Springer Series in Computational Mathematics 50

Francisco-Javier Sayas

Retarded Potentials and Time Domain Boundary Integral Equations

A Road Map

Springer

Springer Series in Computational Mathematics

50

More information about this series at http://www.springer.com/series/797

Francisco-Javier Sayas

Retarded Potentials and Time Domain Boundary Integral Equations

A Road Map

Springer

Francisco-Javier Sayas
Department of Mathematical Sciences
University of Delaware
Newark, DE, USA

ISSN 0179-3632 ISSN 2198-3712 (electronic)
Springer Series in Computational Mathematics
ISBN 978-3-319-26643-5 ISBN 978-3-319-26645-9 (eBook)
DOI 10.1007/978-3-319-26645-9

Library of Congress Control Number: 2015955727

Mathematics Subject Classification (2010): 65M38, 65M12, 65R20, 65J08, 45A05, 45B05, 35L20, 35D30, 35Q61

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Dedicated to my parents
(Elena y Antonio, mamá y papá)
for their unconditional support
of my crazy American adventure

Preface

In these notes I try to show the way through the relatively difficult theory of retarded layer potentials and integral operators for the acoustic wave equation in two and three dimensions. I will also introduce convolution quadrature techniques for the time discretization of potentials and integral equations, giving the reader a taste of their challenging but exciting theory and their huge and partly unexplored potentialities. Part of the aim of these notes has been to set a clear path to learn the mathematical techniques needed to understand time domain boundary integral equations. This is part of a joint effort with Antonio Laliena (Universidad de Zaragoza, Spain), Lehel Banjai's research group (formerly at the Max-Planck-Institut in Leipzig, Germany, and at the time of finishing these notes at Heriot-Watt University, Edinburgh, UK), and my own group at the University of Delaware. This monograph is intended as a learning tool, and that is why the tone will be somewhat colloquial. Apart from some more narrative sections (those with less mathematical rigor), everything else will be duly divided into paragraphs—in the LaTeX sense of the word—(plus propositions and their proofs) so that at each moment we know where we are. A more computational approach to convolution quadrature for wave propagation can be found in the notes written in collaboration with Matthew Hassell [54].

An informal table of contents. Here is a guide of what each of the ten chapters contains:

1. We start with an informal presentation of the retarded layer potentials. We then derive the corresponding boundary integral calculus based on a few elements (the potentials and a uniqueness theorem) and its use for scattering problems. This chapter is, again, informal.
2. We introduce the basic tools for vector-valued distributions and their Laplace transforms. (Not all the proofs will be given here, but all steps will be duly sketched.) We next give the distributional form of the problem of scattering by an obstacle. We study the Laplace transform of the single layer potential and operator and prove estimates depending on the Laplace transform parameter for all of them.

3. We start this chapter by giving formulas for (strong) inversion of the Laplace transform and the differentiation theorem and then use them to delimit a precise class of symbols (Laplace transforms) and their time domain distributional counterparts. Convolution operators with this class of distributions are the setting for the remainder of the theory. We show how layer potentials, boundary integral operators, and their inverses (whenever they exist) are in this class. A rigorous proof of Kirchhoff's formula (the integral representation theorem for causal acoustic waves) gives the necessary justification for the Calderón-type calculus we had introduced in the first chapter. We finally have a look at how causality, finite speed of propagation, and some kind of coercivity are hidden in the Laplace transform of the potentials and operators.
4. Of the two classes of convolution quadrature methods, we introduce here the one that is based on multistep methods. We present an almost finished portrait of the theory of these methods applied to the class of convolution operators and equations that was introduced in the previous chapter and detail the kind of results that are derived in the case of scattering by a sound-soft obstacle.
5. We go back to the single layer retarded potential and go as far as we can with the Laplace domain techniques to prove estimates for the full discretization (Galerkin in space, convolution quadrature in time) of the model equation that solves the scattering problem by a sound-soft obstacle using an indirect formulation. Once we have finished with the single layer operator, we will repeat the process for the double layer potential representation of the scattering by sound-hard obstacles.
6. This chapter is a simplified introduction to a class of abstract differential equations of the second order in Hilbert spaces. The hypotheses are much reduced with respect to what is common in the Hille-Yosida theory, but they will be those that we will meet later on. All the results of this chapter are proved in Appendix B using quite rudimentary arguments of separation of space-time variables, arguments which are related to the discrete spectrum of a given unbounded operator.
7. The techniques of Chapter 6 are now used to prove again all the estimates for the single layer retarded potential and operator (as well as general Galerkin semidiscretization-in-space for the associated equation) using time domain tools. We will develop a streamlined way of proving the time domain results, by working on a cut-off domain and identifying the resulting solution with the beginning of the evolution of the potential solution.
8. We now repeat all the time domain arguments on the double layer potential and its use for an indirect formulation of the scattering problem by a sound-hard obstacle. As the reader will easily realize when reading this chapter, the arguments end up being very similar in each particular situation, and we will only have to take care of whatever is different in each concrete problem.
9. We next mix the time domain theory with convolution quadrature and, case by case, prove new estimates for the fully discrete methods for one model problem. This chapter shows how classical techniques for the numerical analysis of low-order time discretizations can be easily extended to the much more complicated situation we are dealing with.

