Tepper L. Gill · Woodford W. Zachary

Functional Analysis and the Feynman Operator Calculus

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Preface

Two approaches to the mathematical foundations of relativistic quantum theory began in the USA. Both evolved from the application of quantum field methods to electron theory in the late 1940s by Feynman, Schwinger, and Tomonaga (see [SC1]).

The first program is well known and was begun in the early 1950s by Professor A.S. Wightman of Princeton University (1922–2013). Following a tradition inspired by Hilbert, the program was called axiomatic field theory. It sought to provide rigorous justification for the complicated and difficult method of renormalization successfully employed by the physics community (see [SW] and [GJ]). Professor Wightman is considered the founding father of modern mathematical physics, but he also strongly influenced a number of other areas in mathematics.

In 1982, Sokal notice some difficulties with the constructive approach to field theory (the concrete version of axiomatic field theory) and conjectured that this approach may not work as expected in four space-time dimensions (see [SO]). His conjecture was later verified by Aizenman and Graham [AG] at Princeton and Fröhlich [FO] at ETS, Zurich. These results have had a damping effect on research in this direction.

In response to the work of Aizenman, Graham, and Fröhlich, a second, less well-known program was initiated by the present authors at

Howard University in 1986. We sought to understand the issues affecting relativistic quantum theory based on a series of problems suggested by Dirac, Dyson, Feynman, Schwinger, and other major architects of quantum field theory. This book is an outgrowth of our investigations into the mathematical issues facing any attempt to develop a reasonable relativistic quantum theory. Our investigations into the physical foundations are the subject of a future project. (However, those with interest in this subject are directed to [GZ5] and [GMK, see Chap. 5] for some partial results in this direction.)

In 1951, Richard Feynman published what became known as the Feynman operator calculus. It served as the basis for his formulation of quantum electrodynamics, for which he shared the Nobel Prize in Physics with Schwinger and Tomonaga. Freeman Dyson introduced this work to the mathematics and physics communities, providing Feynman's theory both the physical and mathematical legitimacy. Dyson also showed that the two competing formulations of quantum electrodynamics were based on different representations of Heisenberg's S-matrix. Using his understanding of both theories, Dyson made fundamental improvements and simplifications. (It is suggested by Schweber [SC1] that Dyson's contribution is also worthy of the Nobel Prize.)

Feynman's basic idea was to first lay out space-time as one would a photographic film. He then imagined the evolution of a physical system appearing as a three-dimensional motion picture on this film; one seeing more and more of the future as more and more of the film comes into view (see [F]). This gives time its natural role in ordering the flow of events as it does in our conscious view of reality. Feynman suggested that time should serve this role in the manipulation of operator-valued variables in quantum field theory, so that operators acting at different times actually commute. He demonstrated that this approach made it possible to write down and compute highly complicated expressions in a fast and effective manner. In one case, he was able to perform a calculation in one night that had previously taken over 6 months (see [SC1]).

Feynman's faith in his operator calculus is expressed at the end of his book on path integrals (with Hibbs [FH]); he states: "Nevertheless, many of the results and formulations of path integrals can be re-expressed by another mathematical system, a kind of ordered operator calculus. In this form many of the results of the preceding chapters find an analogous but more general representation . . . involving noncommuting variables." Feynman is referring to [F], quoted above.

To our knowledge, Fujiwara [FW] is the only physicist other then Dyson who takes Feynman's operator calculus seriously in the early literature (1952). Fujiwara agreed with the ideas and results of Feynman with respect to the operator calculus, but was critical of what he called notational ambiguities, and introduced a slightly different approach. "What is wanted, and what I have striven after, is a logical well ordering of the main ideas concerning the operator calculus. The present study is entirely free from ambiguities in Feynman's notation, which might obscure the fundamental concepts of the operator calculus and hamper the rigorous organization of the disentanglement technique." Fujiwara's main idea was that the Feynman program should be implemented using a sheet of unit operators at every point except at time t, where the true operator should be placed. He called the exponential of such an operator an expansional to distinguish it from the normal exponential so that, loosely speaking, disentanglement becomes the process of going from an expansional to an exponential. (Araki [AK] formally investigated Fujiwara's suggestion.) As will be seen, Fujiwara's fundamental insight is the centerfold of our approach to the problem.

In our approach, the motivating research philosophy was that, the correct mathematical foundation for the Feynman operator calculus should in the least:

(1) Provide a transparent generalization and/or extension of current mathematical theories without sacrificing the physically intuitive and computationally useful methods of Feynman

(2) Provide a rigorous foundation for the general theory of path integrals and its relationship to semigroups of operators and partial differential equations

(3) Provide a direct approach to the mathematical study of time-dependent evolution equations in both the finite and infinite-dimensional setting

(4) Provide a better understanding of some of the major mathematical and physical problems affecting the foundations of relativistic quantum theory

This book is devoted to the mathematical development of the first three items. We also briefly discuss a few interesting mathematical points concerning item (4). (However, as noted earlier, a full discussion of (4) is delayed to another venue.)

While no knowledge of quantum electrodynamics is required to understand the material in this book, at a few junctures, some physical intuition and knowledge of elementary quantum mechanics would be helpful. We assume a mathematical background equivalent to that of a third year graduate student, which includes the standard courses in advanced analysis, along with additional preparation in functional analysis and partial differential equations. A course (or self-study) based on the first volume of Reed and Simon [RS1, see Chap. 1] offers a real advantage. An introduction to probability theory or undergraduate background in physics or chemistry would also be valuable. In practice, unless one has acquired a reasonable amount of mathematical maturity, some of the material could be a little heavy going. (Mathematical maturity means losing the fear of learning topics that are new and/or at first appear difficult.) However, in order to make the transition as transparent as possible, for advanced topics we have provided additional motivation and detail in many of the proofs.

We have three objectives. The first two, the Feynman operator calculus and path integrals and their relationship to the foundations of relativistic quantum theory, occupy a major portion of the book. Our third objective, infinite-dimensional analysis, provides the purely mathematical background for the first two. We have also included some closely related material that has independent interest. In these cases, we also indicate and/or direct the interested reader to the Appendix.

The book is organized in a progressive fashion with each chapter building upon the previous ones. Almost all of the material in Chaps. 2, 3, and 6–8 has not previously appeared in book form. In addition, Chap. 5 is developed using a completely new approach to operator theory on Banach spaces, which makes it almost as easy as the Hilbert space theory.

Chapter 1 is given in two parts. Part I introduces some of the background material, which is useful for review and reference. Basic results and definitions from analysis, functional analysis, and Banach space theory are included and should at least receive a glance before proceeding.

Part II is devoted to the presentation of a few advanced topics which are not normally discussed in the first 2 years of a standard graduate program, but are required for later chapters in the book. The reader should at least review this part to identify unfamiliar topics, so one may return when needed.

Chapter 2 is devoted to the foundations for analysis on spaces with an infinite number of variables. Infinite dimensional analysis is intimately related to the Feynman operator calculus and path integrals and cannot be divorced from any complete study of the subject. Faced directly, the first problem encountered is the need for a reasonable version of Lebesgue measure for infinite-dimensional spaces. However, research into the general problem of measure on infinite-dimensional vector spaces has a long and varied past, with participants living in a number of different countries, during times when scientific communication was constrained by war, isolation, and/or national competition. These conditions have allowed quite a bit of misinformation and folklore to grow up around the subject, so that even some experts have a limited view of the subject. Yamasaki was the first to construct a σ-finite version of Lebesgue measure on $\mathbb{R}^\infty$ in 1980 (see [YA1]), and uniqueness has only been proved recently (2007) by Kirtadze and Pantsulaia [KP2, see Chap. 6]. However, due to the nature of their approach, the work of Yamasaki and Kirtadze and Pantsulaia is only known to specialists in the field.

