Theokritos Kouremenos

The Unity of Mathematics in Plato's *Republic*

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The new theory was not derived from experience. Viktor could see this quite clearly. It had arisen in absolute freedom; it had sprung from his own head. The logic of this theory, its chain of reasoning, was quite unconnected to the experiments conducted by Markov in the laboratory. The theory had sprung from the free play of thought. It was this free play of thought–which seemed quite detached from the world of experience–that had made it possible to explain the wealth of experimental data, both old and new.

The experiments had been merely a jolt that had forced him to start thinking. They had not determined the content of his thoughts.

All this was quite extraordinary…

His head was full of mathematical relationships, differential equations, the laws of higher algebra, number and probability theory. These mathematical relationships had an existence of their own in some void quite outside the world of atomic nuclei, stars and electromagnetic or gravitational fields, outside space and time, outside the history of man and the geological history of the earth. And yet these relationships existed inside his own head.

Vasily Grossman, *Life and Fate*

CONTENTS

3. THE MYTH ABOUT PLATO'S ROLE IN THE DEVELOPMENT OF MATHEMATICS

PREFACE

In the seventh book of his *Republic* Plato says that, before the future philosopher-rulers begin their study of philosophy, they must engage in an intense and prolonged study of mathematics, ultimately in order to grasp the community and kinship of all its branches, its deep unity. He does not explain how the unity of mathematics is supposed to be understood, however, despite the paramount importance he attaches to this feature of mathematics. The first chapter of this monograph, which develops further Kouremenos (2004), attempts to throw some light on Plato's conception of astronomy in the seventh book of the *Republic* as a propedeutic to philosophy by taking into account a possible connection between fourth-century-BC astronomy and solid geometry that could have shaped Plato's view on the unity of mathematics: the solution to the problem of cube-duplication by Eudoxus of Cnidus has not come down to us, but he could have solved this problem with his famous astronomical theory of homocentric spheres. The second chapter argues that Plato conceives of the unity of mathematics exactly in terms of the mutually benefiting links between its branches, not as imparted by one of them to the rest, over which it is somehow privileged and through which it thus runs, just as he conceives of the unity of the state outlined in the *Republic* in terms of the common benefit for all citizens, not in the light of the privileged role accorded to its philosopher-rulers. The third chapter expands Kouremenos (2011) and concerns two well-known stories: that the solutions to the problem of cube-duplication put forth by Greek mathematicians in the fourth century BC had been motivated by Plato's interpretation of a Delphic oracle given to the inhabitants of the island of Delos, and that the philosopher Plato spurred the mathematician and astronomer Eudoxus to come up with his theory of homocentric spheres. All components of these stories, however, including Apollo's relation with mathematics and the contribution of his oracles in the progress of mathematics in Greece, can be easily traced back to passages in the Platonic corpus. We must thus conclude that both stories are nothing but biographical anecdotes (re)constructing episodes in Plato's life from the Platonic corpus.

Fig. 4 is reproduced from Knorr (1993), fig. 5 from Yavetz (1998) and figs. 6–8 from Riddell (1979). The passage from Grossman's *Life and Fate* is quoted in the epigraph from the translation by Robert Chandler.

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I dedicate this monograph to the memory of my colleague Paraskevi Kotzia.

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1. ASTRONOMY IN THE *REPUBLIC*

1.1. INTRODUCTION

Arithmetic is one of the five branches of mathematics which the future philosopher-rulers of the city outlined in the *Republic* will study for a decade before they move on to dialectic, i.e. philosophy, according to book 7, 537b7–c3. It is introduced in book 6 together with another branch, geometry, in the context of the simile of the divided line. Socrates is presented as asking Glaucon, his codiscussant and Plato's brother, to imagine a line divided into two unequal parts, liken one to sensibles and the other to intelligibles and then divide each part in the same proportion. The first section of the "sensible" part contains shadows, images and reflections on all kinds of surfaces; the second contains the objects that cast shadows and are pictured or reflected (509d6–510b1). Arithmetic and geometry are introduced in the description of the contents of the "intelligible" part of the divided line (510b2–511c2):

