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Gioia Carinci
Anna De Masi
Cristian Giardinà
Errico Presutti

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Cristian Giardinà · Errico Presutti

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Gioia Carinci
Delft University of Technology
Delft
The Netherlands

Anna De Masi
Dipartimento di Matematica
Universita di L'Aguila
L'Aquila
Italy

Cristian Giardinà
Dipartimento di Matematica
Università di Modena e Reggio Emilia
Modena
Italy

Errico Presutti
Gran Sasso Science Institute
L'Aquila
Italy

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# Contents

# Chapter 1
# Introduction

We develop here a theory for free boundary problems which applies to a large class of systems arising from problems in various, even distant, areas of research and which share a common mathematical structure. As we shall see in some detail, these are models for heat conduction, queuing theory, propagation of fire, interface dynamics, population dynamics and evolution of biological systems with selection mechanisms. We shall consider models in continuum and interacting particle systems. Their common mathematical features are the following:

> (1) Microscopic particle dynamics stem from interactions of topological nature.
>
> (2) Macroscopic evolution is ruled by a free boundary problem.

In fact in the models we consider the particles move in $d = 1$ dimension so that there is a rightmost and a leftmost particle, called *boundary particles*. The rules of dynamics are the usual ones, particles are either free (independent random walks or Brownian motions) or they have some local interaction (for instance simple exclusion) and on top of that there may be creations of new particles or particles may duplicate via a branching process. In addition, in order to keep (approximatively) constant the total number of particles, boundary particles are subject to a death process.

The topological nature of the interaction refers to the fact that the boundary particles are special as they may disappear at some given rate, being then replaced by new boundary particles, the rightmost and leftmost particles among those which have survived. Thus the "inside particles", i.e. those in between the boundary particles, evolve in the "usual" way, but the inside particles are not fixed a priori and may eventually become boundary particles depending on the evolution itself.

As a consequence of particle evolution, the spatial domain occupied by the particles varies in time. In particular the location of the boundary particles changes in the course of time due to the death process at the boundary. Correspondingly, as we shall



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discuss extensively in this volume, the macroscopic version of the models is provided by a *free boundary problem* for a PDE with Dirichlet condition supplemented by prescribing the boundary flux. As often occurs, one can relate a macroscopic evolution to microscopic dynamics via a scaling limit procedure (hydrodynamic limit).

The basic example that we will study in detail here is given by the linear heat equation

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial^2 \rho}{\partial r^2}$$

in the time varying domain $[0, X_t]$ with some initial condition $\rho(r, 0) = \rho_0(r) \geq 0$ and boundary conditions

$$-\frac{1}{2}\frac{\partial \rho}{\partial r}(0, t) = j > 0, \quad \rho(X_t, t) = 0 .$$

The free boundary $X_t$ (also called the *edge* in this book) is not given a priori but it should be determined in such a way that

$$-\frac{1}{2}\frac{\partial \rho}{\partial r}(X_t, t) = j.$$

Interpreting $\rho$ as a mass density, the last condition states that the mass flux leaving the system at $X_t$ must be equal to $j$, and since $j$ is also the mass flux entering at 0 (as fixed by the boundary condition at 0), the total mass in the system is preserved. From this perspective the free boundary problem becomes a control problem: find an edge evolution $X_t$ in such a way that the total mass is constant in time.

Well known theorems on the Stefan problem yield a local existence theorem for our basic example when we have "classical initial data". We will define here a weak version of the problem and prove global existence and uniqueness of a *relaxed solution* for general initial data. The other models that we will consider in this work have similar structure and the strategies of proof are very close to that in the basic example. The key point in all of them is:

> Construct upper and lower barriers that squeeze the solution we are looking for.

The correct notion of order for these problems is defined by mass transport. Referring to the basic model described above for the sake of definiteness, the barriers are defined in terms of a simplified evolution where we introduce a time grid of length $\delta$ and the evolution is ruled by the heat equation in $\mathbb{R}_+$ in the open intervals $(n\delta, (n+1)\delta)$ with boundary condition

$$-\frac{1}{2}\frac{\partial \rho}{\partial r}(0, t) = j .$$

At the times $n\delta$ we remove an amount of mass equal to $j\delta$ so that at these times the mass conservation is restored. The key point is that we get an upper barrier if we start by removing mass already at time 0 while we get a lower barrier when we remove mass from time $\delta$ on: the order here is in the sense of moving mass to the right. A key step is to prove that the barriers have a unique separating element. Once we have this we conclude by showing that the solution we are considering is trapped in between the barriers which then identifies the solution as the element separating the barriers. As we will see this part of the proof exploits extensively probabilistic ideas and techniques based on the well known relation between heat equation and Brownian motion and between the hitting distribution at the boundaries and the Dirichlet condition in the heat equation.

We think it can be useful for the reader to have one case worked out in all details, so that in Part I we prove the above in the context of our basic example by proving global existence and uniqueness of the relaxed solution of the problem; we also show that this is the limit of the empirical mass density of the associated particle system (in the hydrodynamic limit). In Part II we discuss, in a very sketchy way, several other models, the conjecture being that the results proved for the basic model extend to these other cases. So far the conjecture has been proved for a few cases that we review, referring to the original papers for a full account.