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UNCERTAINTY AND RISK

Mental, Formal, Experimental Representations

Edited by
Mohammed Abdellaoui, R. Duncan Luce, Mark J. Machina and Bertrand Munier

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SERIES C: GAME THEORY, MATHEMATICAL PROGRAMMING AND OPERATIONS RESEARCH

VOLUME 41

Scope: Particular attention is paid in this series to game theory and operations research, their formal aspects and their applications to economic, political and social sciences as well as to sociobiology. It will encourage high standards in the application of game-theoretical methods to individual and social decision making.

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Mohammed Abdellaoui · R. Duncan Luce
Mark J. Machina · Bertrand Munier
(Editors)

Uncertainty and Risk

Mental, Formal, Experimental Representations

With 45 Figures and 49 Tables

Springer

Prof. Mohammed Abdellaoui
Maison de la recherche de l'ESTP
GRID
30 ave du Président Wilson
94230 Cachan
France
abdellaoui@grid.ensam.estp.fr

Prof. Mark J. Machina
University of California
Department of Economics
San Diego
La Jolla, CA 92093-0508
USA
mmachina@ucsd.edu

Prof. R. Duncan Luce
University of California
2133 Social Sciences Plaza A
5100 Irvine, CA 92697
USA
rdluce@uci.edu

Prof. Bertrand Munier
Maison de la recherche de l'ESTP
GRID
30 ave du Président Wilson
94230 Cachan
France
munier@grid.ensam.estp.fr

Library of Congress Control Number: 2007928747

ISSN 0924-6126
ISBN 978-3-540-48934-4 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media

springer.com

Typesetting and production: LE-TEX Jelonek, Schmidt & Vöckler GbR, Leipzig, Germany
Cover design: WMX Design GmbH, Heidelberg

Spin 11924630 Printed on acid-free paper 43/3180/YL - 5 4 3 2 1 0

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Introduction

The concept of uncertainty has much evolved since F. Knight wrote his seminal book on *Risk, Uncertainty and Profit.* Economists have generally reduced the concept to the idea that no probability was available, as opposed to the case of risk. What Knight meant might have been sensibly different: Was not uncertainty the case where probability could not be defined with precision, where there was no consensus measure? In the 1920s, such an imprecision was often sufficient to make any corresponding amounts at stake uninsurable.

Insurance companies, fortunately for us, have since then widely changed their minds as to what can be insured and what cannot be. It would have been amazing and detrimental if researchers had not changed their minds. Already in the 1930s, J. M. Keynes felt the need to deal with another sort of uncertainty. In chapter XII of *The General Theory*, *uncertainty* is defined in a more radical way: it is the situation where *we just don't know.*

The literature about uncertainty deals with different levels of investigation: the neuronal level, giving an account of brain activity; the cognitive level, assessing the role of mental procedures; and finally the choice representation level. At each level uncertainty is modelled in its own way, and that defines rationality in specific ways.

The neuronal level, considering the system of neurons in the brain, as in the newly emerging field of neuroeconomics – which has come to be called *neuronomics* – though promising, has not yet generated enough formal output to be considered in this volume. The reader of this book will rather explore, in virtually each part of the book, the cognitive level, where sources and examinations of *mental* procedures, conjectures and solutions are dealt with, much like in traditional cognitive psychology. Other authors of this book have selected the level of choices, as is more traditional in economics and management science. This last level is the most widely dealt with here. Of the three known levels of study – *neuronal, mental* and *choice pragmatic* – this book focuses on the last two.

At each level of analysis, a researcher has several *tools* at his or her disposal. It is legitimate to rely on previously accepted logical schemes and to develop their consequences formally or to explore the possibilities that they open to rational behavior. At a less general level, decision theoretic and game theoretic tools offer similar ways to proceed. Both types of tools are here characterized as *formal.* A more and more frequent way to proceed, both in psychology and more recently in economics, consists in designing and performing various types of experiments. Such approaches also appear in the present book.

As a result of the newer approaches, today we recognize that uncertainty can be either deeper than the situation described by F. Knight or, in the opposite direction, less radically ignorant of what might happen. Between complete ignorance of the possible futures and mere ambiguity about the probabilities, there is a wide array of different types of uncertainty.

What form do these types take, under which type of conditions, and how can we manage to reach decisions in each type of such really difficult situations? This is the very the topic of the present book.

The papers assembled were given some at the FUR XI conference, organized at GRID (Cachan and Paris, France) in the second half of 2004. They have been selected from some 175 papers, refereed once again, further revised and selected again to form the present set of contributions.

The book is organized in *four parts*: foundational, representational tools, alternative decision rules, and risk attitude modelling.

Part One: Foundations

S. Grant and J. Quiggin propose to explore uncertainty using the language of logic and decision trees, as is often done in artificial intelligence. Needless to say, such an approach opens new avenues of impressive research: The link between their representation and the seminal Max-Min utility model of I. Gilboa and D. Schmeidler (section 9 of the paper) is one of the most fascinating ones.

M. Amarante and F. Maccheroni, in a formal mathematical development, show that a connection to the same seminal model can be found in the very idea that several probability measures have to be considered simultaneously.

J.V. Howard connects the mental and the choice pragmatic levels through the formal representation of finite event trees, thus yielding a new foundation to Bayesian statistics, which assumes, as is well known, the less realistic assumption of countable additivity over a σ-field of events.

Finally, E. Borgonovo and L. Peccati show that, under some relatively modest hypotheses about the structure of the set of the possible states of the world, one can use Sobol's theorem to determine the impact of what they call "parameter uncertainty" on the level of performance of the decision, as evaluated through the relevant utility functional. They have in mind the epistemic notion of uncertainty. They recommend using simulation as a way to let the decision maker concentrate on the quest for information that appears the most important to acquire as a result of this procedure.

Part Two: The Importance of Representational Tools in Understanding Behavior Under Uncertainty and Risk.

A. Guerdjikova argues, on the basis of an experiment, that when diversifying portfolios, the issue is less about whether or not EU or non-EU type of rational behavior obtains than about the use of similarity considerations. The paper offers a generic connection to another tradition derived from artificial intelligence, namely case-based reasoning, specifically linking to the model derived by I. Gilboa and D. Schmeidler within this tradition.