> Σκόπει δὴ αὖ καὶ τὴν τοῦ νοητοῦ τομὴν ᾗ τμητέον.
>
> Πῇ;
>
> Ἧι τὸ μὲν αὐτοῦ τοῖς τότε μιμηθεῖσιν ὡς εἰκόσιν χρωμένη ψυχὴ ζητεῖν ἀναγκάζεται ἐξ ὑποθέσεων, οὐκ ἐπ' ἀρχὴν πορευομένη ἀλλ' ἐπὶ τελευτήν, τὸ δ' αὖ ἕτερον [τὸ] ἐπ' ἀρχὴν ἀνυπόθετον ἐξ ὑποθέσεως ἰοῦσα καὶ ἄνευ τῶν περὶ ἐκεῖνο εἰκόνων, αὐτοῖς εἴδεσι δι' αὐτῶν τὴν μέθοδον ποιουμένη.
>
> Ταῦτ', ἔφη, ἃ λέγεις, οὐχ ἱκανῶς ἔμαθον, ἀλλ' αὖθις <**
>
> **> ἦν δ' ἐγώ· ῥᾷον γὰρ τούτων προειρημένων μαθήσῃ. οἶμαι γάρ σε εἰδέναι ὅτι οἱ περὶ τὰς γεωμετρίας τε καὶ λογισμοὺς καὶ τὰ τοιαῦτα πραγματευόμενοι, ὑποθέμενοι τό τε περιττὸν καὶ τὸ ἄρτιον καὶ τὰ σχήματα καὶ γωνιῶν τριττὰ εἴδη καὶ ἄλλα τούτων ἀδελφὰ καθ' ἑκάστην μέθοδον, ταῦτα μὲν ὡς εἰδότες, ποιησάμενοι ὑποθέσεις αὐτά, οὐδένα λόγον οὔτε αὑτοῖς οὔτε ἄλλοις ἔτι ἀξιοῦσι περὶ αὐτῶν διδόναι ὡς παντὶ φανερῶν, ἐκ τούτων δ' ἀρχόμενοι τὰ λοιπὰ ἤδη διεξιόντες τελευτῶσιν ὁμολογουμένως ἐπὶ τοῦτο οὗ ἂν ἐπὶ σκέψιν ὁρμήσωσι.
>
> Πάνυ μὲν οὖν, ἔφη, τοῦτό γε οἶδα.
>
> Οὐκοῦν καὶ ὅτι τοῖς ὁρωμένοις εἴδεσι προσχρῶνται καὶ τοὺς λόγους περὶ αὐτῶν ποιοῦνται, οὐ περὶ τούτων διανοούμενοι, ἀλλ' ἐκείνων πέρι οἷς ταῦτα ἔοικε, τοῦ τετραγώνου αὐτοῦ ἕνεκα τοὺς λόγους ποιούμενοι καὶ διαμέτρου αὐτῆς, ἀλλ' οὐ ταύτης ἣν γράφουσιν, καὶ τἆλλα οὕτως, αὐτὰ μὲν ταῦτα ἃ πλάττουσίν τε καὶ γράφουσιν, ὧν καὶ σκιαὶ καὶ ἐν ὕδασιν εἰκόνες εἰσίν, τούτοις μὲν ὡς εἰκόσιν αὖ χρώμενοι, ζητοῦντες δὲ αὐτὰ ἐκεῖνα ἰδεῖν ἃ οὐκ ἂν ἄλλως ἴδοι τις ἢ τῇ διανοίᾳ.
>
> Ἀληθῆ, ἔφη, λέγεις.
>
> Τοῦτο τοίνυν νοητὸν μὲν τὸ εἶδος ἔλεγον, ὑποθέσεσι δ' ἀναγκαζομένην ψυχὴν χρῆσθαι περὶ τὴν ζήτησιν αὐτοῦ, οὐκ ἐπ' ἀρχὴν ἰοῦσαν, ὡς οὐ δυναμένην τῶν ὑποθέσεων ἀνωτέρω ἐκβαίνειν, εἰκόσι δὲ χρωμένην αὐτοῖς τοῖς ὑπὸ τῶν κάτω ἀπεικασθεῖσιν καὶ ἐκείνοις πρὸς ἐκεῖνα ὡς ἐναργέσι δεδοξασμένοις τε καὶ τετιμημένοις.
>
> Μανθάνω, ἔφη, ὅτι τὸ ὑπὸ ταῖς γεωμετρίαις τε καὶ ταῖς ταύτης ἀδελφαῖς τέχναις λέγεις.
>
> Τὸ τοίνυν ἕτερον μάνθανε τμῆμα τοῦ νοητοῦ λέγοντά με τοῦτο οὗ αὐτὸς ὁ λόγος ἅπτεται τῇ τοῦ διαλέγεσθαι δυνάμει, τὰς ὑποθέσεις ποιούμενος οὐκ ἀρχὰς ἀλλὰ τῷ ὄντι ὑποθέσεις, οἷον ἐπιβάσεις τε καὶ ὁρμάς, ἵνα μέχρι τοῦ ἀνυποθέτου ἐπὶ τὴν τοῦ παντὸς ἀρχὴν ἰών, ἁψάμενος αὐτῆς, πάλιν αὖ ἐχόμενος τῶν ἐκείνης ἐχομένων, οὕτως ἐπὶ

τελευτὴν καταβαίνῃ, αἰσθητῷ παντάπασιν οὐδενὶ προσχρώμενος, ἀλλ᾽ εἴδεσιν αὐτοῖς δι᾽ αὐτῶν εἰς αὐτά, καὶ τελευτᾷ εἰς εἴδη.