In Sect. 2.1 the Yamasaki version of Lebesgue measure for $\mathbb{R}^\infty$ is constructed in a manner which is very close to the way one learns measure theory in the standard analysis course. In Sect. 2.2, a version of Lebesgue measure is constructed for every Banach space with a Schauder basis (S-basis). In addition, a general approach to probability measures on Banach spaces is developed. The main result in this direction is that every probability measure ν on $\mathfrak{B}[\mathbb{R}]$ with a density induces a corresponding related family of probability measures $\{\nu_{\mathcal{B}}^n\}$ on every Banach space $\mathcal{B}$, with an S-basis, which is absolutely continuous with respect to Lebesgue measure. Under natural conditions, the family converges to a unique measure $\nu_{\mathcal{B}}$. As particular examples, we prove the existence of universal versions of both the Gaussian and Cauchy measures. Section 2.3 is devoted to measurable functions, the Lebesgue integral, and the standard spaces of functions, continuous, L^p, etc. Section 2.4 studies distributions on uniformly convex Banach spaces. Section 2.5 introduces Schwartz space and the Fourier transform on uniformly convex Banach spaces. This allows us to extend the Pontryagin Duality Theorem to uniformly convex Banach spaces in Sect. 2.6. In addition, we provide a direct solution to the diffusion equation on Hilbert space as an interesting application of our universal representation for Gaussian measure. Sections 2.4–2.6 are not required for a basic understanding of the Feynman operator calculus and the theory of path integrals. However, there are natural connections between these subjects. Thus, those with broader concerns and/or interests in other applications will find the study both rewarding and fruitful.

Chapter 3 introduces the Henstock–Kurzweil integral. This is the easiest to learn and best known of those integrals that integrate nonabsolutely integrable functions and extend the Lebesgue integral. Section 3.1 provides a fairly detailed account of the HK-integral and its properties in the one-dimensional case and a brief discussion of the n-dimensional case. Section 3.2 discusses a new class of Banach spaces (KS^p spaces) that are for nonabsolutely integrable functions as the L^p spaces are for Lebesgue integrable functions. These spaces contain the L^p spaces as continuous dense and compact embeddings. Section 3.3 covers some additional classes of Banach spaces associated with nonabsolutely integrable functions which may have future interest. First, we define an important class of spaces $SD^p[\mathbb{R}^n]$, $1 \leqslant p \leqslant \infty$. These spaces contain the test functions of Schwartz [SCH] $\mathcal{D}[\mathbb{R}^n]$, as a dense continuous embedding. In addition, they have the remarkable property that for any multi-index α, $\|D^\alpha \mathbf{u}\|_{SD} = \|\mathbf{u}\|_{SD}$, where D is the distributional derivative. We call them the Jones strong distribution Banach spaces. As an application, we obtain a nice a priori estimate for the nonlinear term of the classical Navier–Stokes initial-value problem. In Sect. 3.4, we introduce a class of spaces in honor of our deceased colleague Woodford W. Zachary. These spaces all extend the class of functions of bounded mean oscillation to include the HK-integrable functions. (Sections 3.3 and 3.4 are not required for the rest of the book.)

Chapter 4 is devoted to a fairly complete account of analysis and operator theory on Hilbert space. The first part introduces the theory of integration of operator-valued functions, and the second part gives a first course in Hilbert space operator theory. The presentation is standard, but an interesting extension of spectral theory is introduced based on the polar decomposition property of closed densely defined linear operators.

Chapter 5 is devoted to operator theory on Banach spaces, with major emphasis on semigroups of operators. Our approach is novel, as it uses the theory of Chap. 4 in a unique manner, showing that the theory on Banach spaces is much closer to the Hilbert space theory then previously known. In the first section we show that, for uniformly convex Banach spaces with a Schauder basis, it is possible to define the adjoint for every closed densely defined linear operator on the space. (This result is extended to a larger class of spaces and operators in the Appendix (Sect. 5.3).) We give a number of examples so that one can see what the adjoint looks like in concrete cases. In the second section, the adjoint is used to give a parallel treatment of semigroups of operators, which is very close to the Hilbert space theory. In the Appendix (Sect. 5.3), in addition to an extension of the adjoint, we extend the spectral theory and provide a complete version of the Schatten classes of compact operators for uniformly convex Banach spaces with a Schauder basis.

Chapter 6 develops infinite tensor product theory for Hilbert and Banach spaces. The Banach space theory is a new subject, which offers a number of advantages for analysis. Our approach generalizes von Neumann's infinite tensor product Hilbert space theory, so we call them spaces of type v. We use infinite tensor products of Hilbert and Banach spaces to construct the mathematical representation for Feynman's physical film. We also introduce the notion of an exchange operator, which will prove important in Chaps. 7 and 8. (Infinite tensor products of Banach spaces are also natural for the constructive study of analysis in infinite-many variables. We have included a few applications and possibilities in the Appendix (Sect. 6.7).)

In Chap. 7, we develop the Feynman operator theory on Hilbert space, as a compromise for the two classes of potential users. Following Fujiwara's idea, we first define what we mean by time-ordering, prove our fundamental theorem on the existence of time-ordered integrals, and extend the basic semigroup theory to the time-ordered setting. This provides, among other results, a time-ordered version of the Hille–Yosida Theorem. We construct time-ordered evolution operators and prove that they have all the expected properties. We define what is meant by the phrase "asymptotic in the sense of Poincaré" for operators. We then develop a general perturbation theory and use it to prove a generalized version of Dyson's second conjecture for quantum electrodynamics, namely, that all theories generated by semigroups are asymptotic in the operator-valued sense of Poincaré. (Dyson conjectured this result for unitary groups.)

In 1955, Hagg [HA, see Chap. 7] investigated the general conditions which a relativistic quantum theory of interacting particles must satisfy in order to be made mathematically rigorous. One of his major conclusions was that the canonical commutation relations need not have unique solutions and that the interaction representation in sharp time does not exist. It has now been experimentally confirmed that there is quantum interference in time (see Chap. 7, [HW]). Thus, Hagg's assumption of sharp time is not physically valid. In this section, we modify Dyson's theory to include an interaction representation which allows time interference of wave packets. Finally, we show that the Fujiwara–Feynman approach to disentanglement can be implemented in a direct manner. This approach also provides a nice extension to the Trotter–Kato perturbation theory. In the last section we develop a general approach to the mathematical foundations for Feynman's sum over paths, which is used in quantum theory.

Chapter 8 provides a few applications of the operator calculus. We first develop a general theory for time-dependent parabolic and hyperbolic evolution equations. We demonstrate that the operator calculus allows us to unify methods and weaken domain requirements.

We then turn to the Feynman path integral. At this time, there is an extensive literature on the development and application of path integral methods in all aspects of physics, chemistry, mathematics, and engineering, and it is impossible to provide a reasonable discussion of these efforts. As a substitute, we provide references to some of the important works on this subject and introduce a number of interesting examples which are not covered in the literature. Our focus is on the mathematical foundations. We first demonstrate that the Kuelbs–Steadman space, $KS^2[\mathbb{R}^3]$, allows us to construct the elementary path integral in exactly the manner suggested by Feynman. Thus, our approach does not encumber physical intuition or computational efficiency. We further show that $KS^2[\mathbb{R}^3]$ is sufficient to provide a rigorous foundation for the Feynman formulation of quantum mechanics.