H. Haller and Sh. Mousavi, in a formal development, show that uncertainty can, under given hypotheses, improve the welfare reached in a Second Best situation such as generated by adverse selection market equilibrium. In the insurance market, the Rothschild and Stiglitz model with adverse selection is used to establish the claim of the authors.

E. Camacho-Cuena and Ch. Seidl investigate experimentally the violation of Lorenz relations in the treatment of an income distribution or of an individual multiple outcome lottery. They show that the nature of responses that are requested from the subjects is the key variable. Merely invoking a framing effect either provides an insufficient explanation, or a quite imprecise one.

B. Sopher and A. Sheth also investigate experimentally inter-temporal choice rationality. By using their design in a variety of cases that have different initial periods, levels of discounting, types of discounting, number of periods, etc. they show that exponential discounting is the clear modal choice pattern of behavior in virtually all cases, even though the tendency toward hyperbolic discounting increases when the compounding rate increases. Their investigation thus confirms the latter point, which has already been found in other samples with other protocols, and seems therefore a rather robust result.

Part Three: The Assessment of Several Alternative Decision Rules

The first alternative rule, the focus of R.M. Hogarth and N. Karelaia, concerns 'simple heuristics that make us smart', paraphrasing the title of the book by Berlin psychologists G. Gigerenzer and P.M. Todd. They examine rules of choice between binary cues (which they emphasize as a limitation of their work). They argue that decision rules succeed according to two factors – aside from error – which they identify as characteristics of choice sets: one is the number of binary cues in the set and the presence or not of a dominance situation; and the other is the way that cues are weighted. The structure of choice sets (also called 'the environment' by the authors) may be separable or compensatory: the more separable, the more effective the simplest heuristics (like "take the best"), the more compensatory the environment, the better performing are the more complex models, like hybrids of different simple cues. But error in the environment makes the predictive ability of any model less and less satisfactory.

J.N. Bearden and R.O. Murphy examine rather sophisticated decision rules that may govern search behavior in the well-known "secretary problem", which they generalize under the name of GSP ("Generalized Secretary Problem"). They show that the existence of suboptimal search behavior (too few rounds of investigation) can be accounted for by a stochastic component in the search policy. To this bias, they oppose the optimal search induced by a dynamic programming procedure which they define and present. They manage to give tentative psychological explanations for the possible biased stopping rules, although they admit that working on what has been called here the – observable – *choice pragmatic* level of investigation is not easy to interpret in terms of the – unobservable – *mental* level.

N.P. Thomas offers an interesting contribution to the literature on collective choice. Using Monte-Carlo simulations he tests two alternative MCDM procedures:

(a) Either each individual evaluates the alternatives at hand; some voting procedure being then started, based on the global scoring of each individual,
(b) Or a voting procedure is organized first on the relevance of each attribute and one can then design a group preference ordering using the attributes selected by the vote.

The paper shows that the second type of procedure can be superior to the first type in cases where value conflicts have emerged in the group of decision makers, whereas, in the other cases, the first procedure leads to a higher welfare.

Part Four: Models of Risk Attitudes Modelling and Methodological Issues

E. Paté-Cornell examines the relationships between the methods of probabilistic risk analysis (PRA), derived from engineering, and those of decision analysis (DA), mainly the expected utility tradition derived from economics and the social sciences. The interest of this paper lies in the experience the author has in both domains, especially PRA. Two PRA cases, taken as benchmarks, make the comparison possible. One is the case of the shuttle's PRA, the other is the case of terrorism prevention. The frequentist and Bayesian concepts of probability are examined. The main conclusions she draws from her analysis are that the risk analyst has to be less specific than the analyst helping the decision maker to actually make a decision, because PRA is, in general, developed before the final decision maker, who will be relying on the PRA model, has been precisely determined. This discussion is quite fascinating and might be echoed in various environments.

H. Grossmann, M. Brocke and H. Holling design a procedure to induce preferences for multi-attribute options. The paper relates finite conjoint measurement and multi-attribute utility analysis. The idea is to set up a computer-based procedure leading the participants gradually to order a given finite set of (multiattribute) alternatives using a single (weak) order. The paper does two things: it shows first that the qualitative information represented by the weak order is sufficient to determine a unique set of numerical utilities; it shows then that the procedure, described in the paper is effective, i.e. that preferences are effectively changed w.r.t. their previous state by the computer interactive program. Two experiments lead to that conclusion.

Odilo W. Huber builds on a line of thought to which he has contributed since the mid-Nineties. The idea is that probabilities are not always actively searched for by corporate executives. Z. Shapira already pointed out some time ago that managers tend to believe that they can change the odds and "get around" issues of risk, although that statement needs some qualification. Precisely, Huber reports some experiments on the topic. They lead to the conclusion that mental representations differ among subjects and among tasks, on one hand; and on another hand, that probabilities which have been actively searched for are better recalled than is pre-existing information.

J. Sounderpandian gives first a quite original survey of the literature on the evolution of risk attitudes and on the diverse approaches which have been taken by researchers on this topic. He then introduces the idea of studying such evolution using simulations, provided that the society under investigation is not too "large" and provided, of course, that individual risk attitudes in that society are interrelated. The paper goes on to derive some understanding of the profile and the fate of a society from this perspective.

Oswald Huber completes part four, as well as the volume, by focusing on the concept of risk-defusing operators (RDOs). He asks whether the practical existence of such RDO's can impact on the decision process and in particular on the search for information. The paper summarizes several experiments indicating that there is, indeed, such an impact. Although extreme interpretations given to the phenomenon are to be questioned (based on the fact that RDO does not get rid of all risk), there are in these findings some undoubtedly important and interesting topics to be further studied by future research.

In summary, this set of papers finds sources of ideas in quite a few disciplines contributing to decision theory. We hope that the reader will also agree that these ideas exhibit a rather unusual degree of originality. May readers enjoy reading this volume.

Part I

Foundations

Conjectures, Refutations and Discoveries: Incorporating New Knowledge in Models of Belief and Choice under Uncertainty

Simon Grant[1] and John Quiggin[2]

[1] Department of Economics, Rice University and Australian National University
e-mail: sgrant@rice.edu
[2] School of Economics, University of Queensland, Australia
e-mail: j.quiggin@uq.edu

Abstract. The purpose of this paper is to develop a model of choice under uncertainty in which individuals do not possess a complete description of the space of states of the world, and in which this description evolves over time. The crucial analytical tool is the description of knowledge in terms of a finite set of propositions.

