Advances in Mechanics and Mathematics 34

Grzegorz Łukaszewicz
Piotr Kalita

Navier–Stokes Equations

An Introduction with Applications

Springer

Advances in Mechanics and Mathematics

Volume 34

More information about this series at http://www.springer.com/series/5613

Grzegorz Łukaszewicz • Piotr Kalita

Navier–Stokes Equations

An Introduction with Applications

Springer

Grzegorz Łukaszewicz
Faculty of Mathematics,
Informatics, and Mechanics
University of Warsaw
Warszawa, Poland

Piotr Kalita
Faculty of Mathematics and Computer
Science
Jagiellonian University in Krakow
Krakow, Poland

ISSN 1571-8689 ISSN 1876-9896 (electronic)
Advances in Mechanics and Mathematics
ISBN 978-3-319-27758-5 ISBN 978-3-319-27760-8 (eBook)
DOI 10.1007/978-3-319-27760-8

Library of Congress Control Number: 2015960205

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

To Renata, Agata, and Jacek, with love
(Grzegorz)

To my beloved wife Kasia
(Piotr)

Preface

Admittedly, as useful a matter as the motion of fluid and related sciences has always been an object of thought. Yet until this day neither our knowledge of pure mathematics nor our command of the mathematical principles of nature have a successful treatment.

–Daniel Bernoulli

Incompressible Navier–Stokes equations describe the dynamic motion (flow) of incompressible fluid, the unknowns being the velocity and pressure as functions of location (space) and time variables. To solve those equations would mean to predict the behavior of the fluid under knowledge of its initial and boundary states. These equations are one of the most important models of mathematical physics. Although they have been a subject of vivid research for more than 150 years, there are still many open problems due to the nature of nonlinearity present in the equations. The nonlinear convective term present in the equations leads to phenomena such as eddy flows and turbulence. In particular the question of solution regularity for three-dimensional problem was appointed by Clay Mathematics Institute as one of the Millennium Problems, that is, the key problems in modern mathematics. This is, on one hand, due to the fact that the problem remains challenging and fascinating for mathematicians and, on the other hand, that the applications of the Navier–Stokes equations range from aerodynamics (drag and lift forces), through design of watercrafts and hydroelectric power plants, to the medical applications of the models of flow of blood in vessels.

This book is aimed at a broad audience of people interested in the Navier–Stokes equations, from students to engineers and mathematicians involved in the research on the subject of these equations.

It originated in part from a series of lectures of the first author given over the past 15 years at the Faculty of Mathematics, Informatics and Mechanics of the University of Warsaw; at summer schools at UNICAMP, Campinas, Brasil; and at Université Jean Monnet, Saint-Etienne, France. The lectures were based on the leading books on the then young theory of infinite dimensional dynamical systems, focused on mathematical physics, in particular, on Temam [220]; Chepyzhov and Vishik [61]; Doering and Gibbon [88]; Foiaş, Manley, Rosa, and Temam [99]; and Robinson [197].

The lectures at the Mathematics Faculty of the University of Warsaw were also attended by students and PhD students from the Faculty of Physics and Faculty of Geophysics, and it became clear that a routine mathematical lecture had to be extended to include additional aspects of hydrodynamics. Some students asked for "more physics and motivation" and "more real applications"; others were mainly interested in the mathematics of the Navier–Stokes equations, and yet others would like to see the Navier–Stokes equations in a more general context of evolution equations and to learn the theory of infinite dimensional dynamical systems on the research level. These several aspects of hydrodynamics well suited the tastes and interests of the lecturer, and also the second author was welcomed to join the project of the book at a later stage.

In consequence, the audience of the book is threesome:

Group I: Mathematicians, physicists, and engineers who want to learn about the Navier–Stokes equations and mathematical modeling of fluids
Group II: University teachers who may teach a graduate or PhD course on fluid mechanics basing on this book or higher-level students who start research on the Navier–Stokes equations
Group III: Researchers interested in the exchange of current knowledge on dynamical systems approach to the Navier–Stokes equations

Although, in principle, all these three groups can find interest in all chapters of the book, Chaps. 2–7 are primarily targeted at Group I, Chaps. 3, 4, 7, 8, 11, and 12 aimed mainly at Group II, and Chaps. 7–16 for Group III.

For a reader with reasonable background on calculus, functional analysis, and theory of weak solutions for PDEs, the whole book should be understandable.

The book was planned to be a monograph which could also be used as a textbook to teach a course on fluid mechanics or the Navier–Stokes equations. Typical courses could be "Navier–Stokes equations", "partial differential equations", "fluid mechanics", "infinite dimensional dynamics systems." To this end many chapters of this book include exercises. Moreover, we did not restrain ourselves to include a number of figures to liven the text and make it more intuitive and less formal. We believe that the figures will be helpful. Special care was undertaken to keep the individual chapters self-contained as far as possible to allow the reader to read the book linearly (in linear portions). That demanded several small repetitions here and there.

To understand the first chapters of this book, just the basic knowledge on calculus, that can be learned from any calculus textbook, should be enough.

The book is planned to be self-contained, but, to understand its last chapters, some knowledge from a textbook like "Partial Differential Equations" by L.C. Evans (which contains all necessary knowledge on functional analysis and PDEs) would be helpful. Each chapter contains an introduction that explains in simple words the nature of presented results and a section on bibliographical notes that will place it in the context of past and current research.

Several people greatly contributed, knowingly or not knowingly, to the creation of the book. Our thanks go to our colleagues and collaborators: Guy Bayada, Mahdi Boukrouche, Thomas Caraballo, José Langa, Pepe Real, James Robinson etc.

The first author is grateful to Guy Bayada who introduced him to the problems of lubrication theory and flows in narrow films during his visits at INSA, Lyon, and to Mahdi Boukrouche with whom he collaborated for several years on this subject. Thomas Caraballo, José Langa, and Pepe Real introduced him to the subject of pullback attractors during his visit at the University of Seville. Thanks for the opportunity to give the summer courses in Campinas and Saint-Etienne go to Marco Rojas-Medar and Mahdi Boukrouche, respectively. Many thanks go also to Chunyou Sun, Meihua Yang, and Yongqin Xie for their kind invitation of the first author to give several lectures at the University of Lanzhou, then at Huazhong University of Science and Technology in Wuhan and at The University in Changsha, China, in June 2013. Inspirational discussions and exchange of ideas with these Chinese friends, including also Qingfeng Ma and Yuejuan Wang, contributed to the form of the last chapters of the book.

The research group at Jagiellonian University with their leader, Stanisław Migórski, greatly motivated the authors as regards contact problems. The second author owes a lot to his colleagues and teachers from Jagiellonian University; he would like to express his thanks especially to Zdzisław Denkowski and Stanisław Migórski. He would also like to thank Robert Schaefer who first introduced to him the topics of fluid mechanics. He is also grateful for inspiring discussions in the field of contact mechanics to his collaborators from the University of Perpignan, Mircea Sofonea and Mikaël Barboteu.

We would like to thank Wojciech Pociecha for his help with the preparation of the figures.

The work was in part supported by the National Science Center of Poland under the Maestro Advanced Project no. DEC-2012/06/A/ST1/00262.

Finally, we express our gratitude to the AMMA Series editor, David Y. Gao, and to Marc Strauss and the editors of Springer Publishing House for their care and encouragement during the preparation of the book.

Warszawa, Poland — Grzegorz Łukaszewicz
Kraków, Poland — Piotr Kalita
October 2015

Contents

1
Introduction and Summary

When you put together the science of movements of water, remember to put beneath each proposition its applications, so that such science may not be without uses.

– Leonardo da Vinci

This chapter provides, for the convenience of the reader, an overview of the whole book, first of its structure and then of the content of the individual chapters.

