Interaction of Mechanics and Mathematics

Serge Preston

Non-commuting Variations in Mathematics and Physics

A Survey

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NOTES ON THE NONCOMMUTING VARIATIONS.

SERGE PRESTON

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1. Preface

The aim of this text is to present and study the method of so-called "**non-commuting variations (shortly, NC-variations)**" in Variational Calculus. To present this method we recall one of the basic rules of Variational Calculus - the rule defining the variations of derivatives $\frac{\partial y^\mu}{\partial x^i}$ of dynamical variables $y^\mu(x)$ (fields in the Field Theory) corresponding to a variation ξ of dynamical variables (fields): "variation of a derivative equals to the derivative of variation". In Mechanics, this rule takes the form $\delta \dot{y}^\mu = \frac{d}{dt}\delta y^\mu$. In Classical Field theory this rule takes the form

$$\delta \frac{\partial y^\mu}{\partial x^i} = \frac{\partial}{\partial x^i}\delta y^\mu.$$

This rule can be formulated as follows: "taking of variations of dynamical variables $y^i(x)$ commute with the taking of derivatives."

This rule was universally adopted in the XVIII and XIX centuries but, as early as in 1887, this rule was questioned by Vito Volterra, see [130, 131]. Studying non-holonomic mechanical systems, V.Volterra noticed that the use of the conventional rule of defining variations of derivatives does not allows us to obtain equations of motion for non-holonomic systems by variational methods. Further developments including works of L.Boltzman,[8] G.Hamel,[57], T.Levi Civita and U.Amaldi,[83] led to the conflicting points of view at the range of applicability of the conventional rule of defining variations (see a Historical Review between Chapters 1 and 2 below). Finally, the status of this, conventional, rule and its relation to the alternative rules - the use of "non-commuting variations" in Non-Holonomic Mechanics were clarified in works of J.Neimark and his coauthors in the 1950s of XX century ([104, 105]) and by A.Lurie in 1961, [88].

Later on, the non-commuting variations were used in the works of B.Vujanovich and T.Atanackovic on dynamical systems with non-conservative forces ([133, 134, 4, 5]), in Elasticity Theory, and in works of H.Kleinert, P.Fiziev and A.Pelster on the dynamics in Cartan-Riemann spaces ([35, 65]).

While studying the application of non-commuting variations in classical field theory we noticed that the usage of non-commuting rules to define variations of derivatives **is equivalent to the use of a non-trivial vertical connections** to modify the procedure of flow prolongation of variational vector fields in the space Y of the configurational bundle $\pi : Y \to X$ of a physical system to the 1-jet bundle $J^1(\pi) \to Y$ over π, [112]. This led us to the study of the geometrical structures underlying the method of non-commuting variations of derivatives in Lagrangian formalism. In particular,a natural variety of questions that arises here is: which of the basic methods of Variational Calculus - Theory of second variations, Hamiltonian systems and Legendre transformation, conservation laws (including Noether theorems), Hamilton-Jacoby Equations, etc. - are preserved in this modified scheme and which parts require modifications to stay true. These and some other related questions are studied in the present work.

We will also show that any system of PDE of the form: "Euler- Lagrange equations with sources"

$$E_j(L) = f_j \tag{1.1}$$

can be realized by the Lagrangian formalism with a conventional action functional $\mathcal{A}(L)$ and the non-commuting variations defined by an appropriately chosen (defined by the sources f_j) tensor K of NC-variations. We show that the basic methods of conventional Lagrangian formalism - Noether Theorem, second variation technique, Hamiltonian equations, Weyl fields preserve their form in Lagrangian formalism with NC-variations. We study the relations between the properties of sources f_j and the curvature $\tilde{R}$ of the vertical connection tensor K.

We demonstrate that a variety of geometrical structures that appeared in the study of dynamics in some physical systems - dissipative potentials, non-holonomic transformations, torsion of zero curvature connections (absolute parallelism), material time and thermasy (= heat displacement), introduced by H.Helmholtz and studied by D.van Dantzig ([135, 136, 110]) are special cases or are closely related to the use of non-commutative variations defined by a vertical connection in the conventional Lagrangian formalism.

Our perspective in this work, supported by the results of the geometrical (bundle) form of Variational Calculus, is that the conventional rules of taking variations of the derivatives of dynamical variables (fields) (underlying the flow prolongation method) have important mathematical advantages (preservation of Cartan distribution, preservation of Lie bracket, etc.) making them more fundamental. Yet, a more general approach allows inclusion into the framework of Variational Calculus, the physical systems that can not be described by the conventional Lagrangian formalism.

In that we adopt the point of view of B. Vujanovich and that of A.Lurie's that in difference to the variations of fields y^i, variations of their derivatives $y^i_{,\mu}$ *are not only kinematical, but dynamical notions* and should be dealt with as such. In particular, this allows us to introduce geometrical factors that have dynamical meaning into the definition of variations of derivatives that have dynamical meaning. This allows us to describe dissipative processes in the system.

These notes are based on the Lectures delivered by author at the 15th Summer School in Global Analysis at Masaruk University,Brno, CZ on August 8-12, 2011. I am using this case to thank participants of this school and, especially, its organizer Professor Demeter Krupka and Dr.Marcella Palese for useful discussions during the school.

In particular, during this school Marcella Palese informed me about a non-conventional procedures of the non-commuting variations introduced by C.Murial and J.Romero in Spain and used by the group of specialists in Spain and Italy with the goal to extend the range of symmetry groups of Lagrangian systems. Their goals were different from ours but their constructions (of λ and μ-prolongations) are similar, but not identical, to our approach of using vertical connections. We have included a condensed exposition of their work and the relations with our scheme in the present text (see Chapter 4).

