SVEN BODO WIRSING

MAXIMAL NILPOTENT SUBALGEBRAS

Nilradicals and
Cartan subalgebras
in associative algebras.
With 428 exercises

I

Anchor Academic Publishing
disseminate knowledge

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For my beloved mother

A mother is special, she's more than a friend.
Whenever you need her, she'll give you a hand.
She'll lead you and guide you in all that you do.
Try all that she can just to see you get through.
Good times and bad times, she's there for it all.
Say head up, be proud, and always stand tall.
She'll love you through quarrels and even big fights,
or heart to heart chats on cold lonely nights.
My mother's the greatest that I've ever known,
I think God made my mother like He'd make his own.
A praiser, a helper, an encourager too,
nothing in this world that she wouldn't do.
To help us succeed she does all that she can,
raised a young boy now into a man.
I want to say thank you for all that you do,
please always know mom, that I love you.

(A true angel by George W. Zellars, February 2006)

Contents

Introduction

Maximal nilpotent are Cartan-subalgebras
as well as the nilradical.
Both will be studied in the magnificient sal
of associated Lie algebras.

(Sven Wirsing, December 2015)

Within the theory of Lie algebras Cartan subalgebras play an important role for the classification of semisimple Lie algebras as well as within the theory of symmetric spaces.

During my time of studying at the Christian-Albrechts-Universität of Kiel Salvatore Siciliano presented his researches in the Oberseminar Algebrentheorie about Cartan subalgebras in Lie algebras associated to associative algebras. His presentation was the starting point for me to study maximal nilpotent substructures in associated Lie algebras of associative algebras. In this work we will present his theory of Cartan subalgebras and enhance it to some special associative algebras (e.g. basic algebras, division algebras, algebras with separable factor algebra by its nilradical). In addition, a second maximal nilpotent substructure is analyzed, its the so-called nilradical of a Lie algebra.

The first chapter introduces some special associative and Lie algebras, monoids and groups. They will be important to visualize and illustrate the general theorems proven within this work. Some applications are also transferred to the exercises at the end of each section or chapter. There are some exercises included enhancing the theory presented so far such that the reader gets a deeper insight. In addition, at the beginning of each exercise series some open-ended topics are included which can be used by the reader – and also by the author – to do additional researches within this theory. The author has included some (manually created) graphics – mostly so called Hasse diagrams – to visualize the results of each section or chapter.

Within chapter 2 basic results about finite subgroups of fields and division algebras are summarized. Some will be proven in details, others will be just presented without a proof. They will play a role later on in the next chapters of this work such that their understanding leads to a better insight of the latter results. In addition, the author includes some proofs of these basic results because of personal interest on the proofs itself. The summary will include the proof that finite subgroups of fields are cyclic, the theorem of Wedderburn about finite division algebras as well as results of Herstein and Amitsur about the classification of finite subgroups of division algebras.

Likewise structured is chapter 3. This chapter focusses on the normal and subnormal subgroup structure of division algebras. We will prove the theorem of Cartan-Brauer-Hua about normal subgroups of division algebras and the theorem of Scott about solvable group of units of division algebras. Finally, the theorem of Stuth about subnormal subgroups is presented (without proving it) enhancing the theorem of Scott.

For an associative algebra the associated Lie algebra can be derived in a natural way. In chapter 4 we analyze the nilradical – the greatest nilpotent ideal – of that Lie algebra and focus our analysis on its associative structure. For this, the center and the nilradical of the associative algebra are of importance: the nilradical is the sum of these two associative substructures. In particular, its an associative subalgebra. For this theorem we assume that the factor algebra by the nilradical of the associative algebra is separable and thus we can use the theorem of Wedderburn-Malcev within the proof.
The analysis begins by determining the nilradical in the case of a solvable associative algebra. For this, results of the so-called generalized Jordan decomposition are used. We demonstrate the theorem on special solvable group algebras based on dihedral and quaternion groups, on the Solomon algebras in characteristic zero, on the Solomon-Tits algebras and on the algebras of upper and lower triangular matrices over an arbitrary field.
In a second step some results of Herstein about simple rings and their associated Lie ring are transferred to simple and semisimple algebras: we will prove that the nilradical is identical to the greatest solvable Lie ideal – the so-called solvable radical. Both structures are identical to the center of the associative algebra. For the semisimple case it is proven that the Lie nilradical of direct products is the direct product of the Lie nilradicals of the corresponding components: there are no diagonals possible.
Both results – for solvable and semisimple associative algebras – are used to determine the nilradical for arbitrary associative algebras.
The chapter is finalized to apply and enhance the theorem for algebra constructions like the tensor product, the adjunction of a unit and matrix algebras over algebras. The idea is to determine the Lie nilradical by the components of the algebra constructions, like by the factors for the tensor product. We will give proofs or counterexamples for these constructions

with respect to this question.