Keywords: unforeseen contingencies, incomplete state-space, propostions

1 Conjectures, Refutations and Discoveries: Incorporating New Knowledge in Models of Belief and Decision under Uncertainty

In any complex decision problem, the usefulness of formal decision procedures is limited by the knowledge that, decisions commonly proved unsatisfactory because of the occurrence of contingencies that were unforeseen, and perhaps unforeseeable, at the time the decisions were taken. This problem is particularly severe in relation to complex environmental problems such as global warming and the sustainable management of large ecosystems.

One popular answer to the question of sustainable design is the 'precautionary principle', namely, that where there is a serious, but unproven, possibility of environmental damage arising from some action or inaction, policy should be designed on the assumption that the risk is in fact real. The opposite position, which may be described as the 'permissive principle', is one which suggests that, in the absence of conclusive proof of danger, the proposed activities of firms and individuals should be given the benefit of any doubt.

Under standard approaches to decision theory, both of these alternatives are rejected in favour of a model in which all possible events (or 'states of nature') are described in advance and assigned a subjective probability. The preferred course of action is the one that maximises the expected return, which may be expressed in monetary terms, or, more generally, in terms of expected utility.

From the decision-theoretic perspective, a strong interpretation of the precautionary principle leads to adoption of a maximin rule, which is excessively conservative and leads to poor average outcomes. Weak interpretations do no more than assert that action should not require full scientific proof of dangers, which is the same as the standard decision-theoretic view. Hence, from the usual decision-theoretic viewpoint, the precautionary principle is either wrong or a restatement of the obvious.

Although highly effective in many contexts, the standard decision-theoretic model has long been criticised for its inability to deal with events for which well-defined probabilities are not available and, even more, for problems where not all possible outcomes can be foreseen. The difficulties associated with the first of these problems were described by Ellsberg (1961) and remained unresolved for many years. Recent work, including that of Epstein (1999), Ghirardato and Marinacci (1999) and Grant and Quiggin (2002d) has resulted in the development of improved characterisations and analytical tools.

The purpose of this paper is to develop a model of choice under uncertainty in which individuals do not possess a complete description of the space of states of the world, and in which this description evolves over time. The crucial analytical tool is the description of knowledge in terms of a finite set of propositions.

2 Epistemology

The problem posed for theories of choice under uncertain by the existence of unforeseen contingencies has been widely recognised. The term 'unknown unknowns', recently used in (widely-derided) remarks by US Defense Secretary Donald Rumsfeld, is commonly used to describe such contingencies. Since all formal theories of decision under uncertainty in widespread use at present rely, implicitly or explicitly, on the availability of a complete description of the state-contingent consequences of actions under consideration, the existence of unforeseen contingencies represents a serious difficulty.

Two main approaches have been adopted. The first has been to choose some form of maximin rule. Such rules have been proposed in a wide range of contexts and arise as a polar case in many different models. In expected utility theory, for example, maximin arises as the polar case for the class of concave utility functions (normally referred to, in this context, as risk-aversion). In rank-dependent models, it is the polar case for a convex probability weighting function (pessimism). Since maximin also arises as a polar case for other models of choice under uncertainty, it is, therefore, not always clear whether a maximin rule is being proposed as a response to lack of knowledge or reflecting an extreme aversion to risk.

The second approach, more directly related to the problem of unforeseen contingency, has been to augment the state space with some form of residual event, aimed at capturing the existence of radically incomplete knowledge. However, it turns out to be quite difficult to implement this approach in such a way that uncertainty about the residual event is non-trivially different from uncertainty about the known set of states of nature.

Both approaches share with expected utility the fact that normative and positive models of choice are derived simultaneously. In the most common approach, the functional form of the model are derived from a set of axioms that are held to be both normatively compelling and descriptively realistic.

It is not strictly necessary, in this approaches, to combine normative and positive models. Some modellers are interested only in description, or only in prescription. Others hold that while some particular set of axioms, typically those of expected utility, is normatively compelling, other axioms yield a more realistic model of observed behavior. Nevertheless, descriptive models are typically constructed in such a way that they can, if desired, be treated as normative models for use by decision-makers. In particular, such models typically refer only to information that is available to decision-makers.

The feature of the state-act model that yields the close fit between normative and positive models is, in the terminology of the rational expectations literature, model-consistency. The information on which individuals are assumed to base decisions is, broadly speaking, the same as that used to model those decisions.

The model-consistent act-state approach has yielded important insights. However, it is inherently unsatisfactory for the case of incomplete knowledge. An adequate external description of the behavior of an individual with radically incomplete knowledge must employ some notion of complete (or at least more extensive) knowledge than that possessed by the individual being described.

Closely related to this is the problem of learning. The concept of learning that has been analyzed most extensively in the literature on decision theory is that of Bayesian updating which is, in a crucial sense, a negative form of learning. The Bayesian decision-maker begins with a prior probability distribution over all states of the world. The occurrence of a particular event amounts to news that a particular subset of states, those making up the complementary event, are no longer possible. Hence the probabilities of these states must be set to zero, while the probabilities of the states that make up the observed event are replaced by their event-conditional probabilities.

If learning in the ordinary sense of the term is to be modelled, it must be possible to represent additions to, as well as subtractions from, the set of possibilities considered by the individual. A natural way of doing this, once the postulate of model-consistency is dropped, is to suppose that, at any given time, the individual has access to a proper subset of some global set of possibilities.