The outline of the book structure is as follows.

Chapter 2 shows the derivation of the Navier–Stokes equations from the principles of physics and discusses their physical and mathematical properties and examples of the solutions for some particular cases, without going into complicated mathematics. This part is aimed to fill in a gap between an engineer and mathematician and should be understood by anybody with basic knowledge of calculus no more complicated than the Stokes theorem.

In Chap. 3, a necessary mathematical background including these parts of functional analysis and theory of Sobolev spaces which are needed to understand modern research on the Navier–Stokes equations is presented. Chapters 4–6 comprise three examples of stationary problems.

Then we smoothly move to the research level part of the book (Chaps. 7–16) which presents the analysis from the point of view of global attractors of the asymptotic (in time) behavior of the velocities being the solutions of the Navier–Stokes system. Roughly speaking we endeavor to show how the modern theory of global attractors can be used to construct the mathematical objects that enclose the seemingly chaotic and unordered eddy and turbulent flows. We tame these flows by showing their fine properties like the finite dimensionality of global attractors, which means that the description of unrestful and turbulent states can be done by finite number of parameters or existence of invariant measures which means that the flow becomes, in statistical sense, stationary.

We deal with non-autonomous problems using the recent and elegant theory of so-called pullback attractors that allows to cope with flows with changing in time sources, sinks, or boundary data. We also solve problems with multivalued boundary conditions that allow to model various contact conditions between the fluid and enclosing object, such as the stick/slip frictional boundary behavior.

G. Łukaszewicz, P. Kalita, *Navier–Stokes Equations*, Advances in Mechanics and Mathematics 34, DOI 10.1007/978-3-319-27760-8_1

The analysis is primarily done for the two-dimensional Navier–Stokes equations, where, as a model, the problem from lubrication theory, that can be reduced to two dimensions is used. The study with various types of boundary conditions, including multivalued ones, is presented.

Some of the results presented in this part are based on the previous and already published work of the authors, but some results, that are the subjects of current research, are yet unpublished elsewhere.

Below we present the content of the book in some more detail.

In Chap. 2 we give an overview of the equations of classical hydrodynamics. We provide their derivation, discuss the associated physical quantities, comment on the constitutive laws, stress tensor, and thermodynamics, finally we present some elementary properties of the derived system and also some cases where it is possible to calculate the exact solutions of the following system of the incompressible Navier–Stokes equations,

$$\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \frac{1}{\rho}\nabla p = f,$$
$$\operatorname{div} u = 0,$$

which are the main subject of the book.

In Chap. 3 we introduce the basic preliminary mathematical tools to study the Navier–Stokes equations, including results from linear and nonlinear functional analysis (e.g., the Lax–Milgram lemma, fixed point theorems) as well as the theory of function spaces (e.g., compactness theorems). We present, in particular, some of the most frequently used in the sequel embedding theorems and inequalities. We discuss the versions of the Gronwall lemma used in the sequel, and provide some necessary background in the theory of Clarke subdifferential and differential inclusions.

In Chaps. 4–6 we consider stationary problems. Chapter 4 is devoted to the stationary Navier–Stokes equations in a bounded three-dimensional domain Q,

$$-\nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in} \quad Q,$$
$$\operatorname{div} u = 0 \quad \text{in} \quad Q,$$

with one of the boundary conditions:

1. $Q = [0, L]^3$ in $\mathbb{R}^3$ and we assume periodic boundary conditions, or
2. Q is a bounded domain in $\mathbb{R}^3$, with smooth boundary, and we assume the homogeneous boundary condition $u = 0$ on ∂Q.

This basic problem serves as an introduction to the mathematical theory of the Navier–Stokes equations. We introduce the suitable function spaces in which we (usually) seek solutions of the stationary problem, then we present the weak formulation. It allows us to use the theories of linear and nonlinear functional analysis (Lax–Milgram lemma and fixed point theorems, respectively) to prove the existence of solutions.

To show typical methods used when dealing with nonlinear problems, we present a number of proofs based on various linearizations and fixed point theorems in standard function spaces. The solutions, due to the nonlinearity of the Navier–Stokes equations are not in general unique, however, under some restriction on the mass force and viscosity coefficient (quite intuitive from the physical point of view), one can prove their uniqueness.

In Chap. 5 we consider the stationary Navier–Stokes equations with friction in the three-dimensional bounded domain Ω. The domain boundary $\partial\Omega$ is divided into two parts, namely the boundary Γ_D on which we assume the homogeneous Dirichlet boundary condition and the contact boundary Γ_C on which we decompose the velocity into the normal and tangent directions. In the normal direction we assume $u \cdot n = 0$, i.e., there is no leak through the boundary, while in the tangent direction we set $-T_\tau \in h(x, u_\tau)$, the multivalued relation between the tangent stress and tangent velocity. This relation is the general form of the friction law on the contact boundary. The existence of solution is shown by the Kakutani–Fan–Glicksberg fixed point theorem and some cut-off techniques.

In Chap. 6 we consider a typical problem for the hydrodynamic equations coming from the lubrication theory. In this theory the domain of the flow is usually very thin and the engineers are interested in the distribution of the pressure therein. Because of the thinness of the domain, in practice one assumes that the pressure would depend only on two and not three independent variables. The pressure distribution is governed then by the Reynolds equation, which depends on the original boundary data. Our aim is to obtain the Reynolds equation starting from the stationary Stokes equations, considered in the three-dimensional domains Ω^ε, $\varepsilon > 0$,

$$-\nu\Delta u + \nabla p = f \quad \text{in} \quad \Omega^\varepsilon,$$
$$\operatorname{div} u = 0 \quad \text{in} \quad \Omega^\varepsilon,$$

with a Fourier boundary condition on the top Γ_F^ε and Tresca boundary conditions on the bottom part of the boundary Γ_C, respectively. We show how to pass, in a precise mathematical way, as $\varepsilon \to 0$, from the three-dimensional Stokes equations to a two-dimensional Reynolds equation for the pressure distribution (see Fig. 1.1).

The passage from the three-dimensional problem to a two-dimensional one depends on several factors and additional scaling assumptions. The existence of solutions of the limit equations follows from the existence of solutions of the original three-dimensional problem. Finally, we show the uniqueness of the limit solution.

In Chaps. 7–10 we consider the nonstationary autonomous Navier–Stokes equations

$$u_t - \nu\Delta u + (u \cdot \nabla)u + \nabla p = f, \tag{1.1}$$
$$\operatorname{div} u = 0, \tag{1.2}$$

in two-dimensional domains. In these chapters the solutions are global in time and unique.

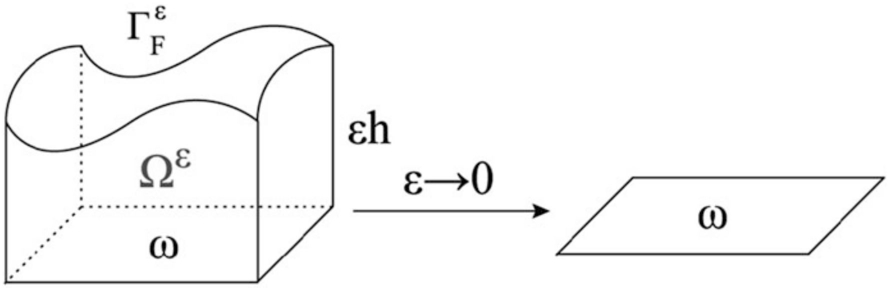

Fig. 1.1 Schematic view of the problem considered in Chap. 6. Incompressible static Stokes equation is solved on the three-dimensional domain Ω^{ε}. The domain thickness is given by εh, where h is a function of the point in the two-dimensional domain ω. As $\varepsilon \to 0$, Reynolds equation on ω is recovered

Chapter 7 is a general introduction to evolutionary two-dimensional Navier–Stokes equations. We prove some basic properties of solutions assuming that the external forces do not depend on time and that the domain of the flow is bounded. The boundary conditions are either periodic or homogeneous Dirichlet ones. In this chapter we introduce the notion of the global attractor, one of the main objects to study also in Chaps. 8–10.