In order to further extend our theory, we introduce some results due to Maslov and Shishmarev on hypoelliptic pseudodifferential operators that allow us to construct a general class of path integrals generated by Hamiltonians, which are not perturbations of Laplacians (see Shishmarev [SH]). We then use the results of Chap. 7 and our sum over path theory to generalized and extend the well-known Feynman–Kac Theorem. Our final result is independent of the space of continuous functions, so that the question of the existence of measures is more of a desire than a requirement. (The strong continuity of the underlying semigroup ensures us that, whenever a measure exists, our theory can be easily restricted to the space of continuous paths.) In the last section, we provide a proof of the last remaining conjecture of Dyson, concerning the cause for the ultraviolet divergency of quantum electrodynamics.

Although our major focus is functional analysis and the Feynman operator calculus, it is clear from the topics covered that the book has much to offer for those with general research interests in both pure and applied mathematics. The book can be used as a text for advanced courses in analysis, functional analysis, operator theory, mathematical physics, mathematical foundations of quantum theory, or special topic seminars in these or related subjects.

Those with advanced training in quantum theory, who mainly work on Hilbert spaces, could study the first part of Sect. 3.2 and the proof of Theorem 3.25 in Chap. 3. A review of the first two subsections of Chap. 6, Sect. 6.5.1 of Sect. 6.5, and Sect. 6.6 would be sufficient to understand Chap. 7 and the main section on path integrals in Chap. 8.

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This book is dedicated to the Reverend LaVerne M. Gill and to the memory of Professors Gerald Chachere, Albert Turner Bharucha-Reid, George R. Sell, and Woodford William Zachary.

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Chapter 1

Preliminary Background

This chapter is composed of two parts: Basic Analysis and Intermediate Analysis.

The first part is a review of some of the basic background that is required from the first 2 years of a standard program in mathematics. There are program differences so that some areas may receive more coverage while others receive less. Our purpose is to provide a reference point for the reader and establish notation. In a few important cases, we have provided proofs of major theorems. In other cases, we delayed a proof when a more general result is proven in a later chapter.

In the second part of this chapter, we include some intermediate to advanced material that is required later. In most cases, motivation is given along with additional proof detail and specific references.

Part I: Basic Analysis

The first part of this chapter is devoted to a brief discussion of the circle of ideas required for advanced parts of analysis and the basics of operator theory. Those with a strong background in theoretical chemistry or physics but little or no formal training in analysis will find Reed and Simon (vol.1) to be an excellent copilot (see below).

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General references for this section are Dunford and Schwartz [DS], Jones [J], Reed and Simon [RS], Royden [RO], and Rudin [RU].

1.1. Analysis

1.1.1. Sets. Let X be a nonempty set, let $\emptyset$ be the emptyset, and let $\mathcal{P}(X)$ be the power set of X (i.e., the set of all subsets of X).

Definition 1.1. Let $A, B, A_n \in \mathcal{P}(X), n \in \mathbb{N}$, then

(1) $A^c = \{a \in X : a \notin A\}$, the compliment of A.

(2) $A \backslash B = A \cap B^c$.

(3) (De Morgan's Laws)

$$\left[\bigcup_{k=1}^{\infty} A_k\right]^c = \bigcap_{k=1}^{\infty} A_k^c, \qquad \left[\bigcap_{k=1}^{\infty} A_k\right]^c = \bigcup_{k=1}^{\infty} A_k^c.$$

We define the lim inf and lim sup for sets by:

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty} A_k \qquad \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k.$$

Theorem 1.2. *Let* $\{A_n\} \subset \mathcal{P}(X), n \in \mathbb{N}$, *then the* lim inf and lim sup *satisfy:*

(1)
$$\liminf_{n\to\infty} A_n \subset \limsup_{n\to\infty} A_n.$$

(2)
$$\limsup_{n\to\infty} A_n = \{a : a \in A_k \text{ for infinitely many } k\}.$$

(3)
$$\liminf_{n\to\infty} A_n = \{a : a \in A_k \text{ for all but finitely many } k\}.$$

(4)
$$(\limsup_{n\to\infty} A_n)^c = \liminf_{n\to\infty} A_n^c.$$

(5) *If* $A_n \supset A_{n+1}$, *then*

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n = \bigcap_{k=1}^{\infty} A_k.$$

(6) *If* $A_n \subset A_{n+1}$, *then*

$$\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n = \lim_{n\to\infty} A_n = \bigcup_{k=1}^{\infty} A_k.$$

Definition 1.3. Let $A, B \subset X$. (We assume they are nonempty.)

(1) The cartesian product, denoted $A \times B$, is defined by

$$A \times B = \{(a, b) : a \in A,\ b \in B\}.$$

In general, $A \times B \neq B \times A$, so that the order matters. If $\{A_k\}$ is a countable collection of subsets of X, we define the cartesian product by:

$$\prod_{k=1}^{\infty} A_k = \{(a_1, a_2, \ldots) : a_k \in A_k\}.$$

Definition 1.4. A map $f : A \to B$ (a function, or a transformation), with domain $D(f) \subset A$ and range $R(f) \subset B$ is a subset $f \subset A\times B$ such that, for each $x \in A$, there is one and only one $y \in B$, with $(x, y) \in f$. We write $y = f(x)$ and call $f(A) = \{f(x) : x \in A\} \subset B$, the image of f and, call $f^{-1}(B) = \{x : f(x) \in B\} \subset A$, the inverse image of B. We say that f is one to one or injective, if for all $x_1 \neq x_2 \in A$, we have that $y_1 = f(x_1) \neq y_2 = f(x_2) \in B$. We say that f is onto or surjective if, for each $y \in B$, there is a $x \in A$, with $y = f(x)$.

1.1.2. Topology. We only consider Hausdorff spaces or spaces with the Hausdorff topology (see below). For an elementary introduction to topology, we recommend Mendelson [ME]. Dugundji [DU] is more advanced, but is also worth consulting.

Definition 1.5. Let X be a nonempty set and let τ be a set of subsets of X. We say that τ defines a Hausdorff topology on X, or that X is Hausdorff, if

(1) X and $\emptyset \in \tau$.

(2) If $O_1, \ldots, O_n$ is a finite collection of sets in τ, then $\bigcap_{i=1}^{n} O_i \in \tau$.

(3) If Γ is a index set and, for each $\gamma \in \Gamma$, there is a set $O_\gamma \in \tau$, then $\bigcup_{\gamma\in\Gamma} O_\gamma \in \tau$.

(4) If $x, y \in X$ are any two distinct points, there are two disjoint sets $O_1, O_2 \in \tau$ (i.e., $O_1 \cap O_2 = \emptyset$), such that $x \in O_1$ and $y \in O_2$.

We call the collection τ the open sets of the topology for X. A set $N \in \tau$ is called a neighborhood for each point $x \in N$, and the set $\tau_x \subset \tau$ of all neighborhoods for x is called a complete neighborhood basis for x. Thus, any set O, containing x, also contains some neighborhood basis set $N(x) \in \tau_x$.

A set P is said to be closed if P^c is open. It follows that, if Γ is any index set and, for each $\gamma \in \Gamma$, there is a closed set $P_\gamma \in \tau$, then by De Morgan's Law, $\bigcap_{\gamma\in\Gamma} P_\gamma$ is also closed. Thus, we can also define the <u>same</u> topology τ, using closed sets.

Let $M \neq \emptyset$, be a subset of X.