10. In this chapter we collect the ideas of Chapters 7 and 8, looking for common patterns that allow us to easily guess what kind of bounds we will obtain in new situations. We also show easy (not to say straightforward) extensions to screen problems and to linear elasticity.

    There are five appendices:

- Appendix A contains some Laplace domain arguments for the Maxwell transient single layer potential
- Appendix B contains the proofs of the results on evolution equations that were presented in Chapter 6.
- Appendix C presents some algorithms for the implementation of convolution quadrature.
- Appendix D contains a precise but very terse introduction of the Sobolev space background material needed for this monograph.
- Appendix E shows some numerical illustrations of two-dimensional scattering problems.

Although the theory of time domain boundary integral equations is far from finished (as its full potential in applications is only partially exploited), let me drop here some names of some of the originators of the current excitement in the area. First of all, the focus of these notes lies in the realm of integral equations for wave propagation, although boundary integral equations are also used for parabolic problems. While over ten years old, Costabel's encyclopedia article [33] contains an excellent introduction to the use of integral equations for evolutionary partial differential equations.

What follows is a highly non-exhaustive list, so please, nobody take offense if their name does not appear here.

- The theory of time domain boundary integral equations (at least, the theory that we numerical analysts use) stems from two papers by Alain Bamberger and Tuong Ha-Duong [9, 10] in 1986. Many other papers were published [7, 23, 49], and even more theses were written (unfortunately much of this material was left unpublished and is now very difficult to locate) in the buoyant French numerical analysis school. The names of Jean-Claude Nédélec (at the Polytechnique) and Alain Bachelot (at the University of Bordeaux) are attached to quite a lot of these doctoral dissertations. Much of this is reported and referenced in the survey paper [50]. Touffic Abboud and Isabelle Terrasse can be held responsible for the practical development of these methods, evolving in their (to the best of my knowledge) only commercial implementation. Interest in research aspects of this approach seems to be back: Abboud and Terrasse together with Patrick Joly (INRIA Rocquencourt, France) and Jerónimo Rodríguez (Santiago de Compostela, Spain) have recently developed one of the few sets of integral transparent boundary conditions [1].
- Convolution quadrature originated as a completely independent tool for approximation of convolutions. It came to age very much at the same time as time domain integral equations, with two articles by Christian Lubich [64, 65] in

1988. This technique was first devised for problems with parabolic structure (exemplified in the operators having Laplace transforms defined on a sector instead of a half-plane). A second family of convolution quadrature methods, based on Runge-Kutta methods, originated in the joint work of Lubich with Alexander Ostermann [67]. Almost at the same time, Lubich applied his ideas to problems with hyperbolic structure, including the single layer potential for the three-dimensional equation [66]. This was only natural, since CQ is based in Laplace transform methods and the theory of Bamberger and Ha-Duong is based on exactly the same principle. The theory of CQ based on RK schemes applied to hyperbolic problems was left unfinished and was only recently completed by Lubich in collaboration with Lehel Banjai and Jens Markus Melenk [18, 19].

- Not being as popular as their frequency domain cousins, time domain boundary integral equations have known a rich development in the engineering community. Galerkin methods (the original ones developed in the French school) are a particular case of marching-on-in-time (MoT) methods for time domain integral equations. The group of Eric Michielssen at the University of Michigan and a large array of researchers in European universities (with a strong group in Ghent) have developed applications to electromagnetism and searched the limits of the known world in computational time domain integral equations [6, 24, 32, 72, 74, 86, 87].
- The convolution quadrature point of view was initially not very well tended by the mathematical boundary integral community, but there was a strong development in the field of applications to elastodynamics, much of it led by the group of Martin Schanz [21, 57, 63, 69, 75, 76] at the Graz University of Technology (Austria). An early account of this development can be found in the monograph [82]. A more recent survey can be found in [22]. Applications to electromagnetism have been developed by the group of Daniel Weile and Peter Monk at the University of Delaware [28, 30, 31, 62, 89], while researchers in all corners of the world have been developing CQ-BEM [29, 46–48].
- Some papers by Stefan Sauter (University of Zurich) with different collaborators [51, 52] re-sparked the interest of numerical analysts in time domain integral equations, specially with a focus on convolution quadrature techniques. Galerkin methods with smooth basis functions have also attracted the interest of the Zurich group [79, 88]. Lehel Banjai and his group are making rapid progress in this direction [8, 11, 12, 14–17, 20]. Myself, working with my then graduate student Antonio Laliena, proved that the Laplace domain contained much more information than we had expected and that convolution quadrature techniques combined perfectly with space Galerkin discretization in many nontrivial situations of scattering of acoustic and elastic waves with penetrable obstacles, including nonhomogeneous obstacles where numerical modeling is carried out with the finite element method [60]. We were happy to find a quite general (and I want to say innovative) approach that has since been applied to electromagnetism or more complicated elastodynamic problems. Contributions of my group in the area of CQ appear in recent papers [13, 20, 43, 56, 73, 81].