In Chap. 8 we prove the existence of invariant measures associated with two-dimensional autonomous Navier–Stokes equations. The invariant measures are supported on the global attractor. Then we introduce the notion of a stationary statistical solution and prove that every invariant measure is also such a solution. Existence of the invariant measures (stationary statistical solutions) supported on the global attractor reveals the statistical properties of the potentially chaotic fluid flow after a long time of evolution when the external forces do not depend on time. The non-autonomous case is considered in Chap. 12.

In Chaps. 9–10 we consider system (1.1) and (1.2) in the domain Ω depicted in Fig. 1.2, with homogeneous condition $u = 0$ on Γ_D, periodic condition $u(0, x_2) = u(L, x_2)$ on Γ_L, and several contact boundary conditions on Γ_C. The motivation for such problem setup comes again from problems in contact mechanics, the theory of lubrication and shear flows in narrow films.

One may look at the domain Ω as a rectification of the ring-like domain considered in the theory of lubrication, where it represents a cross section of an infinite journal bearing. The problem reduces to describing a flow between two cylinders. The outer cylinder is at rest and the inner cylinder rotates providing a driving force to the fluid (lubricant). Since the cylinders are infinitely long it can be assumed in the first approximation that the flow is two-dimensional. Described domain geometry is schematically presented in Fig. 1.3.

The boundary conditions on Γ_C include the following ones.

In Chap. 9 we pose

$$u = Ue_1, \quad U \in \mathbb{R}, \quad U > 0 \quad \text{on} \quad \Gamma_C,$$

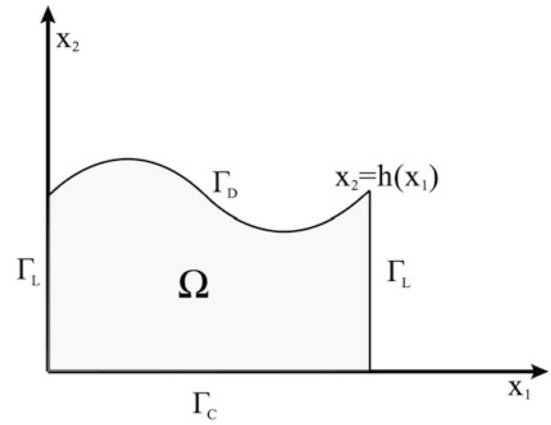

Fig. 1.2 Schematic view of the flow domain and its boundaries in Chaps. 9 and 10

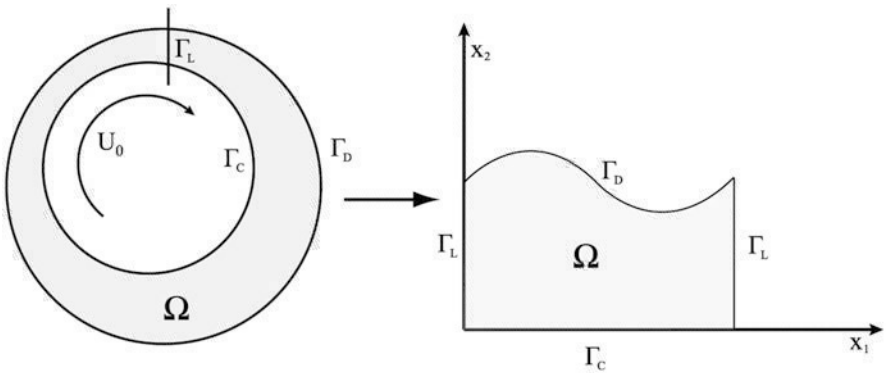

Fig. 1.3 Three-dimensional infinite ring-like domain and its rectification considered in Chaps. 9 and 10

by which we mean that the boundary Γ_C is moving with a constant velocity $U_0 e_1 = (U_0, 0)$ and the velocity of the fluid at the boundary equals the velocity of the boundary.

We prove the existence of a global attractor and estimate from above its fractal dimension in terms of given data and geometry of the domain of the flow.

In Chap. 10 we consider two problems. We assume that there is no flux across Γ_C so that the normal component of the velocity on Γ_C satisfies

$$u \cdot n = 0 \quad \text{on} \quad \Gamma_C,$$

and that the tangential component of the velocity u_τ on Γ_C is unknown and satisfies the Tresca friction law with a constant and positive maximal friction coefficient k. This means that

$$\left.\begin{array}{l} |T_\tau(u,p)| \le k \\ \\ |T_\tau(u,p)| < k \Rightarrow u_\tau = U_0 e_1 \\ \\ |T_\tau(u,p)| = k \Rightarrow \exists \lambda \ge 0 \text{ such that } u_\tau = U_0 e_1 - \lambda T_\tau(u,p) \end{array}\right\} \quad \text{on} \quad \Gamma_C, \tag{1.3}$$

where T_τ is the tangential component of the stress tensor on Γ_C and $U_0 e_1 = (U_0, 0)$, $U_0 \in \mathbb{R}$, is the velocity of the lower surface producing the driving force of the flow.

In the second problem the boundary Γ_C is also assumed to be moving with the constant velocity $U_0 e_1 = (U_0, 0)$ which, together with the mass force, produces the driving force of the flow. The friction coefficient k is assumed to be related to the slip rate through the relation $k = k(|u_\tau - U_0|)$, where $k : \mathbb{R}^+ \to \mathbb{R}^+$. If there is no slip between the fluid and the boundary then the friction is bounded by the threshold $k(0)$

$$u_\tau = U_0 \Rightarrow |T_\tau| \le k(0) \quad \text{on} \quad \Gamma_C, \tag{1.4}$$

while if there is a slip, then the friction force density (equal to tangential stress) is given by the expression

$$u_\tau \ne U_0 \Rightarrow -T_\tau = k(|u_\tau - U_0|) \frac{u_\tau - U_0}{|u_\tau - U_0|} \quad \text{on} \quad \Gamma_C. \tag{1.5}$$

Note that (1.4) and (1.5) generalize the Tresca law (1.3) where k was assumed to be a constant. Here k depends of the slip rate, this dependence represents the fact that the kinetic friction is less than the static one, which holds if k is a decreasing function.

We prove that for both problems above there exist exponential attractors, in particular the global attractors of finite fractal dimension.

In Chaps. 11–13 we consider the time asymptotics of solutions of the two-dimensional Navier–Stokes equations. First, in Chap. 11 we prove two properties of the equations in a bounded domain, concerning the existence of determining modes and nodes. Then we study the equations in an unbounded domain, in the framework of the theory of infinite dimensional non-autonomous dynamical systems, introducing the notion of the pullback attractor.

Chapter 12 presents a construction of invariant measures and statistical solutions for the non-autonomous Navier–Stokes equations in bounded and some unbounded domains in $\mathbb{R}^2$. More precisely, we construct the family of probability measures $\{\mu_t\}_{t\in\mathbb{R}}$ and prove the relations $\mu_t(E) = \mu_\tau(U(t,\tau)^{-1}E)$ for $t, \tau \in \mathbb{R}$, $t \ge \tau$ and Borel sets E in H. The support of each measure μ_t is included in the section $A(t)$ of the pullback attractor. We prove also the Liouville and energy equations. Finally, we consider statistical solutions of the Navier–Stokes equations supported on the pullback attractor.