(1) The interior of M, denoted $\text{int}(M)$, is the union of all $O \in \tau$ such that $O \subset M$. If $x \in \text{int}(M)$, we say that x is an interior point of M.

(2) The closure of M, which we denote by $\overline{M}$, is the set of all $x \in X$ such that, for all $N(x) \in \tau_x$, $N(x) \cap M \neq \emptyset$.

(3) We say that M is dense in X if $\overline{M} = X$. If M is also countable, we say that X is separable.

If M and N are any two subsets of X, then $\overline{M \cup N} = \overline{M} \cup \overline{N}$ and, $\overline{M} = M$ if and only if M is closed.

We say that $x_0 \in X$ is a limit point of $M \subset X$, if $x_0 \in \overline{M\backslash\{x_0\}}$ or equivalently, for every $N(x_0) \in \tau_{x_0}$, there is a $y \in N(x_0)$ and $y \notin M$.

Definition 1.6. Let (X_1, τ_1) and (X_2, τ_2) be two Hausdorff spaces. A function f, with $D(f) = X_1$ and $R(f) \subset X_2$, is said to be continuous at a point $x \in X_1$ if, for each neighborhood basis set $N[f(x)] \in \tau_{2,x}$, there is a neighborhood basis set $N(x) \in \tau_{1,x}$ such that $f[N(x)] \subset N[f(x)]$. In terms of inverse images, this says that $f^{-1}\{N[f(x)]\}$ is open in X_1 for each $N[f(x)]$ in X_2. (A little reflection shows that the above definition may be translated to the one we learned in elementary calculus, using ε's and δ's, when $X_1 = X_2 = \mathbb{R}$.) We say that f is continuous on X_1 if it is continuous at each point of X_1.

The topological space (X, τ) is said to be connected if it is not the disjoint union of two open sets. In a connected space X and $\emptyset$ are the only two sets that are both open and closed.

If Γ is a index set, $\{A_\gamma : \gamma \in \Gamma\} \subset X$ is called a cover of $M \subset X$, if $M \subset \bigcup_{\gamma\in\Gamma} A_\gamma$. If each $A_\gamma \in \tau$, we call $\{A_\gamma : \gamma \in \Gamma\}$ an open cover of M. If in addition Γ is finite, we call it a finite open cover of M.

We say that M is compact if, for every open cover $\{A_\gamma : \gamma \in \Gamma\}$, there always exists a finite subset of Γ, $\gamma_1, \dots, \gamma_n$ such that $M \subset \bigcup_{k=1}^n A_{\gamma_k}$.

Definition 1.7. Let (X_1, τ_1) and (X_2, τ_2) be two topological spaces, with $X_1 \cap X_2 = \emptyset$. The coproduct space $(X, \tau) = (X_1, \tau_1) \oplus (X_2, \tau_2)$ is the unique topological space, with the property that each open set $O \subset X$ is of the form $O = O_1 \cup O_2$, where $O_1 \in \tau_1$ and $O_2 \in \tau_2$.

(X, τ) is also known as the disjoint union space or direct sum space. (If (X_1, τ_1) and (X_2, τ_2) are Hausdorff, then it is easy to see that (X, τ) is Hausdorff.)

1.1.3. σ-Algebras.

Definition 1.8. Let $\mathcal{A} \subset \mathcal{P}(X)$ be a collection of subsets of $X \neq \emptyset$. We say that $\mathcal{A}$ is an algebra if the following holds:

(1) $X, \emptyset \in \mathcal{A}$ and,

(2) If $A, B \in \mathcal{A}$ then $A^c, B^c \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$.
It is easy to verify that:

(3) $A \cap B \in \mathcal{A}$ and $A \setminus B \in \mathcal{A}$.

(4) If n is finite and $\{A_k\} \subset \mathcal{A},\ 1 \leq k \leq n$, then

$$\bigcup_{k=1}^{n} A_k \in \mathcal{A}, \qquad \bigcap_{k=1}^{n} A_k \in \mathcal{A}.$$

Definition 1.9. Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra. We say that $\mathcal{A}$ is a σ-algebra if

$$\bigcup_{k=1}^{\infty} A_k \in \mathcal{A},$$

for any countable family of sets $\{A_k\} \in \mathcal{A}$. It is also easy to see that

$$\bigcap_{k=1}^{\infty} A_k \in \mathcal{A},$$

along with

$$\liminf_{n \to \infty} A_n \in \mathcal{A}$$

and

$$\limsup_{n \to \infty} A_n \in \mathcal{A}.$$

Definition 1.10. If Σ is a nonempty class of subsets of X, the smallest σ-algebra $\mathcal{A}$, with $\Sigma \subset \mathcal{A}$ is called the σ-algebra generated by Σ and is written $\mathcal{A}(\Sigma)$.

Remark 1.11. Since $\Sigma \subset \mathcal{P}(X)$, there is at least one σ-algebra containing Σ.

Lemma 1.12. *If J is an index set and for each $\alpha \in J$, $\mathcal{A}_\alpha$ is σ-algebra, then $\mathcal{A} = \bigcap_{\alpha \in J} \mathcal{A}_\alpha$ is a σ-algebra.*

Definition 1.13. If $\mathcal{A}$ is a σ-algebra of subsets of a nonempty set X, we call the couple $(X, \mathcal{A})$ a measurable space.

Definition 1.14. If $\mathcal{A}$ is a σ-algebra of subsets of a nonempty set X, we call a sequence $\{A_k\} \subset \mathcal{A}$ a partition of X if the sequence is disjoint and $\bigcup_{k=1}^{\infty} A_k = X$.

Definition 1.15. If X is a topological space and Σ is the class of open sets of X, then $\mathcal{A}(\Sigma) = \mathfrak{B}(X)$ is called the Borel σ-algebra of X.

1.1.4. Measure Spaces.

Definition 1.16. Let X be a nonempty set. An outer measure ν^* is a function on $\mathcal{P}(X) \to [0, \infty]$, such that

(1) $\nu^*(\emptyset) = 0$.
(2) If $B \subset A$, then $\nu^*(B) \le \nu^*(A)$.
(3) If $A \subset \bigcup_{k=1}^{\infty} A_k$, then

$$\nu^*(A) \le \sum_{k=1}^{\infty} \nu^*(A_k).$$

If for each sequence of disjoint sets $\{A_k\} \subset \mathcal{A}$,

$$\nu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \nu(A_k),$$

we say that ν is a measure. We also say that ν is σ-additive and call the triple $(X, \mathcal{A}, \nu)$ a measure space.

Definition 1.17. Let $(X, \mathcal{A})$ be a measurable space and let $\nu(A) \in \mathbb{C}$, the complex numbers, for each $A \in \mathcal{A}$. We say that ν is a complex measure if $\nu(\emptyset) = 0$ and for each disjoint countable union $\bigcup_{k=1}^{\infty} A_k$ of sets in $\mathcal{A}$, we have

$$\nu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \nu(A_k),$$

where the convergence on the right is absolute.

Definition 1.18. Let $(X, \mathcal{A}, \nu)$ a measure space.

(1) We say that ν is a finite measure if $\nu(X) < \infty$.

(2) We say that ν is concentrated on a set $A \in \mathcal{A}$, if $A = U^c$ and U is the largest open set with the property that $\nu(U) = 0$. We also call A the support of ν.

(3) We say that ν is a regular measure if given $A \in \mathcal{A}$, for each $\varepsilon > 0$, there is a open set O and a closed set K such that: $K \subset A \subset O$ and $\nu(O \setminus K) < \varepsilon$.