In the previous chapter we have deeply analyzed the Lie nilradical of an associative algebra with respect to its associative structure. The Lie nilradical is a maximal nilpotent substructure, and the Cartan subalgebras are maximal nilpotent, too. They are in focus of the next chapter. They are defined as being nilpotent and self-normalizing Lie subalgebras. The aim of this chapter is the same as for Lie nilradical: their determination and the description of their associative structure. Some results of this chapter are based on an article of Salvatore Siciliano [59], others are enhancements of his theory to other classes of associative algebras like division algebras, simple, semisimple and separable associative algebras, reduced associative algebras or associative algebras with separable factor algebra by their nilradical. Standard examples are investigated in details, in particular group algebras, lower and upper triangular matrices and Solomon-(Tits) algebras for illustrating the developed theory.
The main result of this chapter is the 1:1-connection between maximal tori (maximal commutative separable subalgebras) und Cartan subalgebras. Centralizing maximal tori is a bijection between these structures. The inverse calculates for every Cartan subalgebras a maximal torus by creating the set of fully separable elements of the Cartan subalgebra.
In some cases both sets – maximal tori and Cartan subalgebras – are identical, like for separable associative algebras. Central divisions algebras are separable, too, and we prove a theorem of Salvatore Siciliano (in a different way) that maximal tori and Cartan subalgebras are exactly the maximal separable subfields. We enhance the theorem by proving that these are exactly the separable maximal subfields which is also an alternative proof of a theorem of Emmy Noether. In particular, it is proven that all maximal tori = Cartan subalgebras have the same dimension and are isomorphic as Lie algebras. This theorem is transferred to non-central division algebras.
Solvable associative algebras have the property that maximal tori are exactly the radical complements if the factor algebra by its associative nilradical is separable. This result – proven by Thorsten Bauer in his dissertation [4] and by Salvatore Siciliano in [59] – is proven by a different approach and revised later on in the second to last section of this chapter. As a consequence of our main theorem about Cartan subalgebras and the theorem of Wedderburn-Malcev all maximal tori and Cartan subalgebras are conjugated, and the Cartan-subalgebras are exactly the centralizers of the radical complements.
For basic algebras we transfer the determination of Cartan subalgebras to Cartan subalgebras of maximal solvable substructures. These maximal ones are describable as direct sums of maximal tori and the associative nilradical. The centralizers of the maximal tori of the underlying algebra are identical to the centralizer within these maximal solvable subalgebras. Afterwards we focus on reduced group algebras. In the modular case the terms basic and

solvable are equivalent. For semisimple group algebras the situation is more complex: the group is hamiltonian and the equation $a^2 + b^2 + 1 = 0$ has no solution in special field extension based on roots of unities. Finally, we determine the dimension of the Cartan subalgebras for these group algebras based on the results of chapter 6.
In the second to last section we analyze how the determination of Cartan subalgebras can be done based on separable radical complements. The maximal tori of the radical complement and of the whole algebra are identically. For separable radical complements maximal tori and Cartan subalgebras are identically, too. The centralizers of them are exactly the Cartan subalgebra of the underlying algebra. Based on this result a strategy is developed for determining Cartan subalgebras. For solvable algebras this strategy is used and the determination of Cartan subalgebras is revised in a more transparent way. We apply this strategy also on group algebras of dihedral groups.
The chapter is finalized to apply and enhance the theorem for Cartan subalgebras for algebra constructions like the tensor product, the adjunction of a unit and matrix algebras over algebras. The idea is to determine the Lie nilradical by the components of the algebra constructions, like by the factors for the tensor product. We will give proofs or counterexamples for these constructions with respect to this question.