In Chap. 13 we consider the problem of existence and finite dimensionality of the pullback attractor for a class of two-dimensional turbulent boundary driven flows. We generalize here the results from Chap. 9 to the non-autonomous problem. The new element in our study with respect to that in Chap. 9 is the allowance of the velocity of Γ_C to depend on time, i.e.,

$$u = U(t)e_1, \quad U(t) \in \mathbb{R} \quad \text{on} \quad \Gamma_C.$$

Our aim is to study the time asymptotics of solutions in the frame of the dynamical systems theory. We prove the existence of the pullback attractor and estimate its fractal dimension. We shall apply the results from Chap. 11, reformulated here in the language of evolutionary processes.

Chapters 14–16 are devoted to global in time solutions of the Navier–Stokes equations which are not necessary unique. We introduce theories of global attractors for multivalued semiflows and multivalued processes to include this situation. We study further examples of contact problems in both autonomous and non-autonomous cases.

In Chap. 14 we consider two-dimensional nonstationary Navier–Stokes shear flows in the domain Ω as in Fig. 1.2, with nonmonotone and multivalued boundary conditions on Γ_C. Namely, we assume the following subdifferential boundary condition

$$\tilde{p}(x,t) \in \partial j(u_n(x,t)) \quad \text{on} \quad \Gamma_C,$$

where $\tilde{p} = p + \frac{1}{2}|u|^2$ is the Bernoulli (total) pressure, $u_n = u \cdot n$, $j : \mathbb{R} \to \mathbb{R}$ is a given locally Lipschitz superpotential, and ∂j is a Clarke subdifferential of j.

Our considerations are motivated here by feedback control problems for fluid flows in domains with semipermeable walls and membranes and by the theory of lubrication. We prove the existence of global in time solutions of the considered problem which is governed by a partial differential inclusion, and then we prove the existence of a trajectory attractor and a weak global attractor for the associated multivalued semiflow.

In Chap. 15 we study the three-dimensional problem in a bounded domain Ω. The problem domain is the three-dimensional counterpart of the one presented in Fig. 1.2. The boundary of Ω is divided into three parts: the lateral one Γ_L on which we assume the periodic boundary conditions, the homogeneous Dirichlet one and, finally, the contact one Γ_C on which we consider a general form of multivalued frictional type boundary conditions $-T_\tau \in g(u_\tau)$. We prove the existence of the Leray–Hopf weak solutions and, using the framework of evolutionary systems, existence of the weak global attractor.

Finally, in Chap. 16 we consider further non-autonomous and multivalued evolution problems, this time in the frame of the theory of pullback attractors for multivalued processes. First we prove an abstract theorem on the existence of pullback $\mathscr{D}$-attractor and then apply it to study a two-dimensional incompressible

Navier–Stokes flow with a general form of multivalued frictional contact conditions on Γ_C. We assume that there is no flux across Γ_C and hence we have

$$u_n(t) = 0 \quad \text{on} \quad \Gamma_C,$$

and that the tangential component of the velocity u_τ on Γ_C is in the following relation with the tangential stresses T_τ,

$$-T_\tau(t) \in \partial j(x, t, u_\tau(t)) \quad \text{on} \quad \Gamma_C.$$

In above formula $j : \Gamma_C \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a potential which is locally Lipschitz and not necessarily convex with respect to the last variable, and ∂ is the subdifferential in the sense of Clarke taken with respect to the last variable u_τ.

The tangent conditions on Γ_C in Chaps. 15 and 16 represent the frictional contact between the fluid and the wall, where the friction force depends in a nonmonotone and even discontinuous way on the slip rate, and are a generalization of the conditions considered in Chap. 10. For this case we prove the existence of the attractor.

Most chapters are devoted to two-dimensional problems. Three-dimensional problems are considered only in Chaps. 2, 4, 5, 6, and 15. One reason for that is associated with the character of the Navier–Stokes equations, namely the fact that in the two-dimensional problems it is relatively easy to prove the uniqueness of the solutions which allows us to use the well-developed theory of infinite dimensional dynamical systems for semigroup and processes, while the uniqueness of the three-dimensional Navier–Stokes equations is in general an open question. We also consider the two- and three-dimensional problems without assuming the solution uniqueness in the framework of (more recent) theories of trajectory attractors, multivalued semiflows, evolutionary systems, and multivalued processes.

The other reason to focus on the two-dimensional flows concern the simplicity. Our aim was to test first the more elementary two-dimensional models of some real engineering problems. The word "test" here means not only checking the well posedness of a particular problem. In Chap. 9 we estimate the attractor dimension and show how it depends on the shape of the domain (cf. [24, 26], where the upper bounds of the attractor dimension depend also on the geometry of Γ_D). Assume that the answer to the question on the dependence of the attractor dimension on the geometry of the boundary is such that in the two-dimensional case the estimate from above of the attractor dimension is independent of geometry (for example, on the roughness of the boundary represented by the oscillations of the function $h = h(x_1)$ describing Γ_D). Such a result would be contradictory to our intuition, provided the intuitive hypotheses

attractor dimension $\sim$ *level of chaos in the flow* $\sim$ *geometry of the flow domain*

where "$\sim$" means "is related to," are justified.

Such a contradiction with our intuition could be resolved in the following ways:

1. there is no such contradiction in the “real” three-dimensional case, it appears only in the two-dimensional case (but where lies the difference?),
2. the attractor dimension does not represent the level of chaos in the fluid flow described by the (good) model of the Navier–Stokes equations,
3. the Navier–Stokes equations model is not good enough to give the right answer to the problems of chaotic movement of the classical fluids.

The close correspondence between the level of chaos in the fluid flow and the geometry of the domain is evident as a physical phenomenon (recall observing a flow of water in the river).

To confirm the agreement of the results provided by the modeling with our physical intuition or else to confront the above potential possibilities motivated us to study the problems of the existence and properties of the attractor. There are still many interesting and important problems close to these considered in the book and we were able to touch only a few ones. One example is to further study the relations between the (type of) boundary conditions and the attractor dimension.

Finally, we remark that this book is devoted to incompressible flows, for the mathematical treatise of compressible ones see, e.g., [157, 187].

2
Equations of Classical Hydrodynamics

The neglected borderline between two branches of knowledge is often that which best repays cultivation, or, to use a metaphor of Maxwell's, the greatest benefits may be derived from a cross-fertilization of the sciences.

– John William Strutt, 3rd Baron Rayleigh

In this chapter we give an overview of the equations of classical hydrodynamics. We provide their derivation, comment on the stress tensor, and thermodynamics, finally we present some elementary properties and also some exact solutions of the Navier–Stokes equations.

2.1 Derivation of the Equations of Motion

Fluid flow may be represented mathematically as a *continuous transformation* of three-dimensional Euclidean space into itself. The transformation is parametrized by a real parameter t representing time.

Let us introduce a fixed rectangular coordinate system (x_1, x_2, x_3). We refer to the coordinate triple (x_1, x_2, x_3) as the *position* and denote it by x. Now consider a particle P moving with the fluid, and suppose that at time $t = 0$ it occupies a position $X = (X_1, X_2, X_3)$ and that at some other time t, $-\infty < t < +\infty$, it has moved to a position $x = (x_1, x_2, x_3)$. Then x is determined as a function of X and t

$$x = x(X, t) \quad \text{or} \quad x_i = x_i(X, t)\,. \tag{2.1}$$

If X is fixed and t varies, Eq. (2.1) specifies the *path* of the particle initially at X. On the other hand, for fixed t, (2.1) determines a transformation of a region initially occupied by the fluid into its position at time t.