(4) We say that ν is a σ-finite measure if there is a sequence $\{A_k\} \subset \mathcal{A}$, with

$$X = \bigcup_{k=1}^{\infty} A_k, \text{ and } \nu(A_k) < \infty.$$

(5) We say that ν is a Radon measure, if the set K in (3) can be chosen as compact or the sequence $\{A_k\} \subset \mathcal{A}$ in (4) can be chosen with each A_k is compact.

(6) We say that ν is a complete measure if $A \in \mathcal{A}$, with $B \subset A$ and $\nu(A) = 0$ then $B \in \mathcal{A}$ and $\nu(B) = 0$.

(7) We say that ν is a probability measure if $\nu(X) = 1$.

(8) We say that a complex measure ν is of bounded variation if

$$|\nu|(X) = \sup \sum_{k=1}^{\infty} |\nu(A_k)| < \infty,$$

where the supremum is taken over all partitions of X. We call $|\nu|(X)$ the total variation of ν.

(9) We say that the complex measure ν is a signed measure if both $|\nu| + \nu$ and $|\nu| - \nu$ are real valued. In this case, we define the positive part and the negative part by: $\nu^+ = \frac{1}{2}(|\nu| + \nu)$ and $\nu^- = \frac{1}{2}(|\nu| - \nu)$. We call this the Jordan Decomposition.

Theorem 1.19 (The Hahn Decomposition Theorem)**.** *Let ν be a signed measure on $(X, \mathcal{A})$. Then there exists a partition X_1, X_2 of X such that, for every $A \in \mathcal{A}$:*

$$\nu^+(A) = \nu(A \cap X_1) \textit{ and } \nu^-(A) = -\nu(A \cap X_2).$$

Theorem 1.20 (The Jordan Decomposition Theorem)**.** *Let ν be a signed measure on $(X, \mathcal{A})$. If μ_1 and μ_2 are positive measures and $\nu = \mu_1 - \mu_2$, then $\nu^+ \leq \mu_1$ and $\nu^- \leq \mu_2$.*

Thus, the Jordan decomposition $\nu = \nu^+ - \nu^-$, has the above minimal property. If ν is complex, this decomposition becomes $\nu = \nu_1^+ - \nu_1^- + i(\nu_2^+ - \nu_2^-)$, for two positive measures, ν_1 and ν_2.

Definition 1.21. We say that X is an Abelian group if for each pair $x, y \in X$, $x \oplus y \in X$ and

(1) $x \oplus y = y \oplus x$. (The Abelian property.)
(2) For all $x, y, z \in X$ $(x \oplus y) \oplus z = x \oplus (y \oplus z)$.
(3) There is an element $0 \in X$ called the identity and $x \oplus 0 = 0 \oplus x = 0$, for all $x \in X$.
(4) For each $y \in X$, there is a unique element $y^- \in X$, such that $y \oplus y^- = y^- \oplus y = 0$.
(5) We say that Y is a subgroup of X if $Y \subset X$ and for all $y_1, y_2 \in Y$, $y_1 \oplus y_2 \in Y$, satisfying conditions (1)–(4) above.

The real or complex numbers form an Abelian group with addition (or multiplication if we exclude zero). The rational numbers (real or complex) form a subgroup, with the same exception for multiplication.

When X is an Abelian group (with $\oplus = +$) and $(X, \mathcal{A}, \nu)$ is a measure space, we say that $\mathfrak{T}$ is an admissible translation invariance group for $(X, \mathcal{A}, \nu)$ if $\mathfrak{T}$ is a subgroup of X and $\nu(A - t) = \nu(A)$, for all $t \in \mathfrak{T}$. If $\mathfrak{T} = X$, we say that ν is translation invariant on X.

1.1.5. Integral.

Let $(X, \mathcal{A}, \nu)$ a measure space.

Definition 1.22. Let f be a function on X, $f : X \to K$, where $K = \mathbb{R}$ or $\mathbb{C}$.

(1) We say that f is measurable if $f^{-1}(B) \in \mathcal{A}$, for every set $B \in \mathfrak{B}[K]$, the Borel algebra on K. In this case, we say that $f \in \mathcal{M}[X]$ or $\mathcal{M}$, when X is understood.
(2) We say that two functions f and g are equal almost everywhere and write $f(x) = g(x)$, ν-(a.e.), if they have the same domain and $\nu\{x : f(x) \neq g(x)\} = 0$. In general, a property is said to hold ν-(a.e.) on X if the set of points where this property fails has ν-measure zero.

Definition 1.23. A (nonnegative) simple function s is defined on X by

$$s(x) = \sum_{k=1}^{n} a_k \chi_{A_k}(x),$$

where the $a_k \in [0, \infty)$ and the family of measurable sets $\{A_k\}$ form a (finite) partition of X (i.e., $\nu(A_i \cap A_j) = 0$, $i \neq j$ and $\bigcup_{k=1}^{n} A_k = X$).

(By convention, if need be, we can always add a set A_{n+1} to the collection and define $a_{n+1} = 0$ so that the union is always X.)

Lemma 1.24. *If $0 \leq f \in \mathcal{M}$, then there is a sequence of simple functions $\{s_n\}$, with $s_n \leq s_{n+1}$ and $s_n \to f$ (a.e.) at each point of X, as $n \to \infty$.*

Definition 1.25. If $f : X \to [0, \infty]$ is a measurable function and $A \in \mathfrak{B}(X)$, we define the integral of f over A by:

$$\int_A f(x) d\nu = \lim_{n\to\infty} \int_A s_n(x)\, d\nu,$$

where $\{s_n\}$ is any increasing family of simple functions converging to $f(x)$.

Theorem 1.26. *If f, g are nonnegative measurable functions and $0 \leq c < \infty$, we have:*

(1) *$\int_X f(x) d\nu(x)$ is independent of the family of simple functions used;*
(2) *$0 \leq \int_X f(x) d\nu(x) \leq \infty$;*
(3) *$\int_X cf(x) d\nu(x) = c \int_X f(x) d\nu(x)$;*
(4)
$$\int_X [f(x) + g(x)] d\nu(x) = \int_X f(x) d\nu(x) + \int_X g(x) d\nu(x).$$
(5) *If $f \leq g$, then $\int_X f(x) d\nu(x) \leq \int_X g(x) d\nu(x)$.*

Theorem 1.27 (Fatou's Lemma)**.** *Let $\{f_n\} \subset \mathcal{M}$ be a nonnegative family of functions, then:*

$$\int_X \left(\liminf_{n\to\infty} f_n(x) \right) d\nu(x) \leq \liminf_{n\to\infty} \int_X f_n(x) d\nu(x).$$

Theorem 1.28 (Monotone Convergence Theorem)**.** *Let $\{f_n\} \subset \mathcal{M}$ be a nonnegative family of functions, with $f_n \leq f_{n+1}$. Then:*

$$\lim_{n\to\infty} \int_X f_n(x) d\nu(x) = \int_{\mathcal{B}} \left(\lim_{n\to\infty} f_n(x) \right) d\nu(x).$$

Definition 1.29. If $f \in \mathcal{M}$, we define

$$\int_X f(x) d\nu(x) = \int_X f_+(x) d\nu(x) - \int_X f_-(x) d\nu(x),$$

where $f_+(x) = \frac{1}{2}(|f(x)| + f(x))$ and $f_-(x) = \frac{1}{2}(|f(x)| - f(x))$. We say that f is integrable whenever both integrals on the right are finite. The set of all integrable functions is denoted by $\mathcal{L}^1[X, \mathfrak{B}(X), \nu] = \mathcal{L}^1[X]$.