- Penny Davies and Dougald Duncan in the UK have explored alternatives for MoT schemes with different shapes of basis functions or collocation in time [36–40]. The full Galerkin approach is also being jointly developed by the groups of Ernst Stephan at the University of Hanover (Germany) and Matthias Maischak at Brunel University (England) with a current focus on acoustics. Several researchers in Italy (among them, Alessandra Aimi and Mauro Diligenti at Parma) are also readdressing the full Galerkin method for the acoustic equations [2–5].

Acknowledgments The first draft of these notes (covering roughly Chapters 1 to 5) was written as support for a five-hour lecture series *Retarded boundary integral equations and applications*, delivered in the context of the closure workshop of the special semester on *Theoretical and numerical aspects of inverse problems and scattering theory*. The workshop took place in La Coruña, Spain, on July 4–8, 2011. The organizers of the event are warmly thanked for having thought of me for this occasion. The current version has been expanded using recent results on time domain analysis.

As already mentioned, a great deal of what appears in this document is the result of continuous collaboration with Antonio Laliena, Víctor Domínguez, and Lehel Banjai. Some of my current students (Matthew Hassell, Tonatiuh Sánchez-Vizuet, and Tianyu Qiu) have had a careful look at several chapters of these notes, working out all the exercises and checking proofs. Tainyu Qiu helped me with the appendix on the Maxwell equations. Another one of my students, Allan Hungria, has done an excellent job in proofreading and creating pristine LaTeX TikZ figures. My current research is partially supported by the NSF (DMS 1216356). Three different meetings at the Oberwolfach Mathematical Institute, and in particular the talks I gave there, helped me in the search of a systematic approach to the development of this theory. I am deeply grateful to the organizers of those workshops (Martin Costabel and Ernst Stephan in the first one; Ralf Hiptmair, Roland Hoppe, Patrick Joly, and Ulrich Langer for the last two) for giving me the chance to enjoy the wonders of working, thinking, and discussing in the middle of the Black Forest.

Newark, DE, USA Francisco-Javier Sayas
June 2015

Contents

Chapter 1
The retarded layer potentials

In this chapter we are going to introduce the basic concepts of time domain acoustic layer potentials and how they can be used to represent the solutions of scattering problems. *All notions introduced in this chapter will be given at an intuitive level and with basically no formalization. The reader will find a precise sketch of the theory in the next chapters.* For the sake of clarity, let me remark here:

- Sections 1.1 through 1.3 deal with three-dimensional waves.
- Just by looking at the mathematical expressions therein, it will be clear that Sections 1.4 through 1.6 are dimension-independent.
- Section 1.7 revisits the particular three-dimensional case, using the specific formulas for the Huygens' single layer potential.
- Finally, Section 1.8 will present formulas for the two-dimensional case.

1.1 Acoustic sources and dipoles

Let us start this chapter by having a look at a spherical wave. We consider a function (a signal) $\lambda : \mathbb{R} \to \mathbb{R}$ such that $\lambda(t) = 0$ for all $t < 0$. A function of the time variable that vanishes for $t < 0$ will be always referred to as a **causal function**. We now choose $\mathbf{x}_0 \in \mathbb{R}^3$ and consider the function

$$u(\mathbf{x}, t) := \frac{\lambda(t - c^{-1}|\mathbf{x} - \mathbf{x}_0|)}{4\pi|\mathbf{x} - \mathbf{x}_0|}. \qquad (1.1)$$

A more or less boring computation shows that

$$c^{-2}\frac{\partial^2 u}{\partial t^2} = \Delta u \qquad \forall \mathbf{x} \neq \mathbf{x}_0 \quad \forall t > 0,$$

F.-J. Sayas, *Retarded Potentials and Time Domain Boundary Integral Equations*, Springer Series in Computational Mathematics 50,
DOI 10.1007/978-3-319-26645-9_1

as long as $\lambda \in \mathcal{C}^2(\mathbb{R})$, where the Laplace operator Δ is taken in the space variables. (The result is actually true for less smooth λ, but we are not going to worry about regularity at this point.) It is interesting to notice the following facts.

- The function u moves on spherical surfaces. Actually,

$$u(\mathbf{x}, t) = \frac{\lambda(t - c^{-1} r)}{4\pi r} \qquad |\mathbf{x} - \mathbf{x}_0| = r. \tag{1.2}$$

  This shows that points on a sphere centered at $\mathbf{x}_0$ perceive the same solution at the same time.
- The previous formula shows also that for a point at distance r of the point source, we need to wait $c^{-1} r$ time units to start perceiving any signal. Apart from this delay, the entire signal is received at speed c (and with a damping factor $4\pi r$). The signal goes through, exactly as emitted.