Preliminary results on the NC-variations Lagrangian formalism where published in the Proceedings of the GCM2008, [112].

2. Introduction.

In Chapter I we give a short sketch of classical Lagrangian formalism. Here we tried to make a presentation as simple as possible. Yet, we introduce in the beginning of Chapter 1, some basic invariant notions, whose more detailed description reader can find in the Appendix (and, in more details, in the literature refered there to. We define the configuration bundle, $\pi : Y \to X$, one-jet bundle $J^1(\pi)$ as the domain of Lagrangian functions and the total derivatives d_μ on the space $J^1(\pi)$ used to write down the Euler-Lagrange equations in an invariant way.

In Chapter 2 we define the non-commutative variations for an action functional of a Lagrangian $L(x^\mu, y^i, y^i_{,x^\mu})$ of the first order. Non-commutative variations are defined using tensors $K^\mu_{i\nu}$ in the configurational space Y. Variations of derivatives defined with the help of tensor K will be called **K-twisted variations** and the Euler-Lagrange equations obtained using variations defined this way will be called K-twisted EL-equations.

We get the Euler-Lagrange equations

$$EL(L)_\mu = f_\mu,\ \mu = 1, \ldots, m$$

with the sources f_j defined by a "tensor K", formulate corresponding Noether Theorem (proved in Appendix III), present the canonical Energy-Momentum balance law.

A variety of examples of $EL + NV$ systems and classes of such systems are presented here.

Using Legendre transformation we construct corresponding Hamilton equations with sources and compare them with the "metriplectic or "double bracket" systems.

Then we show the form taken by the second variation formalism (sufficient conditions, Jacoby equation, etc.) in the case of NC-variations. At the end of this Chapter we show that this approach to the Lagrangian formalism can be readily extended to the higher order Lagrangian problems and to the case of "degenerate Lagrangians" where the source terms are of higher order then the Lagrangian itself.

In Chapter 3, we show that the procedure of K-twisted prolongation of a variation $\xi = \xi^i \partial_{y^\mu}$ of dynamical fields y^μ to the 1-jet bundle $J^1(\pi)$ is lacking two basic properties of the conventional flow prolongations of variational vector fields: conservation of Lie vector fields brackets and preservation of Cartan distribution in the 1-jet space. While the second property is valid only if tensor K vanishes, obstruction to the preservation of the Lie brackets is determined. It consists in two parts - curvature form tensor $\tilde{R}$ and the "skew-symmetric bracket" presenting the deformation of the Lie bracket of the vector fields.

Next, we show that the tensor K defining the Non-Commuting Lagrangian formalism has the form of the vertical component of an (Ehresmann) connection ω on the affine bundle $\pi_{10} : J^1(\pi) \to Y$ - the component responsible for the term of the form $a^i_\mu \partial_{y^i_\mu}$ of the K-vertical lift of a vector field $\xi = \xi^\mu \partial_{x^\mu} + \xi^i \partial_{y^i}$.

More then this, vertical/vertical component of the curvature $R(\omega)$ coincide with the tensor $\tilde{R}$ mentioned above. It is shown that one can define the covariant flow prolongation of vector fields from Y to $J^1(Y)$ so that the K-twisted prolongation of a vector field coincide with the modified by K flow prolongation.

In Sec.20, we consider the case where tensor K does not depend on the derivatives of dynamical fields $K^i_{\mu j} \in C^\infty(Y)$. We calculate different quantities characterizing K-twisted prolongations for this case and study the relation between the form of source terms, f_j, and the properties of the "curvature" $\tilde{R}$.

In Chapter 4 we present a short description of works of the group os spanish and italian mathematicians developed the Theory of twisted prolongations of vector fields to the jet bundles in many respects similar to our scheme. Their works had different goal - to construct, using the twisted prolongations of vector fields, alternative classes of symmetry groups of differential equations and systems of differential equations. their "theory of λ and μ -prolongations and symmetries has an important property - vector fields obtained by the prolongations to the jet bundles preserves, in some modified sense, the Cartan distributions and contact formes. This property has an elegant form and probably can be useful in further development of Geometrical Theory of Differential Equations.

In Chapter 5, we discuss several situations when the non-commuting variations were used explicitly or implicitly in the variational description of some physical systems. For some time a "geometrization" of a mechanical system, i.e., presentation equations of motion of such a system as the geodesic motion with respect to some linear connection in the configurational space Q of this system was a very popular problem in Mechanics. In Sec.40, it is shown, following the work of B.Vujanovich,[134, 133] that the same result can be achieved without changing the geometry of the space Q but, instead, by using conventional Lagrangian of this system and redefining the variations of velocities in the tangent space $T(Q)$. In Sec.41, the short review of variational approach to the non-holonomic mechanical systems is presented. Using the approach of L.Boltzman we construct the equations of motion in non-holonomic systems with line non-holonomic relations. We notice that the Bolzmann tensor defining the non-commutativity tensor K of the variations is defined here by the torsion of the zero curvature connection corresponding to the non-holonomic frame (see Appendix I, Sec.66).

In Sections 42,43 we present the use of non-holonomic (gauge) transformations for constructing Variational principle with non-commuting variations defined by the torsion of the (absolute parallelism) connection given by this transformation. First example of such scheme (see Section 42) is the one that was developed by H.Kleinert and his collaborators P.Fiziev and A.Pelster [35, 65] to describe Mechanics in spaces with metrics and connections (Cartan spaces). In Section 43 we present a short review of properties of **Uniform Materials**. Uniform materials were defined by K.Kondo (1955) and developed by a variety of specialists including E.Kroner, B.Bilby, C.C.Wang, C.Truesdell in 60th of XX century and by many specialists later on. We refer to the monographs [22, 137] for the detailed description of "Uniform materials"" theory.