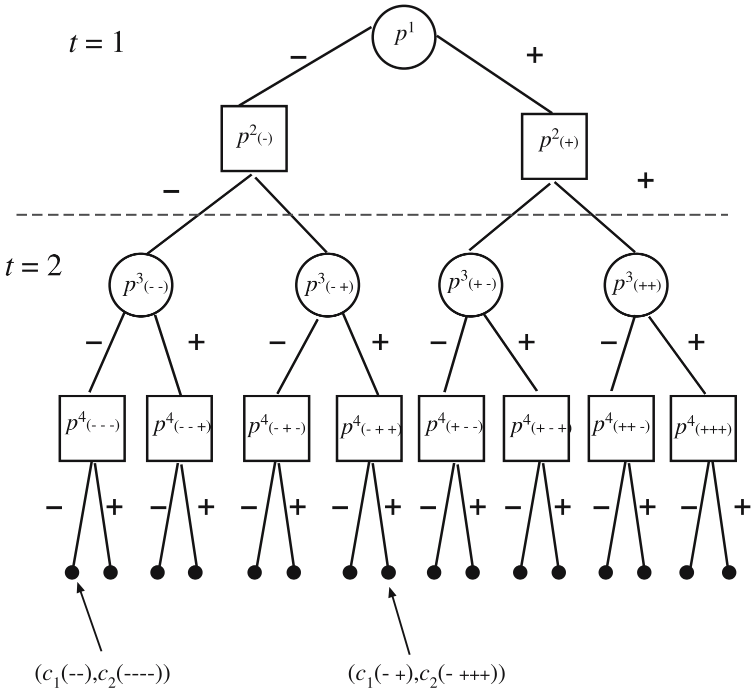

Fig. 1. Decision tree with full information

In this paper, we will argue that, rather then seeking to work directly with a generalized concept of the state space, it is preferable to consider a specification of knowledge in terms of propositions, and to postulate that, at any given time, individuals are equipped to consider only a finite subset of a potentially infinite set of such propositions. The propositional approach may be related back to the state space approach, since any state of nature is characterised by the set of propositions that are true in that state.

3 Example

Consider a standard decision tree with two decision nodes and two chance nodes, as illustrated in Fig. 1. As illustrated there are chance nodes (decisions by Nature) at $t = 1$ and $t = 3$, and decision nodes at $t = 2$ and $t = 4$. Thus there are a total of $2^4 = 16$ possible terminal nodes. We will assume that payoffs are received at times $t = 2$ and $t = 4$ after decisions are made. At each node, we will denote a move to the left by 0 $(-)$ and a move to the right by 1 $(+)$.

This representation is natural for a fully informed decision-makers or outside observers. In decision theory, however, we are normally concerned with decision-makers who are not fully informed. In Fig. 2 we illustrate the case

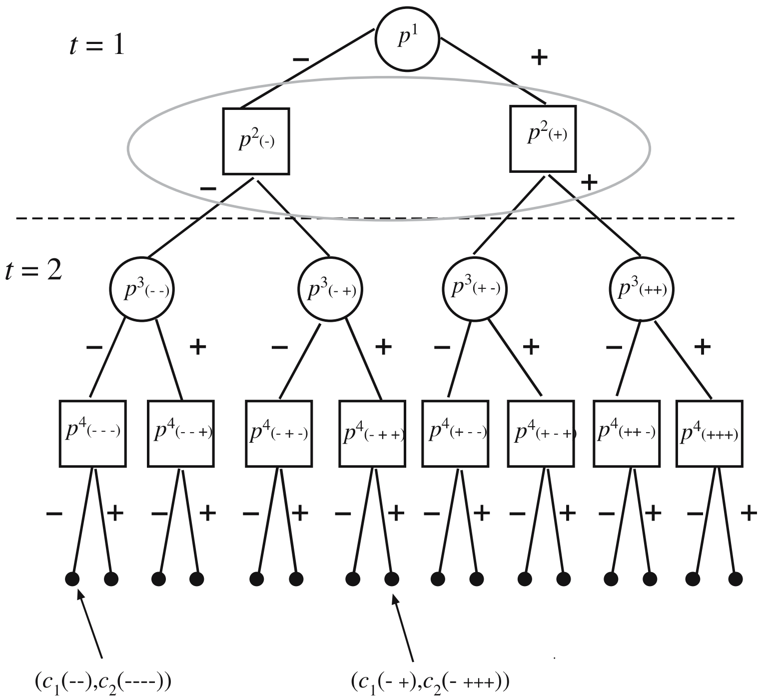

Fig. 2. Decision tree with state contingent uncertainty

where the decision at $t = 2$ must be taken without knowledge of the act of Nature at $t = 1$ ...

We now consider a more radical form of uncertainty, in which the decision at $t = 4$ is taken without the decision-maker even being aware of the chance node at $t = 3$. Thus, the decision-maker cannot distinguish between the nodes 000 and 001, 010 and 011.

In these circumstances, the decision-maker not only does not know what outcome will arise if, say, decision 1 is taken at $t = 4$, but also does not possess a complete state-contingent description of the possible outcomes.

In considering problems of this general kind, the main focus of attention has been on the question of whether, given sufficient information on preferences over decisions, it may be possible to infer a state-contingent model consistent with those preferences. The most promising approach begins with the work of Kreps (1992) and has been developed by Dekel, Lipman and Rusticchini (2001) and Epstein and Marinacci (2006). For these purposes it is more convenient to adopt the logically equivalent representation in Fig. 3 in which the choice node at $t = 3$ is eliminated, and attention is focused on the coarsely specified consequences of the decision at $t = 4$.

The focus of this paper is very different. We are primarily concerned with describing the structure of beliefs like those illustrated in Fig. 4 and the