"Look now at how the intelligible part must be divided."

"How?"

"In this manner: the soul is forced to study one part of it from hypotheses, using things that were imitated earlier on as images, not ascending to a starting point but descending to an endpoint, but with regard to the other part, it ascends from a hypothesis to an unhypothetical starting point and approaches it without its images, with and through the forms themselves."

"I did not get what you just said," he replied. "But again <**."

"**>," I said. "You will understand my point more easily after the following. As you know, I am sure, the students of geometry, arithmetic and the like lay down odd and even, figures, three kinds of angle and other things akin to these in each field, and as if they knew these things, turning them into hypotheses, they do not deign to give either to themselves or to others an account of what is hypothesized, assuming that it is clear to everybody, but start from their hypotheses and go through the subsequent stages to arrive consistently at what they set out to investigate."

"I certainly know this," he said.

"So you also know that they use visible shapes and argue about them, but actually do not think about them but about those things that the visible shapes resemble, their proofs concerning the square itself and the diagonal itself, not that diagonal they draw, and so on–that is, they use as images the shapes they make up and draw, of which there are also shadows and reflections in water, in their attempt to see those things themselves that one can see with no other means than thought."

"It is true," he said.

"These were the intelligibles I was talking about in whose study the soul is forced to rely on hypotheses without ascending to a starting point, since it cannot transcend its hypotheses, but using visible images that are considered to be clearer than the originals and thus prized."

"I see," he said, "that you are talking about geometry and its kindred fields."

"So you can see that the other section of the intelligible part I was talking about is what reason itself grasps with the power of dialectic, employing hypotheses not as starting-points but as genuine hypotheses, let us say as footholds and launchers, so as to reach what is unhypothetical, the principle of all, and then, having gotten hold of it, turn back and, grasping what depends on it, descend in this manner to the end-point, using no sensibles whatsoever but the forms themselves through themselves to themselves, and end up with forms."

The first section of the "intelligible" part of the line contains the objects studied in mathematics via their visible images and problematic definitions, "hypotheses";[1] the second contains the forms studied in philosophy without such aids.

Plato seems to view what is studied in mathematics as forms approached in a particular way. Below in *R*. 7, 533a10–c6, he has Socrates say that mathematics sees beings in a dream via unclear hypotheses for which not accounts are given, not in the state of wakefulness, as dialectic does. Here he has Socrates give the square *itself* with the diagonal *itself* as example of an object studied in geometry. Forms have been introduced as the only beings at the end of *R*. 5, in the description of the philosophers (473e5–480a13) after the claim that, unless philosophers rule or rulers philosophize, humankind's troubles will not end (473c11–e4). Philosophers want to learn about forms such as the beautiful *itself*, the intelligible and unchanging

1 For hypotheses in the divided-line simile as definitions see Bostock (2009) 13.

objects of knowledge, each of which is unique but, since it is associated with the changeable sensibles, appears everywhere as many, e.g. beautiful things: the latter resemble their form but are subject to change, and thus cannot possibly be objects of knowledge but only of opinion, though according to non-philosophers they are the only existents.[2] Forms are not sensible because that they are immaterial.[3] They seem to be conceived as eternal or atemporal entities not existing in space.[4] In terms of the traditional ontological categories, they are usually thought to be abstract properties, not definable in observational terms.[5] As mathematical objects, forms are best regarded as abstract particulars since in mathematics what does not look like a thing, e.g. a function, is regularly treated as such.[6]

If mathematical objects are forms, the sections of the "intelligible" part of the divided line do not answer to two different kinds of intelligibles, one studied by mathematics, the other by philosophy, each discipline approaching its objects in its own way, but to the distinct ways in which philosophy and mathematics approach intelligibles of a single type, forms; if so, the sections of the "sensible" part of the divided line similarly do not correspond to two kinds of sensibles but to two distinct ways in which sensibles are approached, and forms can be objects of belief and sensibles of knowledge insofar as they are related to forms.[7]

We can restore to mathematics its own objects, intelligible ones distinct from forms but similar to them in two crucial respects that explain the use of the same terminology for the description of both kinds of entities, if we rely on the testimony of Aristotle. According to Aristotle, between forms and sensibles Plato wedged mathematical objects as a third kind of existents. These are similar to forms in two respects, hence intelligible, and to sensibles in another: the so-called intermediates are similar to forms, and differ from sensibles, in that they are eternal and cannot move or suffer any change, but also resemble sensibles, and differ from forms, in that for each of them there are many alike (*Metaph*. A 6, 987b14–18).