In Sec.44, we discuss the relation between the method of non-commuting variations and the use of **dissipative potentials** (special case of which are Rayleigh dissipative function) in Lagrangian formalism.

In Chapter 6, we present an application of Lagrangian formalism with the NC-variations to the description of irreversible evolution of a continuous media with heating and structural changes.

In Sec. (46) we introduce thermasy, a scalar variable introduced by H. Helmholtz and later on, used by D.van Dantzig in his study of thermodynamics of moving matter, see [135], A.Green and P.Nahdi in thermoelasticity, see [53, 54] and G.Maugin and V.Kalpakides in Continuum Thermodynamics, [92]. Using thermasy, whose time derivative is absolute temperature one can formulate **entropy balance of a thermodynamical system as the Euler-Lagrange Equation**. We present modified and simplified version of this variational system and write down corresponding energy balance and the heat propagation equation that has the form of Cattaneo heat propagation law, see [100].

Then we introduce the model of material metric space-time (P, G) that was introduced by A.Chudnovsky and the author in order to model the aging processes in the materials, [15, 16]. In this model, evolution of the material is presented by smooth embedding of the material space-time into the Galilean space-time and the material metric G describes the structural properties of material. In particular, the rate S of the proper (=material) time τ relative to the physical type: $d\tau = Sdt$ is the characteristic of the entropy production in the material (if entropy production is zero, $S = 1$). We show that the entropy balance in a thermodynamical system obtained as the Euler-Lagrange for thermasy using the NC-variations defined by the rate of material time S coincide with the Euler-Lagrange Equation for thermasy **obtained using the material time τ instead of physical time** t and **conventional variations instead of NC-variations**.

This duality shows that the usage of NC-variations allows us to model complex irreversible phenomena that is impossible to do using conventional Lagrangian approach.

Appendix:

In the Appendix I we present a short review of geometrical notions used in the text: manifolds, fiber bundles, connections and their curvature, linear connection and its rorsion, prolongation of vector fields from Y to the jet bundles, absolute parallelism. In appendix II we define jet bundles, their mappings, total derivatives, contact structure of jet bundles, connections in jet bundles, Lie vector fields, properties of vertical connections. In Appendix III we define the symmetries and infinitesimal symmetries of the differential systems and the Lagrangian action, define the Noether formal;ism and probe the the First Noether Theorem. In the case od Euler-lagrange equation with sources, Noether equations corresponding to the symmetry Lie groups are **balance equations** rather then the conservation laws. This referees, in particular, to the energy-momentum balance laws.

Part I. Non-commuting variations - elementary topics.

Chapter 1. Basics of the Lagrangian Field Theory.

In this Chapter, we introduce the basic notions of Classical Lagrangian Field Theory of the first order - configurational bundle, action functional, Euler-Lagrange equations in the volume necessary in the main text. We will keep this presentation as short as possible for two reasons: the first reason is that this, classical material is well presented in a variety of well known sources - see [44, 118] for a classical introduction to the Variational Calculus. The second reason is that we prefer to introduce some notions (second variation, Hamilton-Jacobi equation) in the sections where we can compare them with the form they take in a case of non-commuting variations. For more advanced exposition of geometrical structure of classical field theory, including Lagrangian Field Theory we refer to the sources [33, 45, 46, 47, 106].

3. Configurational bundle , 1-jet bundle and the Lagrangian action.

3.1. **Configurational bundle** (Y, π, X)**.** Dynamical variables of Classical Field Theory typically appear to be tensor or tensor density fields defined in the domain of a physical, material or mezoscopic space-time (for the last one, see [96] and other works of W. Muschik).

To organize these fields and their derivatives into a natural geometrical picture it is convenient to introduce the **configurational fiber bundle** - a triple (Y, π, X) with the base space $X, \dim X = n$ and the total space $Y, \dim Y = n + m$, where the smooth mapping $\pi : Y^{n+m} \to X^n$ is onto and of constant rank (see Appendix I for a short introduction of geometrical and topological notions used here). For most situations studied in this book, it is sufficient to assume that the space Y is the product of the base X and the typical fiber F and that both X and F are either Euclidian vector spaces of dimension n and m respectively, or open domains in Euclidian spaces of corresponding dimension. In the general case, spaces X, Y, F are differentiable manifolds, sometimes, with the boundary. See Appendix I where these notions are defined and examples of manifolds are presented.

For a point $x \in X$, the sub-manifold $Y_x = \pi^{-1}(x)$ is called the fiber over x. Fibers Y_x of the bundle π are assumed to be connected m-dim manifolds diffeomorphic to the fixed manifold F^m (see Appendix I).

We will be using local coordinates in the bundle Y, adjusted to the bundle structure - "fibred charts" $(V, x^i, i = 1, \ldots, n; y^\alpha, \alpha = 1, \ldots, m)$ in Y defined in a domain $V \subset Y$. Here x^i are coordinates in the domain $\pi(V) \subset X$ and y^α are fiber coordinates in the fibers Y_x.

Below we will be using shortened notations for partial derivatives in the fiber charts: ∂_i for $\frac{\partial}{\partial x^i}$, and ∂_μ for $\frac{\partial}{\partial y^\mu}$.