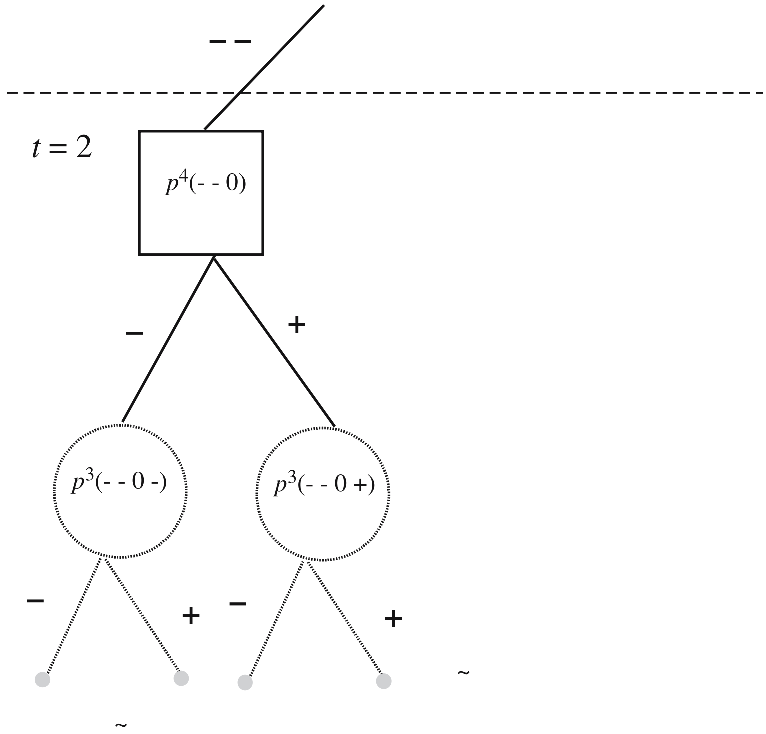

Fig. 3. Decision tree with unreconised contingencies

way in which such beliefs may evolve over time, in the light of both acts of nature and decisions taken by individuals. The natural way to represent both decisions and beliefs, we claim, is in terms of binary propositions. That is, in terms of Figs. 1–4 we impose the (previously implicit) restriction that both chance and decision nodes should have exactly two branches.

4 Propositions

Let the set of states of the world be Ω. We focus on the representation

$$\Omega = 2^{\mathbf{N}} ,$$

where $\mathbf{N} = \{1, 2, \ldots n, \ldots\}$ is supposed to be a finite or countably infinite set, indexing a family of 'elementary' propositions $p^1, p^2 \ldots p^n \ldots$ about the world. Each proposition is a statement such as 'The winner of the 2008 US Presidential election is Hillary Clinton'. An exhaustive description of the state of the world therefore consists of an evaluation of each of the propositions $p^n, n \in N$. As will be shown in more detail below, the elementary propositions may be used to generate a larger set of propositions $\mathbf{P}$.

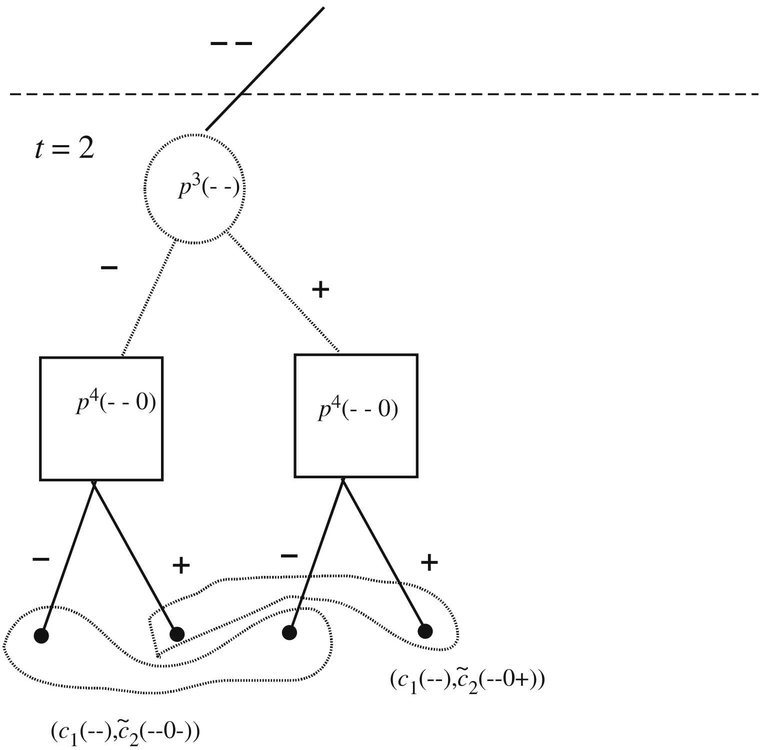

Fig. 4. Decision tree with coarse contingencies

With each proposition and each possible state of the world, a fully informed observer can associate a truth value t^n, which will be denoted 1 (True) or -1 (False). From the viewpoint of a fully informed observer, any state of the world can therefore be described by a real number $\omega \in \Omega \subseteq [0,1]$ [1], given by

$$\omega = \sum_{n \in \mathbf{N}} 2^{-(n+1)} \left(t^n + 1\right) .$$

An elementary proposition p^n is true in state ω if and only if $\omega_n = 1$, where $\omega_n \in \{0,1\}$ is the nth element in the *binary* expansion of ω. Note that, since the mapping $p^n(\omega) = \omega_n$ is defined from the viewpoint of a fully informed observer, the truth value $p^n(\omega)$ does not vary over time.

From this external viewpoint of the model any proposition p^n corresponds to a event $E_n \subseteq \Omega$. More precisely we have

[1] If some propositions may be true in all states of the world, Ω may be a proper subset of $[0,1]$. Alternatively, Ω may be set equal to $[0,1]$ with some states having zero probability in all evaluations.

$$E_n = \{\omega \in [0,1] : \omega_n = 1\} \subset \Omega\,.$$

4.1 Decision-Makers and Decisions

Decision-makers are finitely rational individuals who are not, in general, able to formulate all the propositions in $\mathbf{P}$, or even the elementary propositions $p^n, n \in \mathbf{N}$ and therefore not able to give an exhaustive specification of the state space. We will assume more concretely that, at time t, each individual i is able to conceive a finite set of propositions $\mathbf{P}_t^i$, all of which are generated by a set of elementary propositions $p^n, n \in \mathbf{N}_t^i$ which will be derived below. Note that the elements of the set E_n are not in general, accessible to a decision-maker, even if the proposition p^n is accessible. More generally, proper subsets of E_n are not in general, accessible to a decision-maker.

Example 1. Suppose that the elements of $\mathbf{N}$ are two statements about possible winners of the Melbourne Cup which is a horse race that is run in Melbourne, Australia on the first Tuesday after the first of November. [The winner in 1861 was Archer. The defeated favourite in 1931 was Phar Lap.]

p^1: The winner of the 1861 Melbourne Cup is Archer
p^2: The winner of the 1931 Melbourne Cup is Phar Lap

A decision-maker in October 1861 might be expected to have beliefs about p^1 but not about p^2. However, from the external viewpoint, we have

$$E_1 = \{10, 11\}$$

so that any state of the world consistent with p^1 gives a truth value to p^2.