There are other kinds of solutions of the wave equation that can be understood as traveling on spherical surfaces. If u is a sufficiently smooth solution of the wave equation, so are the three components of ∇u and therefore, so is $\nabla u \cdot \mathbf{n}_0$, where $\mathbf{n}_0$ is a fixed vector. With this idea, and starting in (1.1), we can create a new family of solutions to the wave equation:

$$\begin{aligned} u(\mathbf{x}, t) &:= \nabla_{\mathbf{x}_0} \left(\frac{\varphi(t - c^{-1}|\mathbf{x} - \mathbf{x}_0|)}{4\pi |\mathbf{x} - \mathbf{x}_0|} \right) \cdot \mathbf{n}_0 \\ &= -\nabla_{\mathbf{x}} \left(\frac{\varphi(t - c^{-1}|\mathbf{x} - \mathbf{x}_0|)}{4\pi |\mathbf{x} - \mathbf{x}_0|} \right) \cdot \mathbf{n}_0 \\ &= \varphi(t - c^{-1}|\mathbf{x} - \mathbf{x}_0|) \frac{(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}_0}{4\pi |\mathbf{x} - \mathbf{x}_0|^3} + c^{-1} \dot\varphi(t - c^{-1}|\mathbf{x} - \mathbf{x}_0|) \frac{(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}_0}{4\pi |\mathbf{x} - \mathbf{x}_0|^2}. \end{aligned} \tag{1.3}$$

We will assume that $\mathbf{n}_0$ is a unit vector. This formula bears some similitude with (1.1). For instance, at time t, al points on the surface $|x - x_0| = r$ receive information from the signal φ emitted at time $t - c^{-1} r$. However, the points on the surface do not only observe the value $\varphi(t - c^{-1} r)$ but also its trend $\dot\varphi(t - c^{-1} r)$. The main difference between the wave (1.3) and the spherical wave (1.1) is directionality. While points seeing the source in the direction $\mathbf{n}_0$ get to perceive the signal, all points such that $(\mathbf{x} - \mathbf{x}_0) \cdot \mathbf{n}_0 = 0$ are in a deaf spot and miss the entire signal. Actually, if the angle between $\mathbf{x} - \mathbf{x}_0$ and $\mathbf{n}_0$ is θ, then

$$u(\mathbf{x}, t) = \frac{1}{4\pi r} \left(\frac{\varphi(t - c^{-1} r)}{r} + \frac{\dot\varphi(t - c^{-1} r)}{c} \right) \cos\theta \tag{1.4}$$

The points $\mathbf{x}_0 \pm r\mathbf{n}_0$ (respective North and South pole of the sphere with axis $\mathbf{n}_0$) get the signal with the same amount of attenuation, but mirrored. The reader is encouraged to check the physical dimensions of all the elements in formulas (1.2) and (1.4) to recognize that the respective transmitted signals (λ and φ) have different dimensions. (We can understand the dimensional mismatch by noticing the differentiation in the space variables in (1.3)–(1.4), which seems to require some kind of compensation.)

Another way of motivating the directional spherical wave (1.3) uses the physical idea of dipole. Take two source points

$$\mathbf{x}_0 \pm \tfrac{h}{2}\mathbf{n}_0,$$

separated a distance h in the direction $\mathbf{n}_0$. The upper point $\mathbf{x}_0 + \frac{h}{2}\mathbf{n}_0$ emits a signal $h^{-1}\varphi$ and simultaneously the point $\mathbf{x}_0 - \frac{h}{2}\mathbf{n}_0$ emits the signal $-h^{-1}\varphi$. The receiver gets to hear the signal

$$\frac{1}{h}\Big(\frac{\varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0-\frac{h}{2}\mathbf{n}_0|)}{4\pi|\mathbf{x}-\mathbf{x}_0-\frac{h}{2}\mathbf{n}_0|}-\frac{\varphi(t-c^{-1}|\mathbf{x}-\mathbf{x}_0+\frac{h}{2}\mathbf{n}_0|)}{4\pi|\mathbf{x}-\mathbf{x}_0+\frac{h}{2}\mathbf{n}_0|}\Big)$$

which in the limit $h \to 0$ turns into (1.3).

1.2 Acoustic layer potentials

The single layer potential can be understood as the (continuous) superposition of spherical waves (1.1) being emitted from points on a surface Γ:

$$(\mathcal{S} * \lambda)(\mathbf{x},t) := \int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}). \tag{1.5}$$

The causal signal $\lambda(t)$ has been substituted by a density distribution of causal signals $\lambda(\mathbf{y},t)$, i.e., $\lambda : \Gamma \times \mathbb{R} \to \mathbb{R}$ such that $\lambda(\,\cdot\,,t) \equiv 0$ for $t < 0$. The convolution sign in the notation of this is purely formal for the time being. This also applies to the symbol for the double layer potential that we will define shortly.