G. Łukaszewicz, P. Kalita, *Navier–Stokes Equations*, Advances in Mechanics and Mathematics 34, DOI 10.1007/978-3-319-27760-8_2

We assume that the transformation (2.1) is *continuous* and *invertible*, that is, there exists its inverse

$$X = X(x,t), \quad (\text{or} \quad X_i = X_i(x,t)) \,.$$

Also, to be able to differentiate, we assume that the functions x_i and X_i are sufficiently smooth.

From the condition that the transformation (2.1) possess a differentiable inverse it follows that its Jacobian

$$J = J(X,t) = \det\left(\frac{\partial x_i}{\partial X_j}\right)$$

satisfies

$$0 < J < \infty \,. \tag{2.2}$$

The initial coordinates X of the particle will be referred to as the *material coordinates* of the particle. The *spatial coordinates* x may be referred to as its *position*, or *place*. The representation of fluid motion as a *point transformation* violates the concept of the *kinetic theory* of fluids, as in this theory the particles are molecules, and they are in random motion. In the theory of *continuum mechanics* the state of motion at a given point x and at a given time t is described by a number of functions such as $\rho = \rho(x,t)$, $u = u(x,t)$, $\theta = \theta(x,t)$ representing density, velocity, temperature, and other *hydrodynamical variables*.

Due to the transformation (2.1), each such variable f can also be expressed in terms of material coordinates

$$f(x,t) = f(x(X,t),t) = F(X,t) \,. \tag{2.3}$$

The *velocity* u at time t of a particle initially at X is given, by definition, as

$$u(x,t) = U(X,t) = \frac{d}{dt}x(X,t)\,, \quad (x = x(X,t)) \,. \tag{2.4}$$

Above, X is treated as a parameter representing a given fixed particle, and this is the reason that we use the ordinary derivative in (2.4).

Having the velocity field $u(x,t)$, we can (in principle) determine the transformation (2.1), solving the ordinary differential equation

$$\frac{d}{dt}x(X,t) = u(x(X,t),t),$$

with $x(X,0) = X$, where X is a parameter.

We shall always write

$$\frac{d}{dt}F(X,t) \quad \text{and} \quad \frac{\partial}{\partial t}f(x,t)\,,$$

where F and f are related by (2.3). We have thus

$$\frac{d}{dt}F(X,t) = \frac{d}{dt}f(x(X,t),t) = \frac{\partial f}{\partial x_i}(x(X,t),t)\frac{dx_i}{dt} + \frac{\partial f}{\partial t}(x(X,t),t)\,,$$

so that by (2.4) we obtain a general formula

$$\frac{d}{dt}F(X,t) = \frac{D}{Dt}f(x,t)\,, \tag{2.5}$$

where $\frac{D}{Dt}f(x,t) \equiv \frac{\partial f}{\partial t}(x,t) + u(x,t)\cdot\nabla f(x,t)$ is called the *material derivative* of f.

Transport Theorem Let $\Omega(t)$ denote an arbitrary volume that is moving with the fluid and let $f(x,t)$ be a scalar or vector function of position and time. The *transport theorem* states that

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)\,dx \tag{2.6}$$

$$= \int_{\Omega(t)}\left\{\frac{\partial f}{\partial t}(x,t) + u(x,t)\cdot\nabla f(x,t) + f(x,t)\,\mathrm{div}\,u(x,t)\right\}\,dx\,.$$

For the proof consider the transformation

$$x:\Omega(0)\to\Omega(t),\quad x = x(X,t)\,,$$

as in (2.1). Then

$$\int_{\Omega(t)} f(x,t)\,dx$$

$$= \int_{\Omega(0)} f(x(X,t),t)J(X,t)\,dX = \int_{\Omega(0)} F(X,t)J(X,t)\,dX\,,$$

so that

$$\frac{d}{dt}\int_{\Omega(t)} f(x,t)dx = \frac{d}{dt}\int_{\Omega(0)} F(X,t)J(X,t)\,dX \tag{2.7}$$

$$= \int_{\Omega(0)}\left\{\frac{d}{dt}F(X,t)J(X,t) + F(X,t)\frac{d}{dt}J(X,t)\right\}\,dX\,.$$

By (2.5) we have

$$\int_{\Omega(0)} \frac{d}{dt} F(X,t) J(X,t)\, dX$$

$$= \int_{\Omega(0)} \left\{ \frac{\partial f}{\partial t}(x(X,t),t) + u(x(X,t),t) \cdot \nabla f(x(X,t),t) \right\} J(X,t)\, dX$$

$$= \int_{\Omega(t)} \left\{ \frac{\partial f}{\partial t}(x,t) + u(x,t) \cdot \nabla f(x,t) \right\} dx\,.$$

To prove (2.6) it remains to prove the *Euler formula*

$$\frac{d}{dt} J(X,t) = \operatorname{div} u(x(X,t),t) J(X,t)\,, \tag{2.8}$$

the proof of which we leave to the reader as an exercise.

The fluid is called *incompressible* if for any domain $\Omega(0)$ and any t,

$$\text{volume } (\Omega(t)) = \text{volume } (\Omega(0))\,.$$

From (2.7) with $f(x,t) \equiv 1$ we have

$$\frac{d}{dt}\text{volume } (\Omega(t)) = \frac{d}{dt} \int_{\Omega(t)} dx = \int_{\Omega(0)} \frac{d}{dt} J(X,t) dX\,,$$

hence by (2.8), (2.2), and the arbitrariness of choice of the domain $\Omega(t)$ via $\Omega(0)$ a necessary and sufficient condition for the fluid to be incompressible is

$$\operatorname{div} u(x,t) = 0\,.$$

Exercise 2.1. Prove that the transport theorem can be written in the form

$$\frac{d}{dt} \int_{\Omega(t)} f(x,t)\, dx = \int_{\Omega(t)} \frac{\partial f}{\partial t}(x,t)\, dx + \int_{\partial\Omega(t)} f(x,t) u(x,t) \cdot n(x,t)\, dS\,,$$

where $n(x,t)$ is the outward unit normal to $\partial\Omega(t)$ at $x \in \partial\Omega(t)$.

Equation of Continuity Let $\rho = \rho(x,t)$ be the mass per unit volume of a fluid at point x and time t. Then the mass of any finite volume Ω is

$$m = \int_{\Omega} \rho(x,t)\, dx\,.$$

The *principle of conservation of mass* says that the mass of a fluid in a material volume Ω does not change as Ω moves with the fluid; that is,

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x,t)\,dx = 0\,.$$

From the transport theorem (2.6) it follows that

$$\int_{\Omega(t)} \left\{\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u)\right\} dx = 0\,,$$

whence

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho u) = 0\,. \tag{2.9}$$

Sometimes the principle of conservation of mass is expressed as follows. Let Ω be a fixed volume. Then

$$\frac{d}{dt}\int_{\Omega} \rho(x,t)\,dx = -\int_{\partial\Omega} \rho u\cdot n\,dS\,, \tag{2.10}$$

that is, the rate of change of mass in a fixed volume Ω is equal to the mass flux through its surface.

We notice also the general formula

$$\frac{d}{dt}\int_{\Omega(t)} \rho f\,dx = \int_{\Omega(t)} \rho\frac{D}{Dt}f\,dx\,. \tag{2.11}$$

Exercise 2.2. Derive (2.9) from (2.10).

Exercise 2.3 (Cf. [212]). Show that in material coordinates the equation of continuity is

$$\frac{d}{dt}\{\rho(X,t)J(X,t)\} = 0\,,$$

or

$$\rho(X,t)J(X,t) = \rho(X,0)\,.$$

Exercise 2.4 (Cf. [5]). Show that if $\rho_0(X)$ is the distribution of density of the fluid at time $t=0$ and $\nabla(\operatorname{div} u)=0$, then

$$\rho(x,t) = \rho_0(X(x,t))\exp\left\{-\int_0^t \operatorname{div} u(x,t)\,dt\right\}\,.$$

Exercise 2.5. Find $\rho(x,t)$ for the motion

$$u_i = \frac{x_i}{1+a_i t} \quad (a_1 = 2\,, a_2 = 1\,, a_3 = 0),$$

if $\rho_0(X)$ is the distribution of density of the fluid at time $t = 0$.