Decisions are modelled by allowing the decision-maker to control (at time t) the truth value of some proposition. A decision is, therefore, the act of determining the truth value of a proposition. In the example above, we might consider elementary propositions such as p^3: Decision-maker i bets on Archer, and p^4 : Decision-maker i bets against Archer. We will denote by $\mathbf{\Delta}_t^i \subseteq \mathbf{N}_t^i$ the set of elementary propositions decidable by decision-maker i at time t.

Note: We need to consider whether a decision-maker can fail to decide on an element of $\mathbf{\Delta}_t^i$ at time t and if so how to represent this.

4.2 Compound Propositions

The individual can also consider compound propositions p. A compound proposition is derived by assigning truth values of 1 or -1 to all p^n where n is a member of some (possibly empty) subset $\mathbf{N}(p) \subseteq \mathbf{N}$, leaving all p^n, $n \subseteq \mathbf{N}(p)$ unconsidered. The set $\mathbf{N}(p)$ is referred to as the *scope* of p, and is the disjoint union of $\mathbf{N}_-(p)$, the set of elementary propositions false under p, and $\mathbf{N}_+(p)$, the set of elementary propositions true under p. The simple proposition p^n has scope $\mathbf{N}(p^n) = \{n\}$. We define the null proposition $p^{\emptyset}$ such that $p_n^{\emptyset} = 0, \forall n$ and do not assign a truth value to $p^{\emptyset}$.

Any (non-null) compound proposition p corresponds, from the external viewpoint, to an event

$$E_p = \{\omega \in [0,1] : \omega_n = 0, \forall n \in \mathbf{N}_-(p)\,;\ \omega_n = 1, \forall n \in \mathbf{N}_+(p)\} \subset \Omega\,.$$

We set

$$E_{p^{\emptyset}} = \emptyset\,.$$

A compound proposition p is true in state ω if $\omega \in E_p$ (that is, if $\omega_n = 0, \forall n \in N_-(p)$; $\omega_n = 1, \forall n \in N_+(p)$) and false otherwise. We denote the truth value of proposition p in state ω by $t(p;\omega)$. That is,

$$t(p;\omega) = \left\{ \begin{array}{l} 1 \text{ if } \omega_n = 0, \forall n \in \mathbf{N}_-(p)\,; \omega_n = 1, \forall n \in \mathbf{N}_+(p) \\ 0 \text{ otherwise} \end{array} \right\}.$$

A numerical representation of compound propositions is possible using ternary numbers, where the value 0 denotes 'not considered'. Denote the truth value of proposition p^n under p by $p_n \in \{-1,0,1\}$.

As already noted, certain propositions are under the control of decision-makers. The set of all decisions available to decision-maker i at time t is denoted $\mathbf{D}_t^i$. Without loss of generality, we will assume that all elements of $\mathbf{D}_t^i$ are compound propositions derived from elementary decisions, that is,$\mathbf{D}_t^i \subseteq \{-1,0,1\}^{\mathbf{\Delta}_t^i}$. Since some combinations of elementary decisions may be inconsistent or unconsidered, we do not assume that $\mathbf{D}_t^i = \{-1,0,1\}^{\mathbf{\Delta}_t^i}$.

A given decision/action may jointly determine the value of a number of propositions - most obviously if the value of a compound proposition p is set to 1, this determines the truth value of all the elementary propositions in $N(p)$, and of any compound propositions derived from these elementary propositions. Not all of these compound propositions are necessarily accessible to the decision-maker. So we want a category of 'conscious action', roughly, a decision-maker i consciously acts to determine proposition p at time t if $p \in P_{it}$ and the action of decision-maker i at time t determines the truth value of p.

4.3 Classes of Propositions

The class of all propositions in the model is denoted by $\mathbf{P} = \{-1,0,1\}^{|\mathbf{N}|}$. It is useful to consider more general classes of propositions $P \subseteq \mathbf{P}$. To any class of propositions P, given state ω, we assign the truth value

$$t(P;\omega) = \sup_{p \in P} \{t(p;\omega)\}\ .$$

That is, P is true if any $p \in P$ is true and false if all $p \in P$ are false. In terms of the logical operations defined below, the set P has the truth value derived by applying the OR operation to its members.

5 Logical Operations from the External Viewpoint

From the external viewpoint, the usual logical operations are available with the standard set-theoretic interpretation. It is usual in decision theory to focus on the set theoretic interpretation and, from the external viewpoint the two are isomorphic. But the propositional interpretation is more satisfactory when describing a decision-maker with only partial awareness.

5.1 Implication

The implication relationship $p \rightarrow p'$ holds if and only if

$$p'_n \in \{-1, 1\} \Rightarrow p'_n = p_n \, .$$

That is, $p \rightarrow p'$ if and only if any elementary proposition p^n that is true (false) under p' is also true (false) under p.

The implication relationship is

(i) reflexive $p \rightarrow p$,
(ii) transitive $p \rightarrow p' \ \& \ p' \rightarrow p'' \Rightarrow p \rightarrow p''$,
(iii) anti-symmetric $p \rightarrow p' \ \& \ p' \rightarrow p \Rightarrow p = p'$.

Observe that $p \rightarrow p^{\emptyset}, \forall p$ and that $p \rightarrow p'$ if and only if $E_p \subseteq E_{p'}$.

With each proposition p, we can associate the class of propositions

$$[p] = \{p' : p' \rightarrow p\} \, .$$

That is, $[p]$ is the class of propositions stronger than p. For an elementary proposition p^n,

$$E_{[p^n]} = E_{p^n} \, .$$

More generally, for any class P of propositions we define $[P]$,

$$[P] = \{p' : \exists p \in P, p' \rightarrow p\} \, .$$

Observe that

$$E_P = E_{[P]} \, .$$

We refer to $[P]$ as the *completion* of P and say that P is *complete* if $P = [P]$.

5.2 Consistency and Logical Independence

Two propositions p and p' are consistent, denoted $p \sim p'$ if there exists p'', $p'' \rightarrow p$ and $p'' \rightarrow p'$. The consistency relationship is reflexive and symmetric, but not transitive. To illustrate the latter point informally, note that the proposition 'Hillary Clinton is the winner of the US Presidential election in 2008' is consistent with 'George Bush is the winner of the US Presidential election in 2004' which in turn is consistent with 'Hillary Clinton is not the winner of the US Presidential election in 2008', but the first and third propositions are inconsistent.