The next chapter is dedicated to the dimension of maximal tori in group algebras. We begin this chapter by proving a result of Salvatore Siciliano connecting this dimension to the sum of degrees of all irreducible complex characters for semisimple group algebras. This sum is identical for all fields such that the group algebra is semisimple. We use this result and some classical and modern results about that sum within the character theory of finite groups to bound this dimension – like by the number of involutions, by the order of the group, by the order of abelian subgroups and by the maximal degree – and determine this sum for several classes of groups – like for Frobenius groups, for direct products, for extra special p-groups, for diverse linear groups, for ambivalent groups such as dihedral and symmetric groups, for meta-cyclic groups, for p-groups, for nilpotent groups and for minimal non-abelian p-groups.

Within chapter 7 we focus on the question whether the dimensions of the maximal tori and of the Cartan subalgebras are unique for associated Lie algebras of finite-dimensional associative unital algebras. For maximal tori we give a positive answer to this question for associative algebras with separable factor algebra by its nilradical by calculating this dimension explicitly. The answer for the Cartan subalgebras is positive, too. In characteristic zero we derive this result by using a classical result on Cartan subalgebras over algebraically closed fields. In the modular case we begin the analysis by proving the uniqueness for associated Lie algebras based on solvable finite-dimensional associative algebras, for separable associative algebras and for finite-dimensional associative algebras possessing a central nilradical. The general case is derived by using a result of Premet (which was later proven by Farnsteiner) for restricted Lie algebras over algebraically closed fields in positive characteristic and by using the result on the dimension for maximal tori. In general, the dimension of Cartan subalgebras can differ for restricted Lie algebras. By using a second approach we extend our theorem for the uniqueness of the dimension of Cartan subalgebras to the solvable and nilpotency class. For this, we prove that all maximal tori and Cartan subalgebras of Lie algebras associated to finite-dimensional associative algebras over an arbitrary algebraically closed field are conjugated. We demonstrates these three invariants – dimension, nilpotency and solvable class – by calculating them for group algebras based on dihedral and quaternion groups.

Chapter 8 is an outlook on the second series about maximal nilpotent substructures. We will focus on the solvable case of an associative algebra in more details as in this first volume. For this, we will extend the topic to all maximal nilpotent substructures and to the connection to the maximal nilpotent subgroups of their group of unit. A graphic illustrates the problems analyzed in series II.

Within the appendix we classify a special class of algebras and analyze their Lie nilpotency. This class of algebras was in focus of the diploma thesis of Armin Jöllenbeck.

Chapter 1

Natural examples

This chapter has a preliminary function by summarizing those monoids, groups, associative and Lie algebras which will arise in this work. They will be used for examples of the proven theorems as well as for exercises in which the reader shall apply the results.

Groups and monoids

Let $n \in \mathbb{N}$, N be a set, M a monoid, G a group, A an associative unitary algebra and q a prime power. We will focus on the following groups and monoids:

· $\mathbb{N}$ - natural numbers

· $\mathbb{N}_0$ - natural numbers containing zero

· $(P(N); \cap)$ - power set of N with operation $\cap$

· $(P(N); \cup)$ - power set of N with operation $\cup$

· $(P(N); \delta)$ - power set of N with operation δ - symmetric difference

· $(P(M); \cdot)$ - power set of M with complex product $\cdot$ as operation

· $(P(G); \cdot)$ - power set of G with complex product $\cdot$ as operation

· D_{2n} - dihedral group of order $2n$

· Q_{4n} - quaternion group of order $4n$

· SD_{2^n} - semi-dihedral group of order 2^n

· S_n - symmetric group of degree n

· A_n - alternating group of degree n

· $GL(n, q)$ - general linear group of degree n over $GF(q)$

· $SL(n, q)$ - special general linear group of degree n over $GF(q)$

· $PSL(n, q)$ - projective special general linear group of degree n over $GF(q)$

· $SP(2n, q)$ - symplectic group of degree $2n$ over $GF(q)$

· $GSP(2n, q)$ - general similitudes group

· $U(n, q)$ - unitary group of degree n over $GF(q)$

· C_n or Z_n - cyclic group of order n

· $E(A)$ - group of units of A

· $Q(A)$ - quasiregular group of A

· $\times$ - direct products of groups

· $\wr$ - regular wreath product of groups

· $\ltimes$ - semidirect product of groups.