The reader who meets this kind of potential expression for the first time is encouraged to have a close look at the relatively bad aspect that it has: there is integration in the space variable $\mathbf{y}$ that somehow got its way into the time variable (through the delay). A particular set of densities is the addition of tensor products of functions of space and time

$$\lambda(\mathbf{y},t) = \sum_{j=1}^{N} \Phi_j(\mathbf{y})\lambda_j(t),$$

producing simpler propagated signals

$$\sum_{j=1}^{N} \int_\Gamma \frac{\Phi_j(\mathbf{y})\lambda_j(t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}).$$

Simplifying even more, we can assume that the surface Γ has been subdivided into N panels $\{\Gamma_1, \ldots, \Gamma_N\}$ and Φ_j is just the characteristic function of the panel Γ_j. This is how the single layer potential looks like now:

$$\sum_{j=1}^{N} \int_{\Gamma_j} \frac{\lambda_j(t - c^{-1}|\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|} \mathrm{d}\Gamma(\mathbf{y}).$$

In any of the above expressions, it is easy to check that if a point is at a distance r from Γ, it will take $T = c^{-1}r$ time units for the signal to reach the point. Apart from very simple configurations, different points $\mathbf{x}$ will perceive different outputs, since the balance of distances $|\mathbf{x} - \mathbf{y}|$ with the spacial distribution of the density is going to differ depending on the point of view.

Another class of signals we can plug into the potential expression are time-harmonic signals. A noncausal time harmonic signal emitted from Γ would be

$$\mathrm{Re}\,(\lambda(\mathbf{y})e^{-\imath\omega t}) \qquad \lambda : \Gamma \to \mathbb{C},$$

which is heard as a time harmonic signal

$$\mathrm{Re}\Big(e^{-\imath\omega t} \underbrace{\int_\Gamma \frac{e^{\imath\omega c^{-1}|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x} - \mathbf{y}|} \lambda(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y})}\Big).$$

The underbraced expression can be recognized as a single layer potential associated with the Helmholtz equation $\Delta + k^2$, ($k = \omega/c$ is the wave number) which is the equation satisfied by the spatial part of a time harmonic solution to the wave equation.

A double layer potential can be defined with the same idea of superposition. The directionality at the point $\mathbf{y} \in \Gamma$ is given by the unit normal vector $\boldsymbol{\nu}(\mathbf{y})$:

$$\begin{aligned}(\mathcal{D} * \varphi)(\mathbf{x}, t) :=& \int_\Gamma \nabla_{\mathbf{y}} \left(\frac{\varphi(\mathbf{z}, t - c^{-1}|\mathbf{x} - \mathbf{y}|)}{4\pi|\mathbf{x} - \mathbf{y}|} \right) \Big|_{\mathbf{z}=\mathbf{y}} \cdot \boldsymbol{\nu}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \\ =& \int_\Gamma \frac{(\mathbf{x} - \mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{y})}{4\pi|\mathbf{x} - \mathbf{y}|^3} \big(\varphi(\mathbf{y}, t - c^{-1}|\mathbf{x} - \mathbf{y}|) \\ &\qquad + c^{-1}|\mathbf{x} - \mathbf{y}|\dot{\varphi}(\mathbf{y}, t - c^{-1}|\mathbf{x} - \mathbf{y}|)\big)\mathrm{d}\Gamma(\mathbf{y}).\end{aligned}$$

Obviously, for this expression to make sense we need an orientable surface with a well-defined normal vector field (almost everywhere, so polyhedra are not a problem).

1.3 Jump relations

Let us try to see some properties of the possible limits of the layer potentials when we get close to the surface.

Continuity of the single layer potentials. A possible way to study the single layer potential is by studying functions of the form

$$w(\mathbf{x},\hat{\mathbf{x}},t) := \int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\hat{\mathbf{x}}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}), \tag{1.6}$$

since $(\mathcal{S}*\lambda)(\mathbf{x},t) = w(\mathbf{x},\mathbf{x},t)$. Let $\mathbf{z}\in\Gamma$. We first take the limit $\hat{\mathbf{x}}\to\mathbf{z}$ in (1.6) and we obtain (formally at least)

$$\int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{z}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}). \tag{1.7}$$

In a second step, we recognize in (1.7) the form of a Coulomb potential (the single layer potential for the Laplacian), which is continuous across Γ. This means that

$$\lim_{\mathbf{x}\to\mathbf{z}\in\Gamma}(\mathcal{S}*\lambda)(\mathbf{x},t) = \int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{z}-\mathbf{y}|)}{4\pi|\mathbf{z}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}) =: (\mathcal{V}*\lambda)(\mathbf{z},t).$$

Discontinuity of the normal derivative of the single layer potential. We next look at directional derivatives of $\mathcal{S}*\lambda$. Let $\boldsymbol{\nu}=\boldsymbol{\nu}(\mathbf{z})$ with $\mathbf{z}\in\Gamma$. Then:

$$\begin{aligned}\big(\nabla_{\mathbf{x}}(\mathcal{S}*\lambda)\cdot\boldsymbol{\nu}\big)(\mathbf{x},t) &= -c^{-1}\int_\Gamma \frac{\dot\lambda(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y})\\ &\quad -\int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}}{|\mathbf{x}-\mathbf{y}|^2}\mathrm{d}\Gamma(\mathbf{y})\\ &=: a(\mathbf{x},t)+b(\mathbf{x},t).\end{aligned}$$