The following lemma (proof left to the reader) characterises consistency in terms of the ternary representation used above:

Lemma 1. *For any p, p', $p \sim p'$ if and only if for all n, such that $p_n \in \{-1, 1\}$ either $p'_n = p_n$ or $p'_n = 0$.*

With each proposition p, we can associate the class of propositions

$$\langle p \rangle = \{p' : p' \sim p\} \ .$$

More generally, for any class P of propositions, we define

$$\langle P \rangle = \{p' : \exists p \in P, p \sim p'\} \ .$$

Observing that $[P] \subseteq \langle P \rangle$, we define the set of propositions logically independent of p as

$$\rangle P \langle = \langle P \rangle - [P] \ .$$

Conjecture $\langle \langle P \rangle \rangle = \langle P \rangle \, , \rangle \rangle P \langle \langle = [P]$.

5.3 OR and AND

For any two classes of propositions, P and P', define

$$P \vee P' = [P] \cup [P'] \ ,$$
$$P \wedge P' = [P] \cap [P'] \ .$$

Observe that

$$E_{P \vee P'} = E_P \cup E_{P'} \ ,$$
$$E_{P \wedge P'} = E_P \cap E_{P'} \ .$$

The distributive laws apply to $\vee$ and $\wedge$. Moreover, for the set of complete classes of propositions $\vee$ and $\wedge$ define a lattice structure.

5.4 Negation

The final logical operation to be considered is that of negation. Define:

$$\neg P = \mathbf{P} - \langle P \rangle \ .$$

That is, the negation of P is the set of propositions inconsistent with all elements of P.

Conjecture $[\neg P] = \neg P$.

For any elementary proposition p^n,

$$\neg [p^n] = [\neg p^n] \ ,$$

where $\neg p^n$ is true if and only if p^n is false. More formally, $\neg p^n$ is a proposition having the value $\neg p^n_m = -1, m = n, \neg p^n_m = 0$ otherwise.

We can see this by observing that

$$\begin{aligned}
[p^n] &= \{p : p_n = 1\} \ , \\
\langle [p^n] \rangle &= \{p : p_n = 1\} \cup \{p : p_n = 0\} \ , \\
[\neg p^n] &= \{p : p_n = -1\} \\
&= \mathbf{P} - \langle [p^n] \rangle \\
&= \neg [p^n] \ .
\end{aligned}$$

More generally, for any p

$$\neg [p] = [\neg p] \ ,$$

where

$$\neg p = \{p' : \exists n, \ \text{s.t. } p_n p'_n = -1\} \ .$$

The following lemma (proof left to the reader) characterises consistency.

Lemma 2. *The negation operation has the following properties*

$$\begin{aligned}
E_{[p^n]} \cup E_{[\neg p^n]} &= \Omega \, , \\
[P] &= \neg [\neg P] \ , \\
\rangle P \langle &= \langle P \rangle \cap \langle \neg P \rangle \ , \\
\rangle P \langle &= \rangle \neg P \langle \ , \\
\mathbf{P} &= [P] \cup [\neg P] \cup \rangle P \langle \ .
\end{aligned}$$

Note that the sets making up the union in the last line are mutually disjoint.

6 The Decision-Maker's Viewpoint

The class of all propositions considered by individual i at time t is denoted P^i_t. The scope of the individual's proposition set is given by

$$\mathbf{N}^i_t = \cup_{p \in P^i_t} \mathbf{N}\left(p\right) \ .$$

For a given set P^i_t, the *definitely false set* is given by

$$\mathbf{N}^{i-}_t = \cap_{p \in P^i_t} \mathbf{N}_-\left(p\right)$$

and the *definitely true set* by

$$\mathbf{N}_t^{i+} = \cap_{p \in P_t^i} \mathbf{N}_+ (p) \ .$$

These sets characterise the elementary propositions that are true (false) for every element $p \in P_t^i$. Combining these yields the *characterising proposition* $\underline{p}_{it}$

$$\underline{p}_{tn}^i = \begin{cases} -1 & n \in N_t^{i-} \\ 1 & n \in N_t^{i+} \\ 0 & \text{otherwise} \end{cases} \ .$$

We assume that $\underline{p}_{tn}^i \in P_t^i$.

The *set of active possibilities* is given by

$$N_t^{i*} = N_t^i - \left(N_t^{i-} \cup N_t^{i+} \right) \ .$$

Thus, $n \in N_t^{i*}$ if and only if there exist $p, p' \in N_t^i$ with $p_n \neq p'_n$. Recall that p_n can take the three values $0, 1, -1$.

6.1 Logical Operations for the Decision-Maker

Logical operations for the decision-maker are applied with respect to the set P_t^i and may be derived with reference only to propositions $p \in P_t^i$ [2]. Thus, for any $P, P' \subseteq P_t^i$

$$\begin{aligned} [P]_t^i &= \left\{ p' \in P_t^i : \exists p \in P_t^i, p' \to p \right\} \ , \\ P \vee_t^i P' &= [P]_t^i \cup [P']_t^i \ , \\ P \wedge_t^i P' &= [P]_t^i \cap [P']_t^i \ , \\ \langle P \rangle_t^i &= \langle P \rangle \cap P_t^i \ , \\ \neg P_t^i &= P_t^i - \langle P \rangle_t^i \ . \end{aligned}$$

7 Changes in Knowledge

In the model set out above, there are four possible states of knowledge for individual i at time t about an elementary proposition p^n

[2] Spelling out the definition of $\langle P \rangle_{it}$ given blow in terms of the definition of $\langle P \rangle$ would require references to propositions that are not in P_{it} which seems unsatisfactory. However, using the direct characterisation of consistency in terms of the ternary truth values, it is possible to derive $\langle P \rangle_{it}$ without reference to unconsidered propositions.

(i) (believed to be) impossible, $n \in \mathbf{N}_t^{i-}$,
(ii) (believed to be) certain, $n \in \mathbf{N}_t^{i+}$,
(iii) active possibility, $n \in \mathbf{N}_t^{i*}$,
(iv) not under consideration $n \in \mathbf{N}_t^{i0} = \mathbf{N} - \mathbf{N}_t^{i}$.

