $C \subset W$ be a (smooth) general complete intersection curve, and U_C the normalization of $\pi^{-1}(C)$, with induced morphisms $\pi_C : U_C \rightarrow C$ and $e_C : U_C \rightarrow X$. By [BLR95, Theorem 2.1'], after replacing C with a finite cover if necessary, we may assume that π_C has reduced fibers. As before, there is a canonically defined \mathbb{Q} -Weil divisor Δ_C on U_C such that $K_{U_C/C} + \Delta_C \sim_{\mathbb{Q}} e_C^*K_{\mathcal{F}}$. Therefore $K_{U_C} + \Delta_C \sim_{\mathbb{Q}} \pi_C^*K_C + e_C^*K_{\mathcal{F}}$ is a \mathbb{Q} -Cartier divisor. For a general point $w \in C$, we identify $(\pi_C^{-1}(w), \Delta_C|_{\pi_C^{-1}(w)})$ with (U_w, Δ_w) , which is log canonical by assumption. Thus, by inversion of adjunction (see [Kaw07, Theorem]), the pair (U_C, Δ_C) has log canonical singularities over the generic point of C . Let $w \in C$ be a general point, and S_w any log canonical center of (U_w, Δ_w) . Then there exists a reduced and irreducible closed subset $S_C \subset U_C$ such that:

- $S_w = S_C \cap U_w$, and
- S_C is a log canonical center of (U_C, Δ_C) over the generic point of C .

Moreover, our current assumption implies that

- $\dim(e_C(S_C)) = \dim(S_C)$.

Thus, by [Dem97, Proposition 7.2(ii)], there exist an ample \mathbb{Q} -divisor A and an effective \mathbb{Q} -Cartier \mathbb{Q} -divisor E on U_C such that:

- $e_C^*(-K_{\mathcal{F}}) \sim_{\mathbb{Q}} A + E$, and
- for a general point $w \in C$, $\text{Supp}(E)$ does not contain any log canonical center of (U_w, Δ_w) .

Therefore $(U_C, \Delta_C + \epsilon E)$ is log canonical over the generic point of C for $0 < \epsilon \ll 1$. Notice that $e_C^*(-K_{\mathcal{F}}) - \epsilon E$ is ample since $e_C^*(-K_{\mathcal{F}})$ is nef and big, and hence

$$-(K_{U_C/C} + \Delta_C + \epsilon E) \sim_{\mathbb{Q}} e_C^*(-K_{\mathcal{F}}) - \epsilon E$$

is ample as well. But this contradicts Theorem 3.13, completing the proof of the proposition. \square

Corollary 3.15. *Let \mathcal{F} be an algebraically integrable Fano foliation on a complex projective manifold, and $(\tilde{F}, \tilde{\Delta})$ its general log leaf. Suppose that \mathcal{F} is locally free along the closure of a general leaf. Then $\tilde{\Delta} \neq 0$.*

Proof. Denote by F the closure of a general leaf of \mathcal{F} . If $\tilde{\Delta} = 0$, then \mathcal{F} is regular along F by Lemma 3.7. Hence \mathcal{F} is induced by an almost proper map $X \dashrightarrow W$, and F is smooth. In particular $(\tilde{F}, \tilde{\Delta})$ is log canonical. But this contradicts Proposition 3.14. This proves that $\tilde{\Delta} \neq 0$. \square

3.4 Foliations on \mathbb{P}^n

The degree $\deg(\mathcal{F})$ of a foliation \mathcal{F} of rank r on \mathbb{P}^n is defined as the degree of the locus of tangency of \mathcal{F} with a general linear subspace $\mathbb{P}^{n-r} \subset \mathbb{P}^n$. By 3.5, a foliation on \mathbb{P}^n of rank r and degree d is given by a twisted q -form $\omega \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^q(q + d + 1))$, where $q = n - r$. Thus

$$d = \deg(K_{\mathcal{F}}) + r.$$

Jouanolou has classified codimension 1 foliations on \mathbb{P}^n of degree 0 and 1. This has been generalized to arbitrary rank as follows.

Theorem 3.16.

- (1) [DC05, Théorème 3.8]. A codimension q foliation of degree 0 on \mathbb{P}^n is induced by a linear projection $\mathbb{P}^n \dashrightarrow \mathbb{P}^q$.
- (2) [LPT13, Theorem 6.2]. A codimension q foliation \mathcal{F} of degree 1 on \mathbb{P}^n satisfies one of the following conditions.
 - \mathcal{F} is induced by a dominant rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}(1^q, 2)$, defined by q linear forms L_1, \dots, L_q and one quadratic form Q ; or
 - \mathcal{F} is the linear pullback of a foliation on \mathbb{P}^{q+1} induced by a global holomorphic vector field.