Remark 1.30. As is carefully discussed in elementary analysis, the functions in $\mathcal{L}^1[X]$ are not uniquely defined. Following tradition, we let $L^1[X]$ denote the set of equivalence classes of functions in $\mathcal{L}^1[X]$ that differ by a set of ν-measure zero. By a slight abuse, we will identify an integrable function f as measurable (in $\mathcal{L}^1[X]$) and its equivalence class in $L^1[X]$. The same convention also applies to functions in $L^p[X]$ and will be used later without further comment.

Theorem 1.31 (Dominated Convergence Theorem)**.** *Let* $f_n \in \mathcal{M}[X, \nu]$, $n \in \mathbb{N}$, $g \in L^1(X)$, *with* $g \geq 0$ *and* $|f_n(x)| \leq g(x)$, ν-(a.e.). *If* $\lim_{n\to\infty} f_n(x)$ *exists* ν-(a.e.), *then* $\lim_{n\to\infty} f_n \in L^1[X]$ *and*

$$\lim_{n\to\infty} \int_X f_n(x) d\nu(x) = \int_X \left(\lim_{n\to\infty} f_n(x) \right) d\nu(x).$$

1.2. Functional Analysis

In this section, we include a few basic background results from functional analysis and Banach space theory. Detailed discussions can be found in Dunford and Schwartz [DS], Hille and Phillips [HP], Lax [L1], Reed and Simon [RS], Rudin [RU], or Yosida [YS].

1.2.1. Topological Vector Spaces.

Definition 1.32. A vector space $\mathfrak{X}$ over $\mathbb{C}$ is an Abelian group under addition that is closed under multiplication by elements of $\mathbb{C}$. That is:

(1) For each $x, y \in \mathfrak{X}$, $x + y \in \mathfrak{X}$.
(2) For all $x, y, z \in \mathfrak{X}$, $x + y = y + x$ and $(x + y) + z = x + (y + z)$.
(3) There is a unique element $0 \in \mathfrak{X}$ called zero and $x + 0 = 0 + x = x$ for all $x \in \mathfrak{X}$.
(4) For all $x \in \mathfrak{X}$, there is a unique element $-x \in \mathfrak{X}$ and $x + (-x) = (-x) + x = 0$.
(5) For all $x, y \in \mathfrak{X}$ and $a, b \in \mathbb{C}$, $ax \in \mathfrak{X}$, $1x = x$, $(ab)x = a(bx)$ and $a(x + y) = ax + ay$. We call $b \in \mathbb{C}$ a scalar.

If $\mathfrak{X}$ is a vector space over $\mathbb{C}$, a mapping $\rho(\cdot) : \mathfrak{X} \to [0, \infty)$ is a seminorm on $\mathfrak{X}$ if:

(1) For each $x,\ y \in \mathfrak{X},\ \rho(x) \geq 0$ and $\rho(x+y) \leq \rho(x) + \rho(y)$.

(2) For each $\lambda \in \mathbb{C}$ and each $x \in \mathfrak{X},\ \rho(\lambda x) = |\lambda|\, \rho(x)$.

Definition 1.33. Let V be a subset of $\mathfrak{X}$.

(1) We say that V is a convex subset of $\mathfrak{X}$ if for each $x,\ y \in V,\ \alpha x + (1-\alpha)y \in V$, for all $\alpha \in [0,1]$.

(2) We say that V is an balanced subset of $\mathfrak{X}$ if for each $x \in V$ and $\alpha \in \mathbb{C}$, with $|\alpha| \leq 1,\ \alpha x \in V$.

(3) We say that V is an absolutely convex subset of $\mathfrak{X}$ if it is both convex and balanced.

(4) We say that V is a absorbent subset of $\mathfrak{X}$ if for each $x \in \mathfrak{X},\ \alpha x \in V$, for some $\alpha > 0$. Thus, every point in $x \in \mathfrak{X}$ is in αV for some positive α.

Definition 1.34. A locally convex topological vector space is a vector space with its topology defined by a family of semi-norms $\{\rho_\gamma\}$, where γ is in some index set Γ. Given any $x \in \mathfrak{X}$, a base of ε-neighborhoods about x is a set of the form $V_{\Gamma_0,\varepsilon}(x)$, where Γ_0 is a finite subset of Γ and

$$V_{\Gamma,\varepsilon}(x) = \{\, y \in \mathfrak{X} : \rho_\gamma(x-y) < \varepsilon,\ \gamma \in \Gamma\,\}.$$

Definition 1.35. A locally convex topological vector space $\mathfrak{X}$ is a Fréchet space if it satisfies the following:

(1) $\mathfrak{X}$ is a Hausdorff space.

(2) The neighborhood base about each $x \in \mathfrak{X}$ is induced by a countable number of seminorms (i.e., Γ is a countable set).

(3) $\mathfrak{X}$ is a complete relative to the family of seminorms.

Theorem 1.36. *The vector space $\mathfrak{X}$ is a Fréchet space if and only if:*

(1) *$\mathfrak{X}$ is a locally convex.*

(2) *There is a metric $d : \mathfrak{X}\times\mathfrak{X} \to [0,\infty)$ such that, for all $x, y, z \in \mathfrak{X},\ d(x+z, y+z) = d(x,y)$.*

(3) *$\mathfrak{X}$ is a complete relative to the metric $d(\cdot,\cdot)$.*

Remark 1.37. If the index Γ for the family of semi-norms is countable, then we can define a metric $d(x,y)$ by:

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\rho_n(x-y)}{1+\rho_n(x-y)}.$$

A sequence $\{x_n\}$ in a metric space $\mathfrak{X}$ converges to a limit $x \in \mathfrak{X}$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$. In this case, by the triangle inequality

$$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x).$$

We say that a sequence satisfies the Cauchy convergence condition, or is a Cauchy sequence if

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0.$$

A metric space is said to be complete if every Cauchy sequence converges to a point in the space.

1.2.2. Separable Banach Spaces. Hilbert and Banach spaces are discussed further in Chaps. 4 and 5. Let $\mathcal{B}$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. We say that $\mathcal{B}$ is separable if it contains a countable dense subset.

Definition 1.38. A norm on a vector space $\mathcal{B}$ is a mapping $\|\cdot\|_{\mathcal{B}} : \mathcal{B} \to [0, \infty]$, such that

(1) $\|x\|_{\mathcal{B}} = 0$ if and only if $x = 0$.
(2) $\|ax\|_{\mathcal{B}} = |a| \, \|x\|_{\mathcal{B}}$ for all $x \in \mathcal{B}$ and $a \in \mathbb{C}$.
(3) $\|x + y\|_{\mathcal{B}} \leq \|x\|_{\mathcal{B}} + \|y\|_{\mathcal{B}}$, for all $x, y \in \mathcal{B}$.
(4) We say that $\mathcal{B}$ is uniformly convex if, for each $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon) > 0$ such that, for all $x, y \in \mathcal{B}$ with

$$\max(\|x\|, \|y\|) \leqslant 1, \quad \|x - y\| \geqslant \varepsilon \Rightarrow \tfrac{1}{2}\|x + y\| \leqslant 1 - \delta.$$

The topology on $\mathcal{B}$ is generated by the metric defined by:

$$d(x, y) = \|x - y\|_{\mathcal{B}},$$

so that $\{x : \|x - y\|_{\mathcal{B}} < r\}$ is an open ball about y of radius r.

The space $\mathcal{B}$ is complete if every Cauchy sequence in the above norm converges to an element in $\mathcal{B}$. A complete normed space is called a Banach space.