A smooth mapping $s : U \to Y$ where U is an open subset of X is called the **section of the bundle** π if the composition of projector π and mapping s is the identity mapping of the domain U: $\pi \circ s = id_U$. In applications (in Physics and in Continuum Mechanics), functions $y^\alpha(x)$ represent the components of tensor fields which are the dynamical variables of the considered theory.

If the configurational bundle, π, is a *vector bundle* (see Appendix I), the fibers F_x are real vector spaces (isomorphic to the fixed vector space F^m) and the transition

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transformations (see Appendix I) are the linear mappings between the fibers at each point of the intersection of the domains of coordinate charts.

Quite often the configuration bundle is trivial so that the space Y is the product $Y = X \times F^m$ of the base X, and the standard fiber F^m and the bundle projection $\pi : Y = X \times F \to X$ is the projection onto the first factor.

Remark 1. Spaces (smooth manifolds) that appear in classical physics are *often* of one of the following types:

(1) An open subset $X \subset R^n$ or an open subset with the boundary ∂X,

(2) A k-dimensional submanifold $X^k \subset R^n$ with the boundary ∂X. A 2-dim surface with the boundary in the Euclidian space E^3 is an example.

(3) Compact manifolds - spheres, torus, etc.

Most base manifolds used in classical physics are of the first and second type. Manifolds of the third type usually appear in the problems of Geometry and of Gauge Field Theory, see [111]. Quite often X is the physical or material space-time and $n = 4$.

It is assumed that **the base manifold X is endowed with the Riemannian or pseudo-Riemannian metric** G. We denote by dv the volume n-form corresponding to the metric G. In local coordinates (U, x^i), metric G is presented by the non-degenerate symmetrical (0,2)-tensor field $G = G_{ij}(x)dx^i dx^j$ and the volume form dv has the form

$$dv = \sqrt{|det(G_{ij}(x))|}dx^1 \wedge \ldots \wedge dx^n.$$

3.2. **First orders Lagrangians and the Action functional.** A **Lagrangian Field Theory of order** 1 **with the configurational bundle** $\pi : Y^{n+m} \to X^n$ is defined by a Lagrangian - a function L on the **first jet bundle** $J^1\pi$ **of the configurational bundle** : $L \in C^\infty(J^1(\pi))$. The 1-jet bundle space $J^1\pi$ is fibred over Y and X ($J^1\pi \to Y \to X$). Jet bundle $J^1\pi$ carries, in its fibers over Y, the information about the first derivatives $y^\alpha_i = \frac{\partial y^\alpha}{\partial x^i}$ of the components y^α of the sections $s = \{y^i(x)\}$ of configurational bundle $\pi : Y \to X$ (see Appendix II).

We refer to Appendix II for the definition and basic properties of the 1-jet bundle of a fiber bundle. We remark that a fibred chart (V, x^i, y^α) in the configurational bundle $\pi : Y \to X$ defines the "lifted fibred chart" $(V^1, x^i, y^\alpha, y^\alpha_i)$ in the domain $V^1 = \pi^{-1}_{10}(V)$ of the 1-jet bundle $J^1(\pi)$

Remark 2. Mathematically, it is more natural to define Lagrangian as an n-form $\lambda = L(x^i, y^\alpha, y^\alpha_i)dv$. Here we mostly use the simple definition that is closer to the applications in physics.

Let $L \in C^\infty(J^1(\pi))$ be a Lagrangian of the first order and let $D \subset X$ be a domain in the base space X.

The **action functional** $\mathcal{A}_D(s)$ on the sections $s : D \to Y$ ($D \subset X$ being a domain in X) is defined by the integral

$$\mathcal{A}_D(s) = \int_D L(j^1 s(x))dv = \int_D L(x^i, y^\alpha(x), y^\alpha_{,x^i}(x))dv. \tag{3.1}$$

The **main postulate of Lagrangian Field Theory** is that the configurations of dynamical fields $s(x) = \{y^\alpha(x)\}$, realized in real situations, are the **critical**

points of the action functional. In many cases, real solution $s(x)$ delivers an extremum to the action functional between all configurations (geometrically - sections s of the bundle π that satisfy some additional conditions (boundary conditions, initial conditions, Lagrange conditions, etc).

In the next section we sketch the basic components of the formalism of Calculus of Variations - we introduce first variations and Euler-Lagrange Equations for the first order theory.

4. First Variation and the Euler-Lagrange system.

Consider a Lagrangian Field Theory of the first order with a Lagrangian $L(x^i, y^\alpha, y^\alpha_i)$, where $L \in C^\infty(J^1(\pi))$ is an infinitely differentiable function on the 1-jet space. Let $s(x) = \{y^\alpha(x)\} : U \to Y$ be a section of the configurational bundle π that delivers an extremum (minimum or maximum) of the action functional $A_U(s)$ (see (3.1)). The extremal property of the section s is determined by comparing the value of the action functional at the section s with its action at neighborhood sections $s(x) + \epsilon\xi(x)$. Thus, we need to introduce variations of a section s in a convenient form.

4.1. **First Variations.** Let $s : U \to Y$ be a section of the the configurational bundle $\pi : Y \to X$ defined in a open subset $U \subset X$. In local fibred coordinates, section $s(x)$ is presented by its components $s(x) = (s^\alpha(x^k),\ \alpha = 1, \dots, m)$.