With arguments similar to those we used in the continuity analysis of $\mathcal{S}*\lambda$, we can prove that a is continuous across Γ. We are now going to give a simplified argument demonstrating that

$$b(\mathbf{z}-\varepsilon\boldsymbol{\nu}(\mathbf{z}),t) - b(\mathbf{z}+\varepsilon\boldsymbol{\nu}(\mathbf{z}),t) \overset{\varepsilon\to 0}{\longrightarrow} \lambda(\mathbf{z},t),$$

which is equivalent to showing that the jump of the normal derivative of $\mathcal{S}*\lambda$ across Γ is λ. Note that when $\mathbf{x}\to\mathbf{z}\in\Gamma$, only a neighborhood of $\mathbf{z}$ in Γ is relevant from the point of view of creating a discontinuity in the integral:

$$b(\mathbf{x},t) = -\int_\Gamma \frac{\lambda(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\mathrm{d}\Gamma(\mathbf{y}).$$

To further simplify the exposition, let us assume that Γ is a flat surface around $\mathbf{z}$. After translation, rotation, and localization, we can assume that

$$\mathbf{z} = \mathbf{0} \qquad \boldsymbol{\nu} = (0,0,1) \qquad \Gamma = \{(\mathbf{y},0) \,:\, \mathbf{y}\in\mathbb{R}^2,\ |\mathbf{y}|<R\} = B(0,R)\times\{0\}.$$

If $\mathbf{x} = \mathbf{z} \pm \varepsilon\boldsymbol{\nu}(\mathbf{z}) = \pm\varepsilon(0,0,1)$, then

$$\begin{aligned} b(\mathbf{0}-\varepsilon\boldsymbol{\nu},t) - b(\mathbf{0}+\varepsilon\boldsymbol{\nu},t) &= \varepsilon\int_{B(0,R)} \frac{\lambda(\mathbf{y},t-c^{-1}|(\mathbf{y},\varepsilon)|)}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y} \\ &= \lambda(\mathbf{0},t)\int_{B(0,R)} \frac{\varepsilon}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y} \\ &\quad + \varepsilon\int_{B(0,R)} \frac{\lambda(\mathbf{y},t-c^{-1}|(\mathbf{y},\varepsilon)|)-\lambda(\mathbf{0},t)}{4\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y}. \end{aligned} \tag{1.8}$$

Note that

$$\int_{B(0,R)} \frac{\varepsilon}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y} = \int_0^R \frac{\varepsilon\, r}{\sqrt{(r^2+\varepsilon^2)^3}}\mathrm{d}r = 1 - \frac{\varepsilon}{\sqrt{R^2+\varepsilon^2}} \overset{\varepsilon\to 0}{\longrightarrow} 1. \tag{1.9}$$

On the other hand, for smooth λ

$$|\lambda(\mathbf{y},t-c^{-1}|(\mathbf{y},\varepsilon)|) - \lambda(\mathbf{0},t)| \le C_1|\mathbf{y}| + C_2\varepsilon, \tag{1.10}$$

$$\int_{B(0,R)} \frac{\varepsilon^2}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y} \overset{\varepsilon\to 0}{\longrightarrow} 0, \tag{1.11}$$

and

$$\begin{aligned} \int_{B(0,R)} \frac{\varepsilon|\mathbf{y}|}{2\pi|(\mathbf{y},\varepsilon)|^3}\mathrm{d}\mathbf{y} &= \int_0^R \frac{\varepsilon r^2}{\sqrt{(r^2+\varepsilon^2)^3}}\mathrm{d}r \\ &= \varepsilon\log(\sqrt{R^2+\varepsilon^2}+R) - \frac{\varepsilon R}{\sqrt{R^2+\varepsilon^2}} - \varepsilon\log\varepsilon \overset{\varepsilon\to 0}{\longrightarrow} 0. \end{aligned} \tag{1.12}$$

Using (1.9), (1.10), (1.11), and (1.12) in (1.8), the result follows. The case of curved boundaries is very similar.

Discontinuity of the double layer potential. The expression for the double layer potential

$$\begin{aligned} (\mathcal{D}*\varphi)(\mathbf{x},t) &= c^{-1}\int_\Gamma \frac{\dot\varphi(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}\mathrm{d}\Gamma(\mathbf{y}) \\ &\quad + \int_\Gamma \frac{\varphi(\mathbf{y},t-c^{-1}|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}\,\frac{(\mathbf{x}-\mathbf{y})\cdot\boldsymbol{\nu}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\mathrm{d}\Gamma(\mathbf{y}) \end{aligned}$$

definitely resembles that of the directional derivative of the single layer potential. We can recognize two terms: the first one is continuous and in the second one, we can use exactly the same arguments to prove that for every $\mathbf{z} \in \Gamma$ such that Γ is flat around $\mathbf{z}$

$$(\mathcal{D} * \varphi)(\mathbf{z} + h\boldsymbol{\nu}(\mathbf{z}), t) - (\mathcal{D} * \varphi)(\mathbf{z} - h\boldsymbol{\nu}(\mathbf{z}), t) \stackrel{\varepsilon \to 0}{\longrightarrow} \varphi(\mathbf{z}, t).$$

Note that the sign of the jump is the opposite to the one of the normal derivative of $\mathcal{S} * \lambda$.