General constructions of algebras

Let A be an algebra, K a field, G a group, I an ideal, M a monoid, $n \in \mathbb{N}$ and $T \subseteq A$. The following general constructions of algebras will be used:

· $\otimes$ - tensor product of algebras

· $\times$ - direct products of algebras

· $\oplus$ - direct sum of algebras

· A/I - factor algebra of A by the ideal I

· KG - group algebra of the group G and the field K

· KM - monoid algebra of the monoid M and the field K

· $A^{n \times n}$ - algebra of $n \times n$-matrices over A

· A° - associated Lie algebra of A

· $\langle T \rangle_K$ - K-linear span of T

· $\langle T \rangle_{\mathcal{A}}$ - subalgebra generated by T

· $\langle T \rangle_{\mathcal{A}_1}$ - unital subalgebra generated by T

· A^K - adjunction of a unit to A

· A^{op} or A^{-} - inverse or opposite algebra of A

· $(A \times A; \odot)$ - zero extension of A

· $gl(n, K)$ - identical to $(K^{n\times n})^{\circ}$

· eAe - identical to $\{eae \mid a \in A\}$ for an idempotent e

· $Aug(KG)$ - augmentation ideal of KG.

Commutative algebras

The following commutative algebras will appear:

· $\mathbb{Z}$ - the set of integers

· $K[t]$ - polynomial algebra over K in one variable t.

Fields and skew fields

Let p be a prime number, $n \in \mathbb{N}$ and $(K; L)$ a field extension. We will focus on the following fields, skew fields and elements:

· $\mathbb{Q}$ - rational number field

· $\mathbb{R}$ - real number field

· $\mathbb{C}$ - complex number field

· $\mathbb{H}$ - real quaternion algebra

· $GF(p^n)$ - finite field with p^n elements

· $GF(q)$ - notation for $GF(p^n)$ and $q = p^n$

· $A(a, b)$ - generalized quaternion algebra

· $K(a)$ - smallest subfield in L containing a and K

· ω_d - primitive dth root of unity

· cyclic division algebras.

(Central) - simple associative algebras

Let K be a field, D a division algebra and $n \in \mathbb{N}$. We will use the following (central)-simple associative algebras:

· $K^{n\times n}$ - $n \times n$-matrices over K

· $D^{n\times n}$ - $n \times n$-matrices over D

· $A(a, b)$ - generalized quaternion algebra.

Semisimple associative algebras

We will use the following semisimple associative algebras:

· $\times$ - direct products of simple algebras

· $A/rad(A)$ - the factor algebra by the nilradical of an associative algebra.

Nilpotent associative algebras

Let A be an associative algebra, K a field, p a prime number, $n \in \mathbb{N}$ and G a p-group. We will focus on the following nilpotent associative algebras:

· $rad(A)$ - nilradical of A

· $J(A)$ - Jacobson radical of A

· $s\delta_{u,n}$ - algebra of strict lower triangular matrices of $K^{n \times n}$

· $s\delta_{o,n}$ - algebra of strict upper triangular matrices of $K^{n \times n}$

· $Aug(KG)$ - augmentation ideal of KG based on a p-group G and $char(K) = p$.

Solvable associative algebras

Let $n \in \mathbb{N}$, p a prime number, G a finite group and K be a field. We will focus on the following solvable associative algebras:

· $K\Pi_n$ - Solomon-Tits algebra (see e.g. [76])

· D_n - Solomon algebra in the case $char(K) = 0$ (see e.g. [4])

· $\delta_{u,n}$ - algebra of lower triangular matrices of $K^{n \times n}$

· $\delta_{o,n}$ - algebra of upper triangular matrices of $K^{n \times n}$

· KG - group algebra based on: $char(K) = p$ and G possesses a normal p-Sylow subgroup with an abelian p'-Hall subgroup.

Chapter 2

Finite subgroups of fields and division algebras

In this chapter we summarize some results of finite subgroups in unit groups of fields and division algebras. For some of them we provide a proof, for the others we reference the corresponding literature. We will use some of these results in the next chapters. Therefor these results provide the reader a deeper insight for understanding these results. In addition, this chapter is included on personal interest of the author for the proofs of these results.