Definition 1. (1) ***A variation*** *of section $s : D \to Y$ is the one-parameter family $t \to s_t(x)$ of local sections of the configurational bundle (defined in D) such that $s_0 = s$.*

(2) *Infinitesimal variation, corresponding to the variation $s_t(x)$ is the π-vertical vector field $\xi = \xi^\mu\partial_\mu$ along the image of section s (i.e. defined at the points $s(D) \subset Y$).*

Here and below, ∂_μ is the shortened notation for the partial derivative

$$\partial_\mu = \frac{\partial}{\partial y^\mu}.$$

$$\xi(s(x)) = \frac{d}{dt}_{|t=0} s_t(x). \tag{4.1}$$

Vice versa, let $\xi = \xi^\mu(x, y)\frac{\partial}{\partial y^\mu}$ be a vertical vector field defined in the domain $V \subset Y$ of a fibred chart (V, x^i, y^μ) and let ϕ^t be the phase flow of this vector field. Then, the association $t \to \phi^t(s(x))$ is the variation of a section $s(x)$.

Yet, in order to define variations of action $A_D(s)$ (see (3.1)), we have to define values of derivatives $\frac{\partial s^\alpha}{\partial x^i}$ for variated sections $s_t(x)$. A natural way to do this (called the flow prolongation) is to define variations of sections $s(x)$ by the phase flow of a vertical vector field $\xi(x, y)$, then **extend (lift) this flow to the 1-jet space** $J^1\pi$ and apply obtained 1-parametrical (local) group of transformations to the 1-jet $j^1 s(x)$ of section $s(x)$.

Let a variation of section s, given by the collection of components $s^\alpha(x)$ (dynamical fields), be defined by a π-vertical vector field $\xi = \xi^\alpha\partial_{y^\alpha} \in V(\pi)(U^0)$ (where $U^0 = \pi^{-1}U \subset Y$). In terms of components, an infinitesimal variation of section $s(x)$ has the form $s(x) = \{s^\alpha(x)\} \to \{s^\alpha(x) + \xi^\alpha(x, y)\epsilon\}$ for small ϵ.

Corresponding **variations of derivatives** entering the Lagrangian L and, therefore, the action functional

$$A_D(s) = \int_D L(j^1 s)dv \tag{4.2}$$

are determined by a **prolongation procedure** lifting vector fields ξ in Y to the vector fields in the 1-jet space $J^1(\pi)$.

$$\xi = \xi^\alpha \partial_\alpha \rightarrow \xi^1 = \xi + \xi^\alpha_i \partial_{y^\alpha_i}.$$

Remark 3. Flow prolongation procedure $\xi \rightarrow Pr^1(\xi)$ has the form (see Appendix II for more details)

$$Pr^1(\xi) = \xi^\alpha \partial_\alpha + \xi^\alpha_i \partial_{y^\alpha_i} = \xi^\alpha \partial_\alpha + d_i \xi^\alpha \partial_{y^\alpha_i}. \tag{4.3}$$

Here $\xi^\alpha_i = d_i \xi^\alpha = \partial_i \xi^\alpha + y^\beta_i \partial_\beta \xi^\alpha$ is the **total derivative** of the coefficients $\xi^\alpha \in C^\infty(J^1(\pi))$ by x^i. The second term in the expression (4.3) can be interpreted as the analog of *the derivative by x^i of the variation of the dynamical field* y^α, a component of $d\delta y^\alpha$.

The 1-jet component $\{\xi^\alpha_i\}$ of the prolonged vector field in $J^1(\pi)$ represent the variation of the derivative - $\delta y^\alpha_{,i}$.

The conventional rule "variation of the derivatives is equal to the derivative of variation" $\delta d y^\alpha = d\delta y^\alpha$ now takes the form of the following condition for a variational vector field $\xi = \xi^\alpha(x^i, y^\alpha, y^\alpha_i)\partial_{y^\alpha} + \xi^\alpha_i \partial_{y^\alpha_i}$:

$$\xi^\alpha_i(z) = \delta y^\alpha_i = d\delta y^\alpha = d_i \xi^\alpha. \tag{4.4}$$

Thus, **the commutativity rule of prolongation of variations to the first jet bundle is equivalent to the statement that the π-vertical variations $\xi = \xi^\alpha \partial_\alpha$ of dynamical fields $y^\alpha(x)$ are prolonged to the 1-jet space by the flow prolongation**.

This rule of prolongation, basic in the Variational Calculus with one-dimensional base (Mechanics), is expressed by the relation $\delta \dot{y}^\alpha = \dot{\delta y}^\alpha$, [44]. The same property is basic in the Lagrangian Field Theory, [44, 45, 106].

It is this rule that has been challenged in the works cited in the Introduction. So, in order to present the methods of B.Vujanovic, H.Kleinert and their coauthors in the geometrical form we have to study a natural modifications of the prolongation procedure (4.3). This includes the prolongation of the vector fields on the base X to the jet spaces which is imperative, for instance, for the study of symmetries of Euler-Lagrange equations and corresponding Noether *balance laws.*

Remark 4. Unfortunately, modifying the rule of variations of jet bundle variables, one has, in general, to sacrifice some properties of these variations that are taken for granted in the Variational Calculus. In Ch.3, Sec.18 we show that the basic properties of the flow prolongation - *Lie algebra morphism* and the preservation of the Cartan distribution are generically lost in the modified picture of lifting $\xi \rightarrow Pr^1_K(\xi)$. We also introduce the geometrical structures responsible for this loss and, in some sense, characterizing it. In Chapter 4 we describe the modification of Cartan distribution preservation property suggested by E.Pucci and G.Saccomandi ([115],). Their condition can be applied to a large class of prolongation procedures defined by the C.Muriel and J.Romano for ordinary differential equations and extended by the group of italian mathematicians to the systems of partial differential equations (see Chapter 4 and references therein).