Continuity of the normal derivative of the double layer potential. Assuming more regularity for the density φ, it is possible to show that the normal derivative of $\mathcal{D} * \varphi$ is continuous across smooth points of Γ. The proof is more involved (tangential integration by parts is involved and finite part integrals make their appearance) and requires a certain amount of patience. Because we will take a different point of view, using Laplace transform techniques and basing our results on well-established properties of layer potentials for elliptic problems, we will just accept this result for the time being.

1.4 A Calderón type calculus

The structure of the boundary integral calculus for the wave equation is very similar to that of elliptic operators, so those accustomed to the many formulas (Green representation theorem, boundary integral identities, Calderón projector, etc) of the boundary integral calculus will recognize here exactly the same basic structure. The main difference is at the analytic level: spaces are much less clear and the theory requires quite some effort to be developed. The boundary integral calculus can be derived in several ways. My favorite is the following. It develops from three concepts:

- a uniqueness theorem for transmission problems,
- a concept of single layer operator,
- a concept of double layer operator.

(The three concepts can be grouped in one: an existence and uniqueness theorem for transmission problems.) Once these elements have been established, the representation theorem (Green's Theorem for steady state problems, Kirchhoff's formula for waves) is a direct consequence of these elements. The boundary integral operators are the averages of the Cauchy data of layer operators and they yield a collection of integral identities satisfied by interior and exterior solutions. Two sound mathematically oriented references on boundary integral equations are the monographs of Hsiao and Wendland [55] and the more numerically oriented presentation of Sauter and Schwab [80]. The many intricacies of the theory of Calderón projectors, boundary integral operators and potentials, are thoroughly explained in McLean's celebrated book [68].

We are going to informally expose this theory. We will need Chapters 2 and 3 to develop a rigorous theory for the main building blocks. The geometric layout is composed of a bounded domain Ω^-, with Lipschitz boundary Γ and exterior $\Omega^+ := \mathbb{R}^d \setminus \overline{\Omega^-}$ (that is supposed to be connected). The restriction (trace) of a function u to the boundary Γ from the interior and exterior of Γ will be denoted $\gamma^- u$ and $\gamma^+ u$, respectively. The normal derivative (with the normal vector pointing outwards) from inside and outside are $\partial_\nu^- u$ and $\partial_\nu^+ u$. Jumps of these two quantities across the interface Γ are denoted

$$[\![\gamma u]\!] := \gamma^- u - \gamma^+ u, \qquad [\![\partial_\nu u]\!] := \partial_\nu^- u - \partial_\nu^+ u.$$

Averages are denoted with double curly brackets

$$\{\!\{\gamma u\}\!\} := \tfrac{1}{2}(\gamma^- u + \gamma^+ u), \qquad \{\!\{\partial_\nu u\}\!\} := \tfrac{1}{2}(\partial_\nu^- u + \partial_\nu^+ u).$$

In the background of this theory there is a class of functions $u(\mathbf{x}, t)$ for which we can take second time derivatives, spatial Laplacian, traces and normal derivatives on the boundary and initial values at $t = 0$. For the moment let us refer to these functions as admissible functions.

The uniqueness result. The first key ingredient is a uniqueness result for a kind of transmission problem of the wave equation. It can be informally stated as follows: if an admissible function u satisfies

$$\begin{aligned}
c^{-2} u_{tt} &= \Delta u && \text{in } \mathbb{R}^d \setminus \Gamma \times (0, \infty),\\
[\![\gamma u]\!] &= 0 && \text{on } \Gamma \times (0, \infty),\\
[\![\partial_\nu u]\!] &= 0 && \text{on } \Gamma \times (0, \infty),\\
u(\,\cdot\,, 0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma,\\
u_t(\,\cdot\,, 0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma,
\end{aligned}$$

then u is necessarily zero.

Those used to frequency domain problems will be wondering where the radiation condition is. This can be dealt with in several ways, such as demanding finite energy for each time, or asking for bounded spatial support for each time, etc. At this level, we assume that the class of admissible functions includes this radiation condition. When we develop the correct theoretical frame, radiation will be part of causality and will not have to be expressed as a separate condition.

A single layer potential. For a function $\lambda : \Gamma \times (0, \infty) \to \mathbb{R}$ in a certain class of functions, there exists an admissible function $u := \mathcal{S} * \lambda$ such that

$$\begin{aligned}
c^{-2} u_{tt} &= \Delta u && \text{in } \mathbb{R}^d \setminus \Gamma \times (0, \infty),\\
[\![\gamma u]\!] &= 0 && \text{on } \Gamma \times (0, \infty),
\end{aligned}$$

$$
\begin{aligned}
[\![\partial_\nu u]\!] &= \lambda && \text{on } \Gamma \times (0,\infty),\\
u(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma,\\
u_t(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma.
\end{aligned}
$$

Obviously, $\mathcal{S} * \lambda$ is the unique solution of this problem.