2.1 Finite subgroups of fields

By $E(A)$ and $K[t]$ we denote the group of units of an associative algebra A and the algebra of polynomials over a ring K based on the single variable t. For a group G and an element g of G let $o(g)$ (more exact: $o_G(g)$) the order of g in G.

The following theorem is proven by various arguments within the literature. It is unknown which mathematician provided the first proof of this result. Our variant is based on the main theorem on finite abelian groups.

Theorem 1 *Every finite subgroup of the group of units of a field is cyclic. In particular, the group of units of a finite field is cyclic.*

Proof. Let K be a field and U a finite subgroup of $E(K)$. By using the main theorem on finite abelian groups we decompose U in cyclic groups of prime power order:

$$U = (G_{1,1} \times \cdots \times G_{1,s_1}) \times \cdots \times (G_{r,1} \times \cdots \times G_{r,s_r}).$$

In this decomposition all groups $G_{i,j}$ are of prime power order with respect to the prime number p_i. We arrange the product such that $G_{i,1}$ is the greatest factor within $G_{i,1} \times \cdots \times G_{i,r_i}$. For every i let g_i a generator of $G_{i,1}$.

We focus on the element $g := g_1 \cdots g_r$. g is of order $o(g) = o(g_1) \cdots o(g_r)$ because all prime numbers $p_1, \cdots, p_r$ are distinct. For every $u \in U$ the identity $u^{o(g)} = 1$ is valid.

All elements of U are roots of the polynomial $t^{o(g)} - 1$, and there are at most $o(g)$ distinct roots. Hence we derive $\mid U \mid \leq o(g)$. All $o(g)$-powers of g are distinct. Therefor U is exactly the set of these powers of g. We conclude that U is cyclic and generated by g.$\diamond$

2.2 Results of Wedderburn, Amitsur and Herstein about division algebras

An unitary algebra is an algebra with a unit. An unital subalgebra of an unitary algebra is a subalgebra containing the unit element of the global algebra. Hence a unital subalgebra is unitary. An unitary subalgebra is a subalgebra which is unitary as an algebra. An unitary subalgebra does not need to be unital as its unitary unit could differ from the unit element of the global algebra. Its unit element is an idempotent of the global algebra. The center of A is denoted by $Z(A)$.
Let G be a group, T a subset of G and $g \in G$. By ^{g} we symbolize the conjugation with g and by $C_G(T)$ resp. $N_G(T)$ the centralizer resp. normalizer of T in G.

Our next focus is the proof of a theorem of Wedderburn about finite division algebras. For this proof we need the following two propositions.

Proposition 1 *Let D be a K-division algebra and T be a unital finite-dimensional subalgebra of D. Then T is a division algebra, too.*

Proof. Let $t \in T$ and assume $t \neq 0$. We consider the right and the left multiplication with t on T. Both functions are injective because D is a division algebra. Hence - using the finite dimension of T - they are surjective, too. In particular, 1 has a pre-image with respect to these functions. Both pre-images are the inverse of t and therefor contained in T. $\diamond$

Proposition 2 *Let G be a finite group and U be a subgroup of G. The the U and G are equal if and only if G is the union of all G-conjugate subgroups of U.*

Proof. If U is a normal subgroup the statement is true. Let U be a non-normal subgroup of G. Hence the statement $G > N_G(U)$ is true. The number of conjugates of U is exactly the index of the normalizer of U in G which is $\frac{|G|}{|N_G(U)|}$. All conjugated of U have at least the unit element in common. Therefor we conclude:

$$| \bigcup_{g \in G} U^g | \leq 1 + \frac{|G|}{|N_G(U)|} \cdot (| U | -1).$$

The right hand side is – because of $U \leq N_G(U)$ – not greater than

$$1 + | G | - \frac{|G|}{|N_G(U)|}.$$

By using $G > N_G(U)$ we derive that this value is smaller than $| G |$.$\diamond$

We will prove the following theorem by usage of the theory of central-simple associative division algebras. For this, let $ind(D)$ (more exact: $ind_K(D)$) the index of a central-simple finite-dimensional associative unitary K-division algebra which is the unique dimension of all maximal subfields of D. A good introduction to this theory can be found [49] and in [39].