Definition 1.39. Let $\mathcal{B}$ be a Banach space and let A be a transformation on $\mathcal{B}$, with domain $D(A)$ (i.e., $A : D(A) \subset \mathcal{B} \to \mathcal{B}$).

(1) We say that A is a linear operator on $\mathcal{B}$, if $A(ax + by) = aAx + bAy$, for all $a, b \in \mathbb{C}$ and all $x, y \in D(A)$.
(2) We say that A is densely defined if $D(A)$ is dense in $\mathcal{B}$.

(3) We say that A is a closed linear operator if and only if the following condition is satisfied: $\{x_n\} \subset D(A), x_n \to x$ and $Ax_n \to z$ always implies that $x \in D(A)$ and $z = Ax$.

(4) We say that A is a bounded linear operator if and only if $D(A) = \mathcal{B}$ and

$$\sup_{\|x\|_{\mathcal{B}} \leqslant 1} \|Ax\|_{\mathcal{B}} < \infty.$$

In this case we define the norm of A, $\|A\|_{\mathcal{B}}$, by the above supremum.

1.2.2.1. *Dual Spaces.*

Definition 1.40. Let $\mathcal{B}$ be a Banach space.

(1) The dual space $\mathcal{B}'$ is the set of all bounded linear operators $x^* : \mathcal{B} \to \mathbb{C}$ (called bounded linear functionals on $\mathcal{B}$). The norm of x^* is defined by:

$$\|x^*\|_{\mathcal{B}'} = \sup_{\|x\|_{\mathcal{B}} \leq 1} |x^*(x)| = \sup_{\|x\|_{\mathcal{B}} \leq 1} |\langle x, x^* \rangle| .$$

With this norm $\mathcal{B}'$ is a Banach space. We write $\mathcal{B}'$ as $\mathcal{B}'_s$ and call it the strong dual. The topology is known as the strong topology.

(2) The weak and weak* topology are defined on $\mathcal{B}$ and $\mathcal{B}'$ respectively in the following manner:

- A sequence $\{x_n\} \subset \mathcal{B}$ is said to converge in the weak topology to $x \in \mathcal{B}$ if and only if, for each bounded linear functional $y^* \in \mathcal{B}'$,

$$\lim_{n \to \infty} y^*(x_n) = y^*(x).$$

We also write $w - \lim_{n \to \infty} x_n = x$.

- A sequence $\{x_n^*\} \subset \mathcal{B}'$ is said to converge in the weak* topology to $x^* \in \mathcal{B}'$ if and only if, for each $y \in \mathcal{B}$,

$$\lim_{n \to \infty} x_n^*(y) = x^*(y).$$

We also write $w^* - \lim_{n \to \infty} x_n^* = x^*$.

(3) If $\mathcal{B} = \mathcal{B}''$, we say that $\mathcal{B}$ is reflexive.

(4) A duality map $\mathcal{J} : \mathcal{B} \mapsto \mathcal{B}'$ is a set

$$\mathcal{J}(u) = \left\{ u^* \in \mathcal{B}' \,\middle|\, u^*(u) = \langle u, u^* \rangle = \|u\|^2 = \|u^*\|^2 \right\}, \text{ for all } u \in \mathcal{B}.$$

Remark 1.41. The following remarks are important.

(1) In the definition, we used x^* to represent an element in $\mathcal{B}'$. The notation used varies with the tradition of the particular topical area. To the extent possible, we will try to be consistent within topics studied and the tradition of the field so that the reader will see some correspondence when consulting references for different topics.

(2) It is easy to see that

$$|y^*(x_n) - y^*(x)| \leq \|x_n - x\|_{\mathcal{B}} \|y^*\|_{\mathcal{B}'}$$

for all $y^* \in \mathcal{B}$, so that norm convergence in $\mathcal{B}$ always implies weak convergence. It is also easy to see that

$$|x_n^*(y) - x^*(y)| \leq \|x_n^* - x^*\|_{\mathcal{B}'} \|y\|_{\mathcal{B}},$$

for all $y \in \mathcal{B}$, so that norm convergence in $\mathcal{B}'$ always implies weak* convergence. However (in both cases), the reverse is not true (see Lax [L1, p. 106]).

(3) It is known that every uniformly convex Banach space is reflexive. Furthermore, when $\mathcal{B}$ is uniformly convex, the duality set $\mathcal{J}(u)$, is single valued and uniquely defined by u. However, if $\mathcal{B}$ is not uniformly convex, the duality set $\mathcal{J}(u)$ can have the power of the continuum.

The following examples will help one see what is possible in concrete cases.

(1) If λ_n is Lebesgue measure on $\mathbb{R}^n$, $u \in L^p[\mathbb{R}^n]$, $1 < p < \infty$ and q is such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\mathcal{J}(u)(x) = \|u\|_p^{2-p} |u(x)|^{p-2} u(x) = u^* \in L^q\mathbb{R}^n],$$

and

$$\langle u, u^* \rangle = \|u\|_p^{2-p} \int_{\mathbb{R}^n} |u(x)|^p \, d\lambda_n(x) = \|u\|_p^2 = \|u^*\|_q^2.$$

Thus, it is easy to see that $(L^p[\mathbb{R}^n])'' = L^p[\mathbb{R}^n]$, so that $L^p[\mathbb{R}^n]$ is reflexive for $1 < p < \infty$.

(2) The space $L^1[\mathbb{R}^n]$ is not reflexive, for if $u \in L^1[\mathbb{R}^n]$, then

$$\mathcal{J}(u)(x) = \{ v \in L^\infty[\mathbb{R}^n]| \ : v(x) \in \{\|u\|_1 \operatorname{sign}[u(x)]\}\},$$

where

$$\operatorname{sign}[u(x)] = \begin{cases} 1, & u(x) > 0, \\ -1, & u(x) < 0, \\ [-1,1], & u(x) = 0. \end{cases}$$

It follows that $\mathcal{J}(u)(x)$ is uncountable for each $u \in L^1[\mathbb{R}^n]$.

The transpose matrix on $\mathbb{R}^n$ or the transpose conjugate matrix on $\mathbb{C}^n$ has its parallel for Banach spaces. In this case, they are known as dual operators. They are also known as adjoint operators, but we will reserve this term for a special class of operators on Banach spaces, discussed in Chap. 5. We will also use adjoint for the same class defined on Hilbert spaces in the next section and explain the distinction.

Definition 1.42. Let $A : D(A) \to \mathcal{B}$ be a closed linear operator on $\mathcal{B}$ with a dense domain $D(A)$. The dual of A, A' is defined on $\mathcal{B}'$ as follows. Its domain $D(A')$ is the set of all $y^* \in \mathcal{B}'$ for which there exists an $x^* \in \mathcal{B}'$ such that

$$\langle Ax, y^* \rangle = \langle x, x^* \rangle ,$$

for all $x \in D(A)$; in this case we define $A'y^* = x^*$.

A proof of the following theorem can be found in [HP] or [YS].

Theorem 1.43. *Let $A : D(A) \to \mathcal{B}$ be a closed linear operator on $\mathcal{B}$ with a dense domain $D(A)$.*

(1) *Then $A' : D(A') \to \mathcal{B}'$ is a closed linear operator on $\mathcal{B}'$ and its domain $D(A')$ is dense in $\mathcal{B}'$.*

(2) *If, in addition, $\|A\|_{\mathcal{B}} < \infty$, then $D(A') = \mathcal{B}'$ and $\|A'\|_{\mathcal{B}'} = \|A\|_{\mathcal{B}}$.*

1.2.2.2. *Hilbert Space.*

Definition 1.44. An inner product on $\mathcal{B} = \mathcal{H}$ is a bilinear mapping $(\cdot,\cdot)_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, such that

(1) $(x,x)_{\mathcal{H}} \geq 0$ and $(x,x)_{\mathcal{H}} = 0$ if and only if $x = 0$.