Just as there is a single form of beauty over the many beautiful sensible things, there is a single form over the many intermediates that are alike. Aristotle contrasts mathematical numbers, each of which contains its predecessor plus one unit, from those numbers that do not each contain their predecessors: mathematical numbers consist of undifferentiated and combinable units, but each number of the other type has its own units, not combinable with those of any other number (*Metaph*. M 6, 1080a12–35). The units in the numbers of either type lack magnitude, are partless and indivisible (cf. *Metaph*. M 6, 1080b16–20, and 8, 1083b8–17). Aristotle calls numbers which are sets of undifferentiated and indivisible units "monadic" (from μονάς, "unit"). Numbers with combinable units are intermediates since Aristotle

2 The discussion of the Good at *R*. 6 contrasts the oneness of an intelligible form with the many sensibles associated with, or "participating" in, it and thus also named after it (507b1–9); for the contrast see also *Phd*. 78c10–79a5.

3 For their immateriality see *Sph*. 246a7–c3.

4 See *Ti*. 48e2–52d1 and the description of beauty itself in *Smp*. 210e2–211b5. On whether forms are timeless or eternal see Sorabji (1983) 108–112.

5 See e.g. Fine (1999) 215 n. 1.

6 See Gowers (2008) 10. For a précis of Platonism in mathematics see Brown (2005) 59–60.

7 See Fine (1999). All forms can thus be only those of mathematical objects; see ch. 2.9.

says that for each one of them there exist infinitely many alike (*Metaph*. M 7, 1081a5–12); numbers consisting of non-combinable units, on the other hand, are said to be forms since a form is unique (*Metaph*. M 7, 1082b24–28). There is no hint in Plato's works that he introduced the distinction between intermediates and forms, just as nothing in *R*. 6, 510b2–511c2, hints that his example of an object studied in mathematics, the square itself with the diagonal itself, is not a form but an intermediate.[8] Not unreasonably, scholars have doubted that Plato had put forth this distinction even in his discussions with members of the Academy.[9]

It is implausible that Aristotle simply foisted it on him, however. An Academic argument for the existence of forms discussed in his *On forms* was based on the objects of the sciences: the objects of a science exist; they are not particulars, for these are infinitely many and undetermined but each object of a science is single and determined; thus there are things that are different from particulars, and these things are forms (Alex. Aphr. *in Metaph*. 79.8–11 Hayduck). Aristotle would agree with Plato that the objects of mathematics are determined in the sense that each of them is what it is since e.g. lines are just lines, without breadth and depth, and straight ones lack any curvature (Euc. *El*. 1 Def. 2 and 4). But he might object that each one of them is not unique: no number of lines etc. is assumed in geometry, so if the argument shows that there must exist some things different from sensible particulars in that they are determined, these things will not be forms, each of which is unique, but form-like in that each of them must be eternal and not subject to change or motion if it is not a sensible particular and what is not a sensible particular is eternal and does not change or move. Assuming that there are other eternal things that do not change or move, the forms, each of which is unique, Aristotle could argue that Plato is committed to intermediates, thereby trying to answer a question raised by the passage from the *Republic* translated above. In it Plato talks about the visible shapes used in geometrical proofs and about which the geometers seem to argue, such as a square drawn with its diagonal, one of a great many such shapes that can be or are drawn or exist in the physical world, and he also distinguishes them from the intelligible objects that are truly studied in geometry, such as the square itself with the diagonal itself. These are described by him in the same way that he describes forms: the square itself with the diagonal itself seems to answer to the intelligible form of beauty, the beautiful itself, a single being that is associated with many sensibles and appears everywhere as square things, such as the figures drawn in the context of geometrical proofs,

8 E.g. Yang (1999) argues that it is an intermediate, Franklin (2012) 494–497 that it is a form.

9 For references see Arsen (2012) 201, who argues in favor of mathematical intermediates. For a survey of older literature against intermediates in Plato's ontology see Brentlinger (1963). He attempts to strike a middle position suggesting that as intermediates, in a weaker sense than that in which the term is employed by Aristotle, Plato must have regarded the objects of the definitions of arithmetic and geometry: definitions are said in *Ep*. 7, 342a7–344d2, to be one of the four means by which everything is knowable, so their objects, which are different from both sensibles and forms, whose representations they are, are indispensible to mathematical knowledge, actually of forms, a crucial fact mathematicians fail to grasp, ending up treating erroneously as objects of mathematical knowledge what are only means to it. Brentlinger does not explain, however, why Aristotle speaks of Plato's intermediates as eternal, like forms.