Theorem 2 *(Wedderburn) Every finite division algebra is a field. In particular, its group of units is cyclic.*

Proof. Let D be a finite division algebra and $K := Z(D)$. K is a field and D a central-simple finite-dimensional associative unitary K-division algebra. All maximal subfields have the same dimension $ind_K(D)$. Hence – by using the finiteness of D – they are of the same order. Based on the finite field theory we know that all maximal subfields are isomorphic. Now we use the theorem of Skolem-Noether[1] and conclude that all maximal subfields are conjugated. Every element d of D is contained in a maximal subfield of D

[1]Thoralf Albert Skolem (born 23 May 1887, died 23 March 1963) was a Norwegian mathematician who worked in mathematical logic and set theory. Although Skolem's father was a primary school teacher, most of his extended family were farmers. Skolem attended secondary school in Kristiania (later renamed Oslo), passing the university entrance examinations in 1905. He then entered Det Kongelige Frederiks Universitet to study mathematics, also taking courses in physics, chemistry, zoology and botany. In 1909, he began working as an assistant to the physicist Kristian Birkeland, known for bombarding magnetized spheres with electrons and obtaining aurora-like effects; thus Skolem's first publications were physics papers written jointly with Birkeland. In 1913, Skolem passed the state examinations with distinction, and completed a dissertation titled Investigations on the Algebra of Logic. He also traveled with Birkeland to the Sudan to observe the zodiacal light. He spent the winter semester of 1915 at the University of Göttingen, at the time the leading research center in mathematical logic, metamathematics, and abstract algebra, fields in which Skolem eventually excelled. In 1916 he was appointed a research fellow at Det Kongelige Frederiks Universitet. In 1918, he became a Docent in Mathematics and was elected to the Norwegian Academy of Science and Letters. Skolem did not at first formally enroll as a Ph.D. candidate, believing that the Ph.D. was unnecessary in Norway. He later changed his mind and submitted a thesis in 1926, titled Some theorems about integral solutions to certain algebraic equations and inequalities. His notional thesis advisor was Axel Thue, even though Thue had died in 1922. In 1927, he married Edith Wilhelmine Hasvold. Skolem continued to teach at Det kongelige Frederiks Universitet (renamed the University of Oslo in 1939) until 1930 when he became a Research Associate in Chr. Michelsen Institute in Bergen. This senior post allowed Skolem to conduct research free of administrative and teaching duties. However, the position also required that he reside in Bergen, a city which then lacked a university and hence had no

because the subalgebra of D generated by $\{d, 1\}$ is a subfield of D (see proposition 1). Therefor D is the union of all maximal subfields of D. From this we derive that $E(D)$ is the union of all groups of units of all maximal subfields and that these subgroups are conjugated. We can apply proposition 2 and conclude that D and one maximal subfield of D are identical. The proof is complete and the add-on is a consequence of this result and of theorem 1.$\diamond$

Let V be a K-linear space and T a subset of V. By $\langle T\rangle_V$ we denote the K-linear span of T in V. $GF(p^n)$ resp. $GF(q)$ symbolize a finite field of order p^n resp. q (Galois field).

By usage of our previous results we derive two theorems proven by Herstein:

Theorem 3 *(Herstein) Every finite abelian subgroup of a division algebra is cyclic.*

Proof. Let G be a finite abelian subgroup of a division algebra D. D is a $Z(D)$-Algebra. We focus on the $Z(D)$-linear span of G in D. By using proposition 1 we obtain that this span is a finite-dimensional unital $Z(D)$-division algebra. G is commutative, and hence $\langle G\rangle_{Z(D)}$ is a field and G a finite subgroup of its groups of units. The proof is finished by using theorem 1.$\diamond$

Theorem 4 *(Herstein) Every finite subgroup of a division algebra in positive characteristics is cyclic.*

Proof. Let D be a division algebra, P the central prime subfield isomorphic to $GF(p)$ and G a finite subgroup of $E(D)$. We focus on the unital P-subalgebra $\langle G\rangle_P$ of the P-division algebra D. This division algebra is by the finiteness of G finite-dimensional. Therefor proposition 1 implies that it is a division algebra over P. P is finite and we conclude that this division algebra is finite, too. By usage of theorem 2 of Wedderburn it is a field. The corresponding theorem 1 for fields implies that G is – as a finite subgroup – cyclic.$\diamond$