(2) $(ax+by,z)_{\mathcal{H}} = a(x,z)_{\mathcal{H}} + b(y,z)_{\mathcal{H}}$ and $(w,ax+by)_{\mathcal{H}} = a^c(w,x)_{\mathcal{H}} + b^c(w,y)_{\mathcal{H}}$.

If $(\cdot,\cdot)_{\mathcal{H}}$ is a inner product, it induces a norm on $\mathcal{H}$ by

$$\|x-y\|_{\mathcal{H}} = \sqrt{(x-y,x-y)_{\mathcal{H}}}.$$

If $\mathcal{H}$ is complete with this norm, we call it a Hilbert space.

If $(\cdot,\cdot)_{\mathcal{H}}$ is the inner product on the Hilbert space $\mathcal{H}$, then the same Cauchy–Schwarz inequality from $\mathbb{R}^n$ still holds, $|(x,y)_{\mathcal{H}}| \le \|x\|_{\mathcal{H}} \|y\|_{\mathcal{H}}$.

The following polarization identity also holds for a general Hilbert space:

$$(x,y)_{\mathcal{H}} = \frac{1}{4}\left(\|x+y\|_{\mathcal{H}}^2 - \|x-y\|_{\mathcal{H}}^2\right),$$

if the field of $\mathcal{H}$ is $\mathbb{R}$ and

$$(x,y)_{\mathcal{H}} = \frac{1}{4}\left\{\left(\|x+y\|_{\mathcal{H}}^2 - \|x-y\|_{\mathcal{H}}^2\right) + i\left(\|x+iy\|_{\mathcal{H}}^2 - \|x-iy\|_{\mathcal{H}}^2\right)\right\},$$

if the field of $\mathcal{H}$ is $\mathbb{C}$.

Definition 1.45. Let $A : D(A) \to \mathcal{H}$ be a closed linear operator on $\mathcal{H}$ with a dense domain $D(A)$. The adjoint of A, A^* is defined on $\mathcal{H}$ as follows. Its domain $D(A^*)$ is the set of all $y \in \mathcal{H}$ for which there exists an $x \in \mathcal{H}$ such that

$$(Ax,y)_{\mathcal{H}} = (x, A^*y)_{\mathcal{H}}.$$

We will always call A^* the adjoint of A when it is defined on the same space and A', the dual of A when it is defined on the dual space. In Chap. 5, we will see that the adjoint is also possible for a certain class of Banach spaces, which include the uniformly convex ones.

Theorem 1.43 can be slightly modified to show that $D(A^*)$ is dense in $\mathcal{H}$ and, if $\|A\|_{\mathcal{H}} < \infty$, then $D(A^*) = \mathcal{H}$ and $\|A^*\|_{\mathcal{H}} = \|A\|_{\mathcal{H}}$.

Recall that, two functions $f, g \in \mathcal{H}$ are orthogonal, if $(f,g)_{\mathcal{H}} = 0$ and they are orthonormal if in addition, $\|f\|_{\mathcal{H}} = \|g\|_{\mathcal{H}} = 1$. A set $\{\phi_n\} \subset \mathcal{H}$ is an orthonormal basis for $\mathcal{H}$ if they are orthonormal and each $x \in \mathcal{H}$ can be written as $x = \sum_{k=1}^{\infty} a_k \phi_k$, for a unique family of scalars $\{a_n\} \subset \mathbb{C}$.

Definition 1.46. Let A be a linear operator defined on $\mathcal{H}$.

(1) We say that A is a projection operator if $A^2x = Ax$ for all $x \in \mathcal{H}$.

(2) We say that A is the self-adjoint if $D(A) = D(A^*)$ and $Ax = A^*x$, for all $x \in D(A)$.

(3) We say that a bounded linear operator A is the compact, if for every bounded sequence $\{x_n\} \subset \mathcal{H}$, the sequence $\{Ax_n\}$ has a convergent subsequence.

(4) We say that a compact operator A is trace class if, for some orthonormal basis $\{\phi_n\}$ of $\mathcal{H}$, the trace of A, $tr[A]$ is finite,

where

$$tr[A] = \sum_{n=1}^{\infty} (A\phi_n, \phi_n).$$

It is easy to check that the trace (if it exists) is independent of the basis used.

1.2.3. The Hahn–Banach Theorem.

Theorem 1.47. *Let $\mathcal{B}$ be a Banach space over $\mathbb{C}$ and let $p : \mathcal{B} \to \mathbb{R}$ be such that, for all $x, y \in \mathcal{B}$*

$$p(ax + by) \leq |a|\, p(x) + |a|\, p(y), \text{ whenever } |a| + |b| = 1. \qquad (1.1)$$

If $\bar{L}$ is a linear functional defined on a subspace $\mathcal{D} \subset \mathcal{B}$, with $\left|\bar{L}(x)\right| \leq p(x)$, for all $x \in \mathcal{D}$, then $\bar{L}$ can be extended to a linear functional L on $\mathcal{B}$ such that $|L(x)| \leq p(x),\ x \in \mathcal{B}$ and $L(x) = \bar{L}(x)$ on $\mathcal{D}$.

Proof. We first assume that the field is $\mathbb{R}$. Suppose that $x \in \mathcal{B}$ but $x \notin \mathcal{D}$. Let $\mathcal{E} = (x, \mathcal{D})$ be the vector space spanned by x and $\mathcal{D}$. If we have an extension L of $\bar{L}$ from $\mathcal{D}$ to $\mathcal{E}$, it must satisfy

$$L\,(ax + by) = \lambda L(x) + \bar{L}(y),\ y \in \mathcal{D}.$$

and from (1.1), $|a| + |b| = 1$ implies that

$$p(ax + by) \leq |a|\, p(x) + |b|\, p(y).$$

Suppose that $y_1, y_2 \in \mathcal{D},\ a, b > 0,\ a + b = 1$. Then

$$a\bar{L}(y_1) + b\bar{L}(y_2) = \bar{L}(ay_1 + by_2) \leqslant p[a(y_1 - \tfrac{1}{a}x) + b(y_2 + \tfrac{1}{b}x)]$$
$$\leqslant ap(y_1 - \tfrac{1}{a}x) + bp(y_2 + \tfrac{1}{b}x).$$

We see that for all $y_1, y_2 \in \mathcal{D}$ and all $a, b > 0,\ a + b = 1$, we have

$$\tfrac{1}{a}\left[-p[y_1 - ax) + \bar{L}(y_1)\right] \leqslant \tfrac{1}{b}\left[p(y_2 + bx) - \bar{L}(y_2)\right].$$

It now follows that we must be able to find a number c such that for all $a > 0$,

$$\sup_{y \in D} \tfrac{1}{a}\left[-p[y - ax) + \bar{L}(y)\right] \leqslant c \leqslant \inf_{y \in D} \tfrac{1}{a}\left[p(y + ax) - \bar{L}(y)\right].$$

We can define $L(x) = c$. It is easy to check that $L(x) \leq p(x)$, for all $x \in \mathcal{E}$. We now appeal to Zorn's Lemma (see Yosida [YS, p. 2]), to show that $\bar{L}$ can be extended to all of $\mathcal{B}$, when the field is $\mathbb{R}$.