A double layer potential. For a function $\varphi : \Gamma \times (0,\infty) \to \mathbb{R}$ in a certain class, there exists an admissible function $u := \mathcal{D} * \varphi$ such that

$$
\begin{aligned}
c^{-2}u_{tt} &= \Delta u && \text{in } \mathbb{R}^d \setminus \Gamma \times (0,\infty),\\
[\![\gamma u]\!] &= -\varphi && \text{on } \Gamma \times (0,\infty),\\
[\![\partial_\nu u]\!] &= 0 && \text{on } \Gamma \times (0,\infty),\\
u(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma,\\
u_t(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma.
\end{aligned}
$$

There is an inherent compatibility condition between the three classes of functions where u, λ, and ϕ take values. It can be expressed as follows: given u in the class of pressure (wave) fields, the quantities $\lambda := [\![\partial_\nu u]\!]$ and $\varphi := [\![\gamma u]\!]$ can be used as respective inputs of the single and double layer potentials.

First consequence: Kirchhoff's formula. If u is a solution of the wave equation around Γ

$$
\begin{aligned}
c^{-2}u_{tt} &= \Delta u && \text{in } \mathbb{R}^d \setminus \Gamma \times (0,\infty),\\
u(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma,\\
u_t(\,\cdot\,,0) &= 0 && \text{in } \mathbb{R}^d \setminus \Gamma,
\end{aligned}
$$

then

$$
u = \mathcal{S} * [\![\partial_\nu u]\!] - \mathcal{D} * [\![\gamma u]\!]. \tag{1.13}
$$

This is a direct consequence of the definitions of layer potentials and the uniqueness theorem for transmission problems.

New definitions: boundary integral operators. The properties of potentials

$$
[\![\gamma(\mathcal{S} * \lambda)]\!] = 0 \qquad \text{and} \qquad [\![\partial_\nu(\mathcal{D} * \varphi)]\!] = 0 \tag{1.14}
$$

allow us to define the following four operators:

$$
\begin{aligned}
\mathcal{V} * \lambda &:= \{\!\{\gamma(\mathcal{S} * \lambda)\}\!\} = \gamma^-(\mathcal{S} * \lambda) = \gamma^+(\mathcal{S} * \lambda),\\
\mathcal{K}^t * \lambda &:= \{\!\{\partial_\nu(\mathcal{S} * \lambda)\}\!\},
\end{aligned}
$$

$$\mathcal{K} * \varphi := \{\!\{\gamma(\mathcal{D} * \varphi)\}\!\},$$
$$\mathcal{W} * \varphi := -\{\!\{\partial_\nu(\mathcal{D} * \varphi)\}\!\} = -\partial_\nu^-(\mathcal{D} * \varphi) = -\partial_\nu^+(\mathcal{D} * \varphi).$$

The superscript t in $\mathcal{K}^t$ is not the time variable, but a sort of transposition symbol, that is difficult to explain at this moment. We will come back to this issue in Chapter 2. Since

$$[\![\partial_\nu(\mathcal{S} * \lambda)]\!] = \lambda \qquad \text{and} \qquad [\![\gamma(\mathcal{D} * \varphi)]\!] = -\varphi,$$

the definitions imply that

$$\partial_\nu^\pm(\mathcal{S} * \lambda) = \mp\tfrac{1}{2}\lambda + \mathcal{K}^t * \lambda \qquad \text{and} \qquad \gamma^\pm(\mathcal{D} * \varphi) = \pm\tfrac{1}{2}\varphi + \mathcal{K} * \varphi. \tag{1.15}$$

The collection of all these formulas is often referred to as **the jump relations** of potentials.

Boundary integral identities. Starting at the representation theorem (Kirchhoff's formula)

$$u = \mathcal{S} * [\![\partial_\nu u]\!] - \mathcal{D} * [\![\gamma u]\!]$$

and using the jump relations, we can write, for instance,

$$\begin{bmatrix} \{\!\{\gamma u\}\!\} \\ \{\!\{\partial_\nu u\}\!\} \end{bmatrix} = \begin{bmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{W} & \mathcal{K}^t \end{bmatrix} * \begin{bmatrix} [\![\gamma u]\!] \\ [\![\partial_\nu u]\!] \end{bmatrix}. \tag{1.16}$$

Exterior solutions: direct method. The previous presentation was carried out for solutions of transmission problems. The reader might wonder what to do when we only have an exterior solution at our disposal, i.e., a solution of

$$\begin{aligned} c^{-2}u_{tt} &= \Delta u && \text{in } \Omega^+ \times (0,\infty), \\ u(\,\cdot\,,0) &= 0 && \text{in } \Omega^+, \\ u_t(\,\cdot\,,0) &= 0 && \text{in } \Omega^+. \end{aligned}$$

The simplest thing to do is to consider that $u \equiv 0$ in $\Omega^- \times (0,\infty)$ naturally completes the exterior solution. Then

$$[\![\gamma u]\!] = -\gamma^+ u, \quad \{\!\{\gamma u\}\!\} = \tfrac{1}{2}\gamma^+ u, \quad [\![\partial_\nu u]\!] = -\partial_\nu^+ u, \quad \{\!\{\partial_\nu u\}\!\} = \tfrac{1}{2}\partial_\nu^+ u.$$

Therefore, Kirchhoff's formula (1.13) for this u is rewritten as

$$u = \mathcal{D} * \gamma^+ u - \mathcal{S} * \partial_\nu^+ u, \tag{1.17}$$