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research library, so that he was unable to keep abreast of the mathematical literature. In 1938, he returned to Oslo to assume the Professorship of Mathematics at the university. There he taught the graduate courses in algebra and number theory, and only occasionally on mathematical logic. Skolem's Ph.D. student Øystein Ore went on to a career in the USA. Skolem served as president of the Norwegian Mathematical Society, and edited the Norsk Matematisk Tidsskrift (The Norwegian Mathematical Journal) for many years. He was also the founding editor of Mathematica Scandinavica. After his 1957 retirement, he made several trips to the United States, speaking and teaching at universities there. He remained intellectually active until his sudden and unexpected death. For more on Skolem's academic life, see Fenstad (1970).

Remark 1 The previous theorem 4 is wrong in characteristic zero. In the real quaternion algebra the quaternion group of order 8 is a finite but non-cyclic subgroup of the group of units.$\diamond$

Herstein and Amitsur[2] have classified the finite subgroups of division algebras. A first results deals with so-called meta-cyclic groups. These groups are characterized by possessing a cyclic normal subgroup whose factor group is cyclic, too. A group having only cyclic Sylow subgroups is called a Z-group. It can be proven that Z-groups are meta-cyclic.

Theorem 5 *(Herstein) Every p-subgroup with respect to a prime number $p \neq 2$ of the group of units of a division algebra is cyclic. In particular, every subgroup of uneven order of the group of units of a division algebra is a Z-group.*

Proof. We use theorem 5.3.7 in [63] to derive that a p-group of uneven order is cyclic if it possesses exactly one subgroup of order p. This precondition is with respect to 5.3.8 in [63] valid if every abelian subgroup is cyclic. This was proven within theorem 3. The add-on follows as all Sylow subgroups are cyclic.$\diamond$

Remark 2 The previous theorem 5 fails for $p = 2$. In the real quaternion algebra the quaternion group of order 8 is a finite but non-cyclic subgroup of the group of units. All of its subgroups are cyclic.$\diamond$

By C_n or Z_n we denote a cyclic group of order $n \in \mathbb{N}$. If n, m are integers, then let $o_n(m) := o_{Z/nZ}(mZ)$. We formulate the classification of finite subgroups of division algebras (but we will not prove it here) in characteristic zero:

Theorem 6 *(Amitsur) Every finite subgroup of the group of units of a division algebra in characteristic zero is isomorphic one of the following groups:*

(i) C_n

[2]Shimshon Avraham Amitsur (born August 26, 1921, died September 5, 1994) was an Israeli mathematician. He is best known for his work in ring theory, in particular PI rings, an area of abstract algebra. Amitsur was born in Jerusalem and studied at the Hebrew University under the supervision of Jacob Levitzki. His studies were repeatedly interrupted, first by World War II and then by the Israel's War of Independence. He received his M.Sc. degree in 1946, and his Ph.D. in 1950. Later, for his joint work with Levitzki, he received the first Israel Prize in Exact Sciences. He worked at the Hebrew University until his retirement in 1989. Amitsur was a visiting scholar at the Institute for Advanced Study from 1952 to 1954. He was an Invited Speaker at the ICM in 1970 in Nice. He was a member of the Israel Academy of Sciences, where he was the Head for Experimental Science Section. He was one of the founding editors of the Israel Journal of Mathematics, and the mathematical editor of the Hebrew Encyclopedia. Amitsur received a number of awards, including the honorary doctorate from Ben-Gurion University in 1990. His students included Avinoam Mann, Amitai Regev, Eliyahu Rips and Aner Shalev.