4.2. **Euler-Lagrange Equations and natural boundary conditions.** Form the variation of the action $A_D(s)$ using this **flow prolongation** (4.3) of a vertical variational vector field $\xi = \xi^\alpha(x,y)\partial_\alpha$. Infinitesimal variations of fields and their derivatives (independent variables x^μ are not variated) have the form

$$\begin{cases} y^\alpha \to y^\alpha + \epsilon\xi^\alpha(x,y), \\ y^\alpha_i \to y^\alpha_i + \epsilon d_i\xi^\alpha. \end{cases}$$

Calculate the first variation of action:

$$\begin{aligned} &\Delta A_D(s)(\epsilon\xi) = A_D(s_\epsilon) - A_D(s) = \\ &= \int_D [L(x, s^\alpha(x) + \epsilon\xi^\alpha(x,s(x)), s^\alpha_{,i}(x) + \epsilon(d_i\xi^\alpha(x,s(x)))) - L(x,s(x),s^\alpha_{,i}(x))]dv = \\ &= \epsilon\int_D \left[\frac{\partial L}{\partial y^\alpha}\xi^\alpha(x,s(x)) + \frac{\partial L}{\partial y^\alpha_i}(d_i\xi^\alpha(x,s(x)))\right]dv + O(\epsilon^2) = \\ &= \epsilon\int_D \left[\frac{\partial L}{\partial y^\alpha} - d_i\left(\frac{\partial L}{\partial y^\alpha_i}\right)\right]\xi^\alpha(x,s(x))dv + \epsilon\int_{\partial D}\frac{\partial L}{\partial y^\alpha_i}\xi^\alpha n_i dS + O(\epsilon^2) \end{aligned} \tag{4.5}$$

First variation of the action $A_D(s)$ at a section $s(x) = (x^i, y^\mu(x))$ has the form

$$\delta A_D(s) = \int_D [\frac{\partial L}{\partial y^\alpha} - d_i\left(\frac{\partial L}{\partial y^\alpha_i}\right)]\xi^\alpha(x,s(x))\epsilon dv + \epsilon\int_{\partial D}\frac{\partial L}{\partial y^\alpha_i}\xi^\alpha n_i dS. \tag{4.6}$$

where the boundary term $\int_{\partial D}\frac{\partial L}{\partial y^\alpha_i}\xi^\alpha dS^{n-1}$ appears after integrating by parts.

If the variational vector field $\xi = \xi^\alpha\partial_\alpha$ vanishes on the boundary ∂D, the last term in the previous expression for the variation vanish and the arbitrariness of the variations ξ^α lead to the system of Euler-Lagrange equations in the form

$$E_\beta(L) = \frac{\partial L}{\partial y^\beta} - d_i\left(\frac{\partial L}{\partial y^\beta_i}\right) = 0,\ \beta = 1,\dots,m. \tag{4.7}$$

Here $d_i = \frac{\partial}{\partial x^i} + y^\mu_i\frac{\partial}{\partial y^\mu}$ is the total derivative by x^i (see Appendix II).

If, having this equation, we omit the condition that the variational vector fields ξ vanish on the boundary ∂D of the domain D together with its normal derivatives, we extend the class of admissible variations. If, in such a case, we integrate by parts in the expression for variation of action (4.2), the boundary integral in the sum (4.6)

$$\int_{\partial D} L_{,y^\alpha_i}(s)\xi^\alpha dS_i$$

appears to be nonzero.

. As a result, using at first, the variational vector fields that vanishes on the boundary ∂D, and then the general variations we get, in addition to the **Euler-Lagrange equations (4.7)**, the **natural boundary conditions**

$$L_{,y^\alpha_i}(s)\cdot n_i = \pi^i_\mu n_i = 0,\ \alpha = 1,\dots,m, \tag{4.8}$$

where $n_* = \{n_i\}$ is the covariant vector corresponding to the unit normal vector n^i on the boundary ∂D and $\pi^i_\mu = L_{,y^\alpha_i}(s)$ is the momentum (1,1)-tensor

. Thus, the natural boundary condition requires that the normal component of the momenta vanish.

4.3. **Symmetries and Noether Theorem.** Let $L(x^i, y^\mu, y^\mu_i)$ be a first order Lagrangian and let $G \subset Diff(Y)$ be a finite-dimensional Lie group of diffeomorphisms of the space Y that is, at the same time, the group (geometrical) symmetries of the Lagrangian L (see Appendix III, Sec. 77) where definitions and properties of groups of **variational** and **divergent** symmetries are presented). In particular, A Lie group G of (diffeomorphic) transformations of the manifold Y is a **group of divergent symmetries of Lagrangian** L if the infinitesimal condition (78.7) (see Appendix III, Sec.78) is fulfilled.

Locally, in terms of fibred coordinates (x^i, y^μ), transformations ϕ of the space Y, corresponding to the elements $g \in G$, have the form

$$\phi : (x^i, y^\mu) \to (\bar{\phi}^i(x^j, y^\nu),), \phi^\mu(x^j, y^\nu)), \tag{4.9}$$

with $\bar{\phi}^i, \phi^\mu(x, y)$ being smooth functions of corresponding variables.

An important special case of geometrical transformations is the case of **projectable** transformations ("automorphisms of the bundle $\pi : Y \to X$"). In fibred coordinates, projectable transformations, are characterized by the condition $\bar{\phi}^i = \bar{\phi}^i(x^j)$ in (4.9).

Infinitesimally, Lie algebra $\mathfrak{g}$ of the group G of geometrical transformations is formed by the vector fields in Y, $\zeta = \zeta^i(x, y)\partial_{x^i} + \zeta^\mu(x, y)\partial_{y^\mu}$, while the Lie algebra of a group of automorphisms of π formed by the projectable vector fields $\zeta = \zeta^i(x)\partial_{x^i} + \zeta^\mu(x, y)\partial_{y^\mu}$. In the case of projective transformations, transformations $\phi \in G$ generate (by taking projections) the group $\bar{G}$ of transformations $\bar{\phi}$ of the base X.