Exercise 2.6. Prove (2.11).

Principle of Conservation of Linear Momentum We assume that the forces acting on an element of a continuous medium are of two kinds. *External*, or *body, forces*, such as gravitation or electromagnetic forces, can be regarded as reaching into the medium and acting throughout the volume. If f represents such a force *per unit mass*, then it acts on an element Ω as

$$\int_\Omega \rho f\, dx\,.$$

The *internal*, or *contact, forces* are to be regarded as acting on an element of volume Ω through its bounding surface. Let n be the unit outward normal at a point of the surface $\partial\Omega$, and t_n the force *per unit area* exerted there by the material volume outside $\partial\Omega$. Then the surface force exerted on the volume Ω can be expressed by the integral

$$\int_{\partial\Omega} t_n\, dS\,.$$

The *Cauchy principle* says that t_n depends at any given time only on the position and the orientation of the surface element dS; in other words,

$$t_n = t_n(x,t,n)\,.$$

The *principle of conservation of linear momentum* says that the rate of change of linear momentum of a material volume equals the resultant force on the volume

$$\frac{d}{dt}\int_{\Omega(t)} \rho u dx = \int_{\Omega(t)} \rho f\, dx + \int_{\partial\Omega(t)} t_n\, dS\,, \tag{2.12}$$

where f is assumed to be known.

By (2.11), (2.12) yields

$$\int_{\Omega(t)} \rho\frac{Du}{Dt} dx = \int_{\Omega(t)} \rho f\, dx + \int_{\partial\Omega(t)} t_n\, dS\,. \tag{2.13}$$

From this equation we derive a very important fact, namely, that the vector t_n (called *normal stress*) can be expressed as a linear function of n, in the form

$$t_n(x,t,n) = n(x,t)T(x,t)\,, \tag{2.14}$$

where $T = \{T_{ij}\}$ is a matrix called the *stress tensor*. This will allow us to pass from the integral form (2.13) of the equation of conservation of linear momentum to a differential one.

Let l^3 be the volume of $\Omega = \Omega(t)$. Dividing both sides of (2.13) by l^2 and letting the volume tend to zero we obtain

$$\lim_{|\Omega|\to 0} l^{-2} \int_{\partial\Omega} t_n \, dS = 0\,, \tag{2.15}$$

that is, the stress forces are in local equilibrium.

Let Ω be a domain containing a fluid, and consider a regular tetrahedron with vertex at an arbitrary point $x \in \partial\Omega$, and with three of its faces parallel to the coordinate planes. Let the slanted face have normal $n = (n_1, n_2, n_3)$ and area Σ. The normals to the other faces are $-e_1$, $-e_2$, and $-e_3$, and their areas are $n_1\Sigma$, $n_2\Sigma$, and $n_3\Sigma$. Applying (2.15) to the family of tetrahedrons obtained by letting $\Sigma \to 0$, we obtain

$$t(n) + n_1 t(-e_1) + n_2 t(-e_2) + n_3 t(-e_3) = 0\,, \tag{2.16}$$

where $t(n) = t_n = t_n(x,t,n)$, $t(-h) = t_{-h}$ for $h \in \{e_1, e_2, e_3\}$, and $n_i > 0$. By a continuity argument, (2.16) holds for all $n_i \geq 0$, and then we prove easily that $t(e_i) = -t(-e_i)$, $i = 1,2,3$, and that it holds for all n. This means that $t(n)$ may be expressed as a linear function of n; that is, we can write it in the form (2.14). Thus, by (2.13) and by the Green theorem we obtain

$$\int_{\Omega(t)} \rho \frac{Du}{Dt} \, dx = \int_{\Omega(t)} (\rho f + \operatorname{div} T) \, dx\,,$$

whence, by the arbitrariness of the domain of integration,

$$\rho \frac{Du}{Dt} = \rho f + \operatorname{div} T\,, \tag{2.17}$$

or

$$\rho \left(\frac{\partial}{\partial t} u_i + \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j} u_i \right) = \rho f_i + T_{ji,j}, \quad i = 1,2,3\,.$$

This is the general *Cauchy equation of motion* in differential form.

Exercise 2.7. Give a physical interpretation of the components of the stress tensor.

Notice that we have not specified T yet, that is, we have not made any assumptions concerning the nature of forces acting on surface elements. These forces depend on the kind of fluid, or, more generally, on the kind of medium under consideration.

In the simplest model the contact forces act perpendicularly to the surface elements. We have then

$$t(n) = -p(x)n\,,$$

and call p the *pressure*. The minus sign is chosen so that when $p > 0$, the contact forces on a closed surface tend to compress the fluid inside; p represents the pressure exerted from outside on a surface of the element of the fluid.

In particular, all fluids at rest exhibit this stress behavior, namely that an element of area always experiences a stress normal to itself, and this stress is independent of the orientation. Such stress is called *hydrostatic*.

We call this idealized model a *perfect fluid*. The equation of motion for perfect fluids is

$$\rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) = \rho f - \nabla p\,,$$

where

$$(u\cdot\nabla)u_i = \sum_{j=1}^{3} u_j\frac{\partial}{\partial x_j}u_i\,,\quad i = 1,2,3\,.$$

All real fluids when in motion can exert tangential stresses across surface elements, in which case the tensor T is not diagonal.

The stress tensor may always be written in the form

$$T_{ij} = -p\delta_{ij} + P_{ij}\,.$$

In this case P_{ij} is called the *viscous stress tensor*.

In classical fluid dynamics it is assumed that the stress tensor is *symmetric*, that is,

$$T_{ij} = T_{ji}\,.$$

This assumption has very important consequences. It may be also considered as a theorem if we assume a specific form of the equation of conservation of angular momentum. We shall discuss this in Sect. 2.2.

Exercise 2.8 (Cf. [5]). Show that the Cauchy equation of motion can be written as

$$\frac{\partial}{\partial t}(\rho u_i) = \rho f_i + (T_{ji} - \rho u_j u_i)_{,j}\,,$$

and interpret it physically.

Exercise 2.9 (Cf. [5]). Show that if F is any function of position and time, then

$$\int_{\partial\Omega} F T_{ji} n_j \, dS = \int_{\Omega} \left[T_{ji} F_{,j} + \rho F \left(\frac{Du_i}{Dt} - f_i \right) \right] dx$$

(theorem of stress means).

Equation of Energy *The first law of thermodynamics* in classical hydrodynamics states that the increase of total energy (we shall consider here only kinetic and internal energies) in a material volume is the sum of the heat transferred and the work done on the volume. We denote by q the *heat flux* (then $-q \cdot n$ is the heat flux into the volume) and by E the *specific internal energy*. Then the balance expressed by the first law of thermodynamics is, cf. [5],

$$\frac{d}{dt} \int_{\Omega(t)} \rho \left(\frac{1}{2} |u|^2 + E \right) dx \tag{2.18}$$

$$= \int_{\Omega(t)} \rho f \cdot u \, dx + \int_{\partial\Omega(t)} t_n \cdot u \, dS - \int_{\partial\Omega(t)} q \cdot n \, dS \, .$$

The first integral on the right-hand side is the rate at which the body force does work, the second integral represents the work done by the stress, and the third integral is the total heat flux into the volume.