3.17. Let \mathcal{F} be a codimension q foliation of degree 1 on \mathbb{P}^n .

In the first case described in Theorem 3.16(2), \mathcal{F} is induced by the q -form on \mathbb{C}^{n+1}

$$\begin{aligned} \Omega &= \sum_{i=1}^q (-1)^{i+1} L_i dL_1 \wedge \cdots \wedge \widehat{dL}_i \wedge \cdots \wedge dL_q \wedge dQ + (-1)^q Q dL_1 \wedge \cdots \wedge dL_q \\ &= (-1)^q \left(\sum_{i=q+1}^{n+1} L_j \frac{\partial Q}{\partial L_i} \right) dL_1 \wedge \cdots \wedge dL_q \\ &\quad + \sum_{i=1}^q \sum_{j=q+1}^{n+1} (-1)^{i+1} L_i \frac{\partial Q}{\partial L_j} dL_1 \wedge \cdots \wedge \widehat{dL}_i \wedge \cdots \wedge dL_q \wedge dL_j, \end{aligned}$$

where L_{q+1}, \dots, L_{n+1} are linear forms such that L_1, \dots, L_{n+1} are linearly independent. Thus, the singular locus of \mathcal{F} is the union of the quadric $\{L_1 = \cdots = L_q = Q = 0\} \cong \mathbb{Q}^{n-q-1}$ and the linear subspace $\{\frac{\partial Q}{\partial L_{q+1}} = \cdots = \frac{\partial Q}{\partial L_{n+1}} = 0\}$.

In the second case described in Theorem 3.16(2), the singular locus of \mathcal{F} is the union of linear subspaces of codimension at least 2 containing the center \mathbb{P}^{n-q-2} of the projection.

3.5 Foliations on \mathbb{Q}^n

In this subsection we classify del Pezzo foliations on smooth quadric hypersurfaces.

Proposition 3.18. *Let \mathcal{F} be a codimension q del Pezzo foliation on a smooth quadric hypersurface $\mathbb{Q}^n \subset \mathbb{P}^{n+1}$. Then \mathcal{F} is induced by the restriction of a linear projection $\mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^q$.*

Proof. If $q = 1$, then the result follows from [AD14, Theorem 1.3]. So we assume from now on that $q \geq 2$.

By [AD13, Proposition 7.7], \mathcal{F} is algebraically integrable, and its singular locus is nonempty by [AD14, Theorem 6.1].

Let $x \in \mathbb{Q}^n$ be a point in the singular locus of \mathcal{F} , and consider the restriction $\varphi : \mathbb{Q}^n \dashrightarrow \mathbb{P}^n$ to \mathbb{Q}^n of the linear projection $\psi : \mathbb{P}^{n+1} \dashrightarrow \mathbb{P}^n$ from x . Let $f : Y \rightarrow \mathbb{Q}^n$ be the blow-up of \mathbb{Q}^n at x with exceptional divisor $E \cong \mathbb{P}^{n-1}$, and $g : Y \rightarrow \mathbb{P}^n$ the induced morphism. Notice that g is the blow-up of \mathbb{P}^n along the smooth codimension 2 quadric $Z = \varphi(\text{Exc}(\varphi)) \cong \mathbb{Q}^{n-2}$. Denote by H the hyperplane of \mathbb{P}^n containing Z , and by F the exceptional divisor of g . Note that $g(E) = H$, and $f(F)$ is the hyperplane section of \mathbb{Q}^n cut out by $T_x \mathbb{Q}^n$. The codimension q del Pezzo foliation \mathcal{F} is defined by a nonzero section $\omega \in H^0(\mathbb{Q}^n, \Omega_{\mathbb{Q}^n}^q(q+1))$ vanishing at x . So it induces a twisted q -form $\alpha \in H^0(Y, \Omega_Y^q \otimes f^* \mathcal{O}_{\mathbb{Q}^n}(q+1) \otimes \mathcal{O}_Y(-qE)) \cong H^0(Y, \Omega_Y^q \otimes g^* \mathcal{O}_{\mathbb{P}^n}(q+2) \otimes \mathcal{O}_Y(-F))$. The restriction of α to $Y \setminus F$ induces a twisted q -form