Notice that the defining property of a one-parameter group ϕ^t to be the group of symmetries of Lagrangian L is that the action of fase field transformations $\phi^t,\ t \in R$ transform solutions of Euler-Lagrange Equations to the other solutions of the same Euler-Lagrange system.

Therefore, infinitesimal (phase) fields of such one-parameter groups act as the infinitesimal variations of the action $A_D(s)$ corresponding to L. Thus, in order to realize this symmetry of the action $A_D(s)$, geometrical symmetries (acting in Y) should be lifted to the 1-jet bundle $J^1(\pi)$ by the same procedure $\xi \to Pr^1(\xi)$ as the variational vector fields.

Let $\zeta \in \mathfrak{g}$ be an arbitrary element of Lie algebra $\mathfrak{g}$. Let $Pr^1(\zeta)$ be the flow prolongation of vector field ζ to the 1-jet bundle $J^1(\pi)$. Then, the vector field ζ is the **infinitesimal divergent symmetry** of Lagrangian, that generates the (at least local) one-parameter group of symmetries of L - phase flow) if and only if there exists a horizontal 1-form $B = B_k(x^i, y^\mu, y^\mu_i)dx^i$ in $J^1(\pi)$ such that

$$pr^1(\zeta)L + Ldiv(\bar{\zeta}) = div(B), \tag{4.10}$$

see (80.90) or [106], Chapter 4 for more details.

Divergence here is defined using the volume form dv defined by the metric g in the base X.

Now we formulate the Noether Theorem for the variational and the divergent symmetry groups. See Appendix III for the proof of this Theorem.

Theorem 1. *Let L be a Lagrangian of order k and let*

$$E_{\alpha}(L) = 0,\ \ \alpha = 1, \ldots, m, \tag{4.11}$$

be an Euler-Lagrange system with the Lagrangian L of order k. Let ξ be a vector field in Y i an infinitesimal variational symmetry of Lagrangian L.

Let $Pr^k(\xi)$ be the prolongation of vector field ξ of order k - to the k-jet bundle $J^k\pi$.Then there exist an n-tuple of the smooth functions P^i such that for some functions $A \in C^{\infty}(J^k\pi$ the following equality (Noether conservation law) is fulfilled

$$Div(A + L\xi) = -Q^{\mu} E_{\mu}(L). \tag{4.12}$$

As a result, for all solutions y of the Euler Lagrange system of equations,

$$Div(P)(y) = 0. \tag{4.13}$$

For the proof of this theorem, see [106], Ch.4 or here, Appendix III, Section 79. par

If a Lie group acting on the space Y is a group of *divergent symmetries* (see Appendix III), relation (4.12) is replaces by the relation

$$Div(A + L\xi) = +Q^{\mu} E_{\mu}(L) = Div(B), \tag{4.14}$$

for the n-tuple of functions $B_i \in C^{\infty}(J^k\pi)$.

As a result, conservation law (4.13) in Theorem 1 for the solutions of Euler Lagrange system holds with the following modification:

$$P = B - A - L\xi. \tag{4.15}$$

corresponding **Noether conservation law** for the solution of Euler-Lagrange equations has the form (comp. (79.7).)

$$Div(P) = 0, \text{where } P^i = \zeta^{\mu} L_{,y^{\mu}_i} + \zeta^i L - \zeta^j y^{\nu}_j L_{,y^{\mu}_i} - B. \tag{4.16}$$

As an example, illustrating the Noether method, to associating conservation (or balance) laws to the symmetry Lie groups of transformations, we present the Stress-Energy-Momentum balance law for the Lagrangian Field Theory.

4.4. **Energy-Momentum balance law.** Here we write down the canonical stress-energy-momentum (CEM) balance law corresponding to a Lagrangian $L \in C^{\infty}(J^k(\pi))$. We will be using the approach of [106], modified to produce the balance law, see Appendix III.

We take the base space $X = R^4$ to be the physical space-time endowed with standard Euclidian or standard Lorentz metric.

In this case $\xi = \xi_k = \partial_{x^k}$ lifted to Y by a connection Γ in the bundle π: $\partial_{x^k} \to \hat{\xi}_k = \partial_{x^k} + \Gamma^{\mu}_k \partial_{\mu}$. Then, the characteristic of the vector field $\hat{\xi}_k$ has the components $Q^{\mu} = \Gamma^{\mu}_k - y^{\mu}_k$ and we define the Energy-Momentum Tensor in its standard form [80].

$$T^i_k = L\delta^i_k - (\Gamma^{\mu}_k - y^{\mu}_k) L_{,y^{\mu}_i}. \tag{4.17}$$

As a result the balance equation 81.4 corresponding to this vector field (*stress-energy-momentum "balance" law*) has the form

$$d_i T^i_k = d_i \left(L\delta^i_k - y^{\mu}_k L_{,y^{\mu}_i} \right) = -\frac{\partial L}{\partial x^k}_{expl.},\ \ k = 0, 1, 2, 3. \tag{4.18}$$

If L does not depend explicitly on x^k, this balance law becomes the "conservation law".

In particular, for $k=0$ we get the energy balance law (or conservation law if $\frac{\partial L}{\partial x^0}_{expl.}=0$)

$$d_i T^i_0 = d_i\left(L\delta^i_0 - y^\sigma_0 L_{,y^\sigma_i}\right) = -\frac{\partial L}{\partial x^0}_{expl.}. \tag{4.19}$$

Notice the linear dependence of the second term in T^i_0 on the *velocities* y^σ_0 and on the *momenta* $-\pi^i_\sigma = L_{,y^\sigma_i}$.