We shall write this equation in another form. From the theorem of stress means (Exercise 2.9) we have, with $F = u_i$,

$$\int_{\partial\Omega(t)} u_i T_{ji} n_j \, dS = \int_{\Omega(t)} \left(T_{ji} u_{i,j} + \rho u_i \frac{Du_i}{Dt} - \rho f_i u_i \right) dx \, .$$

Rearranging the terms and using the transport theorem, we obtain

$$\frac{d}{dt} \int_{\Omega(t)} \rho \frac{1}{2} |u|^2 \, dx = \int_{\Omega(t)} \rho \frac{1}{2} \frac{D}{Dt} |u|^2 \, dx \tag{2.19}$$

$$= \int_{\Omega(t)} \rho f_i u_i \, dx - \int_{\Omega(t)} T_{ji} u_{i,j} \, dx + \int_{\partial\Omega(t)} u_i (t_n)_i \, dS \, .$$

Thus the rate of change of kinetic energy of a material volume is the sum of three parts: the rate at which the body forces do work, the rate at which the internal stresses do work, and the rate at which the surface stresses do work.

From (2.18), (2.19), the transport theorem, and the Green theorem we obtain

$$\int_{\Omega(t)} \left(\rho \frac{DE}{Dt} + \nabla \cdot q - T : (\nabla u) \right) dx = 0 \, ,$$

where $T : (\nabla u)$ is the dyadic notation for $T_{ji} u_{i,j}$, the scalar product of T and ∇u.

Thus

$$\rho \frac{DE}{Dt} = -\nabla \cdot q + T : (\nabla u) \, .$$

Conservation Laws of Classical Hydrodynamics Above we obtained the following system of conservation laws of classical hydrodynamics

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot u \, , \tag{2.20}$$

$$\rho \frac{Du}{Dt} = \nabla \cdot T + \rho f \, , \tag{2.21}$$

$$\rho \frac{DE}{Dt} = -\nabla \cdot q + T : (\nabla u) \, . \tag{2.22}$$

They are laws of conservation of mass, momentum, and energy, respectively.

If we assume the *Fourier law* for the conduction of heat,

$$q = -k \nabla \theta \quad (k \geq 0) \, , \tag{2.23}$$

where k is the *thermal conductivity* of the fluid then the energy equation takes the form

$$\rho \frac{DE}{Dt} = \nabla \cdot (k \nabla \theta) + T : (\nabla u) \, .$$

2.2 The Stress Tensor

In the classical hydrodynamics the stress tensor T is defined by

$$T_{ij} = (-p + \lambda u_{k,k}) \delta_{ij} + \mu (u_{i,j} + u_{j,i}) \, . \tag{2.24}$$

If we define the *deformation tensor*

$$D_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{2.25}$$

then the above formula takes the form

$$T_{ij} = (-p + \lambda u_{k,k}) \delta_{ij} + 2\mu D_{ij} \, . \tag{2.26}$$

Remark 2.1. Formula (2.26) is a consequence of a number of postulates, coming originally from G. Stokes, about the fundamental properties of fluids. These postulates can be formulated as follows (cf. [5, 212]):

(a) The stress tensor T is a continuous function of the deformation tensor D and the local thermodynamic state, but independent of other kinematic quantities.

(b) The fluid is homogeneous; that is, T does not depend explicitly on x.
(c) The fluid is isotropic; that is, there is no preferred direction.
(d) When there is no deformation ($D = 0$), and the fluid is incompressible ($u_{k,k} = 0$), the stress is hydrostatic ($T = -pI$, I is the unit matrix).

Fluids that satisfy these postulates are called *Stokesian*. It can be proved (cf. [5, 212]) that the most general form of the stress tensor in this case is

$$T = (-p + \alpha)I + \beta D + \gamma D^2\,,$$

where p, α, β, γ are some functions that depend on the thermodynamic state, α, β, γ being dependent as well on the invariants of the tensor D.

Moreover, when the dependence of the components of T on the components of D is postulated to be *linear*, the stress tensor can be written as

$$T = (-p + \lambda\ \operatorname{div} u)I + 2\mu D\,,$$

which coincides with (2.24). Such linear Stokesian fluids are called *Newtonian*. Fluids that are not Newtonian are called *non-Newtonian*. One important example of the latter are the micropolar fluids [92, 159].

The Stress Tensor and the Law of Conservation of Angular Momentum Looking at the form of the equation of conservation of linear momentum

$$\frac{d}{dt}\int_{\Omega(t)} \rho u\, dx = \int_{\Omega(t)} \rho f\, dx + \int_{\partial\Omega(t)} t_n\, dS\,,$$

and recalling the definition of angular momentum in mechanics of mass points or rigid particles, it seems natural to *assume* the following form of the *law of conservation of angular momentum*:

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x \times u)\, dx = \int_{\Omega(t)} \rho(x \times f)\, dx + \int_{\partial\Omega(t)} x \times t_n\, dS\,. \tag{2.27}$$

In fact, this form of the law of conservation of angular momentum holds if we assume that all torques arise from macroscopic forces. This is the case in most common fluids, but a fluid with a strongly polar character, e.g., a polyatomic fluid, is capable of transmitting stress torques and being subjected to body torques. We call such fluids *polar*.

Theorem 2.1. *For an arbitrary continuous medium satisfying the continuity equation (2.9) and the dynamical equation (2.17) the following statements are equivalent:*

(i) *the stress tensor is symmetric,*
(ii) *equation (2.27) holds.*

Remark 2.2. In classical hydrodynamics the stress tensor is symmetric, and the law of conservation of angular momentum is defined by Eq. (2.27). In consequence, in classical hydrodynamics the law of conservation of angular momentum can be derived from the law of conservation of mass and the law of conservation of linear momentum, and as such adds nothing to the description of the fluid.

Proof. Let us assume (ii), and we shall deduce (i). Applying formula (2.11), we have from (2.27)

$$\frac{d}{dt}\int_{\Omega(t)} \rho(x \times u)\,dx \tag{2.28}$$

$$= \int_{\Omega(t)} \rho \frac{D}{Dt}(x \times u)\,dx = \int_{\Omega(t)} \rho \left(x \times \frac{Du}{Dt}\right) dx$$

$$= \int_{\Omega(t)} \rho(x \times f)\,dx + \int_{\partial\Omega(t)} x \times t_n\,dS\,.$$

By the Green theorem,

$$\int_{\partial\Omega(t)} x \times t_n\,dS = \int_{\Omega(t)} (x \times (\nabla \cdot T) + T_x)\,dx\,, \tag{2.29}$$

where $\nabla \cdot T$ is another notation for $\operatorname{div} T$, and T_x is the vector $\epsilon_{ijk}T_{jk}$ (ϵ_{ijk} is the alternating tensor of Levi-Civita), so that by (2.28)

$$\int_{\Omega(t)} x \times \left(\rho\frac{Du}{Dt} - \rho f - \nabla \cdot T\right) dx = \int_{\Omega(t)} T_x\,dx\,.$$

The left-hand side vanishes identically by the Cauchy equation; hence the right-hand side vanishes for an arbitrary volume, and so $T_x = 0$. However, the components of T_x are $T_{23} - T_{32}$, $T_{31} - T_{13}$, $T_{12} - T_{21}$, and the vanishing of these implies $T_{ij} = T_{ji}$, so that T is symmetric.

We leave to the reader the proof that (i) implies (ii). □

2.3 Field Equations

Substituting the stress tensor (2.24) into the system (2.20)–(2.22) we obtain the system of *field equations* of classical hydrodynamics

$$\frac{D\rho}{Dt} = -\rho\nabla \cdot u\,, \tag{2.30}$$

$$\rho\frac{Du}{Dt} = -\nabla p + (\lambda + \mu)\nabla \operatorname{div} u + \mu\Delta u + \rho f\,, \tag{2.31}$$

$$\rho \frac{DE}{Dt} = -p \operatorname{div} u + \rho \Phi - \nabla \cdot q \,, \tag{2.32}$$

where

$$\rho \Phi = \lambda (\operatorname{div} u)^2 + 2\mu D : D \tag{2.33}$$

is the *dissipation function* of mechanical energy per mass unit.