(ii) *A Z-group of the form $C_m \ltimes C_4$ for which C_4 acts per inversion on C_m and m is uneven.*

(iii) *A Z-group of the form $T_0 \times \cdots \times T_s$ in which the orders of these factors are pairwise prime to each other, T_0 is cyclic, every T_i, $i \in \underline{s}$ is non-cyclic of the form $C_{p^a} \ltimes (C_{q_1^{b_1}} \times \cdots \times C_{q_r^{b_r}})$, the prime numbers p, q_i, $i \in \underline{r}$ are distinct, for every $i \in \underline{r}$ the semidirect product $C_{p^a} \ltimes C_{q_i^{b_i}}$ is non-cyclic and is satisfying the following condition: if C_{p^c} is the kernel of the operation of $C_{q_i^{b_i}}$ on C_{p^a}, then one of the following cases are valid:*
($q_i = 2$, $p \equiv -1 \bmod 4$, $c = 1$) or
($q_i = 2$, $p \equiv -1 \bmod 4$, 2^{c+1} does not divide $p^2 - 1$) or
($q_i = 2$, $p \equiv 1 \bmod 4$, 2^{c+1} does not divide $p - 1$) or
($q_i > 2$, q_i^{c+1} does not divide $p - 1$.)
In addition, for every non-cyclic factor $C_{p^a} \ltimes C_{q_i^{b_i}}$ within every factor T_j the statement $q_i \cdot o_{p^c}(p)$ does not divide $o_{|T/T_i|}(p)$ is valid.

(iv) *$C_m \ltimes Q_{2^t}$ in which m is uneven, an element of Q_{2^t} of order 2^{t-1} centralizes the group C_m and an element of order 4 of Q_{2^t} inverts the group C_m.*

(v) *$Q_8 \times Z$ in which Z is a Z-group of order m presented in (i), (ii) and 2 has uneven order in $\mathbb{Z}/\mathbb{Z}m$.*

(vi) *$SL(2,3) \times Z$ in which Z is a Z-group of order m presented in (i), (ii) and 2 has uneven order in $\mathbb{Z}/\mathbb{Z}m$.*

(vii) *The binary octahedral group of order 48.*

(viii) *The binary icosahedral group of order 120.*

Proof. see Amitsur [1], Herstein [20] and Lam [38] $\diamond$

2.3 Exercises

Excercise 1 *Read the article [1] of Amitsur. Determine for all finite subgroups of division algebras suitable division algebra in which they are appearing!*

Excercise 2 *Define meta-abelian and supersolvable groups by a research in the literature.*

Excercise 3 *Prove or disprove the following statements:*

(i) *Every cyclic group is meta-cyclic.*

(ii) *The converse of (i) is valid.*

(iii) *Every abelian group is meta-cyclic.*

(iv) *The converse of (i) is valid.*

(v) *Direct products of meta-cyclic groups are meta-cyclic.*

(vi) *Semidirect products of meta-cyclic groups are meta-cyclic.*

(vii) *Every meta-cyclic group is supersolvable.*

(viii) *Every meta-cyclic group is meta-abelian.*

(ix) *A group for which all Sylow subgroups are cyclic is meta-cyclic.*

(x) *A group of squarefree order is meta-cyclic.*

(xi) *Dihedral groups are meta-cyclic.*

(xii) *Quaternion groups are meta-cyclic.*

(xiii) *Semidihedral groups are meta-cyclic.*

Excercise 4 *Prove the following statements: An unitary algebra is an algebra with a unit element. A unital subalgebra of an algebra A is a subalgebra containing the unit element of A. A unital subalgebra is unitary. A unitary subalgebra is a subalgebra which is unitary as an algebra. A unitary subalgebra is not unital in general. (Tip: idempotent elements)*

Excercise 5 *By using an article [20] of Herstein prove the following statements (p prime number, D a skew field and U a subgroup of $E(D)$):*

(i) *If U is of order p or p^2, then U is cyclic.*

(ii) *If $p \neq 2$ and U is a p-group, then U is cyclic.*

(iii) *Is part (ii) true for $p = 2$?*

(iv) *If the order of U is uneven, then U is meta-cyclic.*

Excercise 6 *True or false: The unit group of an infinite field is cyclic. Is it possible to characterize finite fields by characteristics of their unit group?*

Excercise 7 *Determine all finite subgroups of the multiplicative group of complex numbers! How many non-isomorphic subgroups of order n are existing? Visualize them for $n \in \underline{8}$ on the complex plane!*

Excercise 8 *Are there finite subgroups in the additive group of complex numbers? On what terms do finite subgroups of the additive group of a field exist which are non-trivial? What is the answer for the multiplicative group?*