4.5. **General variations, group of automorphisms of the configurational bundle: case when $\phi_t \in Aut(\pi)$..** In order to define the Euler-Lagrange equations (4.7) and the natural boundary conditions (4.8) corresponding to an action (4.2), it is sufficient to use m *independent vertical variations* generated by one-parameter groups of automorphisms $\phi_t \in Aut(\pi)$ of the configurational bundle $\pi : Y \to X$ (See Appendix I) **acting along the fibers** Y_x. In local fibred coordinates (W, x^i, y^μ) such authomorphisms have the representation $(x,y) \to (x, \phi^\mu_t(x,y))$. Infinitesimal variations of this type have the form

$$\xi = \phi^\mu(x,y)\partial_\mu.$$

Such variations are sometimes called the "**outer variations**", [47], Ch.3.

On the other hand, in order to get conservation laws related to the Noether symmetries of the Lagrangian, one has to use vector fields of as general type as possible, therefore including variations of independent variables x^i - **inner variations**. In particular, the energy-momentum balance law (4.16) appears from applying the variations ∂_i generated by the translation of independent variables - space-time coordinates t, x^1, x^2, x^3.

Thus, it is interesting to see a result of variations of action (4.2) generated by the one-parameter groups ϕ_t of *general automorphisms* of the configurational bundle π:

$$(x,y) \to (\bar\phi^i(x), \phi^\mu(x,y)). \tag{4.20}$$

Transformations of such one-parameter groups act on the sections as follows: $s(x) \to \phi_t s(\bar\phi_{-t})$.

In the infinitesimal form, the variational vector field corresponding to such 1-parameter subgroup is

$$\xi = -\xi^i(x)\partial_i + \xi^\mu(x,y)\partial_\mu, \tag{4.21}$$

and its flow prolongation to the 1-jet bundle $J^1\pi$ is

$$Pr^1(\xi) = \xi + (d_i\xi^\mu - y^\mu_j d_i\xi^j)\partial_{y^\mu_i} = (\xi^i\partial_i - y^\mu_j d_i\xi^j\partial_{y^\mu_i}) + (\xi^\mu\partial_\mu + d_i\xi^\mu\partial_{y^\mu_i}). \tag{4.22}$$

Using such variations of the action (1.1) and the standard flow prolongation (4.3) of vector field ξ to the 1-jet bundle $J^1(\pi)$, we obtain for the first variation of the action

$$\delta A_D(s) = \epsilon\int_D \left[\frac{\partial L}{\partial y^\alpha} - d_i(\frac{\partial L}{\partial y^\alpha_i}\right)]\xi^\alpha(x,s(x))dv + \epsilon\int_{\partial D}\frac{\partial L}{\partial y^\alpha_i}\xi^\alpha n_i dS+$$
$$+\epsilon\int_D (L_{,x^k}(-\xi^k) + L_{,y^\mu_i}y^\mu_j d_i(\xi^j) + L(-\xi^i_{,x^i}))dv. \tag{4.23}$$

Notice that the trace $\xi^i_{,x^i}$ of the Jacobi matrix $\xi^i_{,x^j}$ in the last term appears due to the action of transformation $\bar\phi^i(x)$ on the volume form dv.

Integrating by parts the terms containing derivatives of the components of the vector fields in the last integral we get the expression for the first variation of action A_D for an arbitrary variation $\phi_t \in Aut(\pi)$,

$$\delta A_D(s) = \epsilon \int_D [\frac{\partial L}{\partial y^\alpha} - d_i\left(\frac{\partial L}{\partial y^\alpha_i}\right)]\xi^\alpha(x, s(x))dv + \epsilon \int_{\partial D} \frac{\partial L}{\partial y^\alpha_i}\xi^\alpha n_i dS + \\ + \epsilon \int_D [L\delta^j_i - L_{,x^k}\delta^k_i + d_j\left(y^\mu_i L_{,y^\mu_j}\right)]\xi^i dv - \epsilon \int_D d_i(L_{,y^\mu_i} y^\mu_j \xi^j)dv. \quad (4.24)$$

Recall the form of the energy-momentum tensor (see 4.14) $T^i_j = L\delta^j_i - y^\mu_j L_{,y^\mu_i}$.

Requiring that the first variations of action $A_D(s)$ vanish at any variation of general type, and using the independence of variations ξ^μ and ξ^i, we get, in addition to the Euler-Lagrange equations (4.7) and the natural boundary conditions (4.8), one more equation - the **energy-momentum balance law**-

$$d_j\left(L\delta^j_i - y^\mu_i L_{,y^\mu_j}\right) = -L_{,x^i} \quad (4.25)$$

in the domain D, and the additional condition

$$L_{,y^\mu_i} y^\mu_j \xi^j \cdot n^i = 0 \quad (4.26)$$

on the boundary ∂D. This condition is the consequence of the natural boundary condition (4.8).

We refer to [47],Ch.3 for more details about the properties of **inner** and **outer** variations in the smooth situations, the notion of **inner extremals** and their relations to the usual (outer) extremals. We also refer to the article ([49]) and to the references therein for the exposition of the use of inner and outer variations in the important case of non-smooth (Lipschitz) variations and examples of non-smooth minimizers in applications.
[1]

[1]I would like to thank L.Truskinovsky who has sent to me this paper containing an example of non-commuting inner and outer variations in a situation with Lipschitz variations.