Let us assume that the fluid is *viscous* and *incompressible*, namely, that $\mu > 0$ and

$$\operatorname{div} u = 0 \,, \tag{2.34}$$

that the specific internal energy of the fluid is proportional to its temperature,

$$E = c_r \theta \,, \quad \text{where} \quad c_r = \text{const} > 0 \,, \tag{2.35}$$

and that Fourier's law (2.23) (with $k = const \geq 0$) holds. With (2.34), (2.35), (2.23), and (2.33), system (2.30)–(2.32) becomes

$$\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho = 0 \,, \quad \operatorname{div} u = 0 \,, \tag{2.36}$$

$$\rho \left(\frac{\partial u}{\partial t} + (u \cdot \nabla) u \right) = -\nabla p + \mu \Delta u + \rho f \,, \tag{2.37}$$

$$\rho c_r \left(\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta \right) = 2\mu D : D + k \Delta \theta. \tag{2.38}$$

2.4 Navier–Stokes Equations

Assuming that the density ρ of the fluid is uniform and denoting $\nu = \frac{\mu}{\rho}$, $\kappa = \frac{k}{\rho}$ (ν is called the *kinematic viscosity* coefficient), Eqs. (2.36)–(2.38) reduce to

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \Delta u + f \,, \tag{2.39}$$

$$\operatorname{div} u = 0 \,, \tag{2.40}$$

$$c_r \left(\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta \right) = 2\nu D : D + \kappa \Delta \theta. \tag{2.41}$$

When the body forces f do not depend on temperature, the first two equations of the above system,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \Delta u + f\,, \tag{2.42}$$

$$\operatorname{div} u = 0 \tag{2.43}$$

constitute a closed system of equations with respect to variables u, p, and are called *Navier–Stokes equations* of viscous incompressible fluids with uniform density (we shall call them just the Navier–Stokes equations). The mechanical energy of the flow governed by (2.42) and (2.43) due to viscous dissipation is lost and appears as heat. This can be seen from Eq. (2.41) in which the term $2\nu D : D$ is positive, provided the flow is not uniform. In real fluids, however, density depends on temperature, so that our system (2.39)–(2.41) may be physically impossible. In fact, due to viscosity and high velocity gradients the temperature rises in view of (2.41), and this produces density fluctuations, contrary to our assumption that density is uniform in the flow domain. Thus, reduced problems often play the role of more or less justified approximations. For more considerations of this kind cf. [109, Chap. 1].

When the body forces depend on temperature, $f = f(\theta)$, we have to take into account the whole system (2.39)–(2.41). One of the considered in the literature system of equations of heat conducting viscous and incompressible fluid are the so-called *Boussinesq equations*,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho_0}\nabla p + \nu \Delta u + \frac{1}{\rho_0} g\alpha(\theta - \theta_0)\,, \tag{2.44}$$

$$\operatorname{div} u = 0\,, \tag{2.45}$$

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla\theta = \frac{\kappa}{c_r}\Delta\theta\,, \tag{2.46}$$

where g represents the vertical *gravity acceleration*, α is the *thermal expansion coefficient*, and $\frac{\kappa}{c_r}$ is the *thermal diffusion coefficient*. Moreover, ρ_0 and θ_0 are some reference density and temperature, respectively. In the velocity equation the vertical buoyancy force $\frac{1}{\rho_0} g\alpha(\theta - \theta_0)$ results from changes of density associated with temperature variations $\rho - \rho_0 = -\alpha(\theta - \theta_0)$. This is the only term in the system where changes of density were taken into account. We have also abandoned the viscous dissipation term in the temperature equation.

2.5 Vorticity Dynamics

Taking the *curl* of the equation of motion

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\frac{1}{\rho}\nabla p + \nu \Delta u,$$

we obtain

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = (\omega \cdot \nabla)u + \nu \Delta \omega, \tag{2.47}$$

where the vector field $\omega = \nabla \times u$ is called *vorticity* of the fluid. It has a simple physical interpretation. In the case of two-dimensional motion with

$$u = (u_1(x, y), u_2(x, y), 0),$$

the vorticity reduces to

$$\omega = (0, 0, \omega_3(x_1, x_2)) = \left(0, 0, \frac{\partial u_2(x_1, x_2)}{\partial x_1} - \frac{\partial u_1(x_1, x_2)}{\partial x_2}\right),$$

where the third component represents twice the angular velocity of a small (infinitesimal) fluid element at point (x_1, x_2). The vorticity field is, by definition, divergence free,

$$\operatorname{div} \omega = 0.$$

In the case of two-dimensional motions the Eq. (2.47) reduces to

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = \nu \Delta \omega,$$

and we can see that the vorticity in the fluid is transported by two agents: *convection* and *diffusion*, just as the temperature in the system (2.44)–(2.46).

For inviscid fluids ($\nu = 0$) the vorticity field has very important properties that allow us to imagine behavior of complicated turbulent flows [83]. In this case, vorticity is a *local variable* which means that we can isolate a patch of vorticity and observe how it is transported along the velocity field trajectories with a finite speed. For two-dimensional flows this is evident as then the vorticity equation reduces to

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = 0.$$

For more information, cf. [166].

Exercise 2.10. Vorticity has nothing in common with rotation of the fluid as a whole. Calculate the vorticity of the flows: (a) $u(x_1, x_2, x_3) = (u_1(x_2), 0, 0)$ and (b) $u(r, \phi, z) = (0, k/r, 0)$ for $r > 0$.

2.6 Thermodynamics

Equations of State From the point of view of thermodynamics the state of a homogeneous fluid can be described by some definite relations among a number of certain *state variables*, the most important being the volume V ($V = 1/\rho$), the entropy S, the internal energy E, the pressure p, and the absolute temperature θ, cf. [212].

In such a description one may start with a relation of the form (cf. [212])

$$E = E(S, V) \quad \text{(Gibbs relation)} \tag{2.48}$$

and define p and θ by

$$p = -\frac{\partial E}{\partial V}, \quad \theta = \frac{\partial E}{\partial S}, \tag{2.49}$$

with $p, \theta > 0$ by assumption. In this case, taking the total differential in (2.48) and using (2.49), we obtain

$$dE = \theta\, dS - p\, dV \quad \text{or} \quad dE = \theta\, dS - p\frac{1}{\rho^2}\, d\rho\,. \tag{2.50}$$

A simple phase system is said to undergo a *differentiable process* if its state variables are differentiable functions of time: $V = V(t)$, $S = S(t)$, etc. Assuming such a dependence one usually assumes, together with (2.50), that

$$\frac{DE}{Dt} = \theta\frac{DS}{Dt} - p\frac{DV}{Dt}$$

or

$$\frac{DE}{Dt} = \theta\frac{DS}{Dt} - p\frac{1}{\rho^2}\frac{D\rho}{Dt}\,. \tag{2.51}$$

Relation (2.51) makes it possible to write a definite form of the balance of entropy when we know the laws of conservation of mass and internal energy. We shall use this relation in the sequel.

Second Law of Thermodynamics and Constraints on Viscosity Coefficients Consider the law of conservation of energy (2.32)

$$\rho\frac{DE}{Dt} = \nabla\cdot(k\nabla\theta) - p\,\operatorname{div} u + \rho\Phi\,, \tag{2.52}$$

where Fourier's law is assumed, and $\rho\Phi$ is given by (2.33). We see that the internal energy increases with the influx of heat transfer, compression, and the viscous dissipation.