A crucial feature of the model proposed here is that knowledge can change over time, say from period t to $t+1$ in several different ways. First, some elementary proposition p^n, under consideration at time t, may be verified or falsified by experience at time $t+1$. For the case when p^n is an active possibility at time t, this is analogous to the observation of data in a Bayesian model. However, we allow for the possibility that a proposition treated by the decision-maker as impossible may be verified in period $t+1$ or vice versa.

Next, the state of knowledge may change as a result of inference. For example, the truth value of compound propositions may change as a result of information about elementary propositions. In addition, as will be discussed below, beliefs about active possibilities may be updated in the light of changes in knowledge. The canonical example of such updating is the Bayesian inference procedure in which a posterior distribution is derived from a prior distribution following the observation of data.

The most important, and novel, case treated in the model proposed here is that when a proposition that was previously not under consideration is either verified by experience or becomes an active possibility as a result of inference. Informally, at least, we may distinguish several processes by which this may take place. Surprises arise when an unanticipated event occurs, independently of the actions of the decision-maker, so that some previously unconsidered proposition is verified or falsified. Discoveries are similar, but arise from events that are not fully anticipated, but result from purposive thought and experiment on the part of the decision-maker[3]. Conjectures arise when a previously unconsidered proposition becomes active, typically as a result of formal or intuitive inference.

Symmetrical with the process by which new propositions come under consideration are processes of forgetting, by which propositions previously under consideration cease to be so. Given the finite capacity of human minds, it is reasonable to suppose that, on some appropriate measure of information content, the size of the set of propositions under consideration by any individual remains roughly constant over time. If this measure is approximately equal to the number of elementary propositions under consideration, then the number of propositions forgotten should be equal, on average to the number acquired through discovery and related processes.

[3] At this stage in the project, actions have not been modelled explicitly, and therefore the distinction between surprises and discoveries must remain informal.

8 Inference, Conjecture and Refutation

8.1 Inference

One standard form of discovering new propositions, first considered in formal terms by the ancient Greeks, is that of logical inference. If $p, p' \in P_t^i$ it is natural, in a normative framework, to postulate that $p \vee p'$ and $p \wedge p'$ should be available for inclusion in $P_{(t+1)}^i$ and, further that, if p and p' both have belief values 1 or 0, standard truth-table techniques should apply to determine the belief values of derived propositions such as $p \vee p'$. In a descriptive framework, we must be more cautious. It is well-known that individuals commonly fail to derive all the logical consequences of their beliefs. Furthermore, as the ancient Greek logicians observed when they created lists of common fallacies, individuals frequently attribute incorrect or unjustified beliefs to derived propositions. Nevertheless, the formulation of Π_t^i should give a high probability to the derivation of logical inferences, at least for intelligent individuals with some formal or informal training in logical reasoning.

8.2 Popper and Lakatos on Conjectures and Refutations

Until the early 20th century, most discussion of new knowledge, particularly scientific knowledge, relied either on observation (induction) or inference. The work of Karl Popper, along with the reports of Poincare and others on the process of mathematical and scientific discovery, drew attention to the importance of processes such as conjecture and refutation. Popper drew a sharp distinction between the context of discovery (conjectures) and the context of justification (potential refutation). Whereas previous philosophers of science, and particularly the Vienna school of logical positivism, with which Popper was associated, had focused their attention on evidence that confirmed scientific hypotheses, Popper made the point that the crucial property of a scientific hypothesis was potential refutation. In our terms, the simplest statement of the Popperian model may be stated as one in which no hypothesis can ever be definitely proved (so that no positive proposition can ever be an element of N_t^{i+}) but any non-trivial hypothesis p can be refuted by the observation of some element of $\neg p$, with the result that $p \in N_t^{i-}$ [4]. Subsequent work in the Popperian, such as that of Lakatos has presented a more complex and nuanced view, but has retained a central focus on potential refutation.

Popper's most important contributions to the understanding of conjectures were negative. In the pre-Popperian picture, scientific hypotheses were derived from observed regularities derived from the patient accumulation of observations. This leads naturally to the confirmationist view of justification rejected by Popper. The most useful work on the generation of conjectures

[4] Popper's main point was to deny scientific status to theories like Marxism and Freudian psychology for which, he claimed, there was no possible refutation by evidence.

from previous knowledge is that of Lakatos who shows how a concern with deriving testable implications from a model under challenge leads naturally to the generation of new conjectures.

9 Research Agenda

Thus far, the discussion has been concerned solely with beliefs, to the exclusion of preferences and actions. This is in sharp contrast with the expected-utility approach, where probability beliefs are derived from preferences over actions, considered as mappings from the state space to some outcome space. Other models of choice under uncertainty provide more of a separate role for beliefs, without wholly separating beliefs, preferences and actions. It is clear that a satisfactory account of problems involving uncertainty must encompass preferences and actions.

It is not, as yet clear, how this should best be undertaken within the framework set out. Given the absence of a state-space accessible to the decision-maker, it is not clear that maintaining the separation between the state space and the outcome space, crucial in standard Bayesian decision theory, is appropriate here. It may be more desirable to consider partly or completely probablized subsets of P_t^i as 'possible worlds', each with their own associated outcome space.

As far as preferences are concerned, the most promising approach appears to involve adaptation of ideas developed by Gilboa and Schmeidler. Within possible worlds, preferences may be described by some version of the 'multiple priors' model. When considering actions that have consequences that appear to depend importantly on unforeseeable events, some version of the 'case-based decision theory' model, also proposed by Gilboa and Schmeidler, may be appropriate.

10 Concluding Comments

The problem of unforeseeable events is critical in decision theory. This paper has set out a framework within which this problem can be addressed.

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When an Event Makes a Difference

Massimiliano Amarante[1] and Fabio Maccheroni[2]

[1] Department of Economics, Columbia University, NY, USA
[2] Istituto di Metodi Quantitativi and IGIER, Università Bocconi, Milan, Italy
e-mail: ma734@columbia.edu, fabio.maccheroni@unibocconi.it

Abstract. For (S, Σ) a measurable space, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be convex, weak* closed sets of probability measures on Σ. We show that if $\mathcal{C}_1 \cup \mathcal{C}_2$ satisfies the Lyapunov property, then there exists a set $A \in \Sigma$ such that $\min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) > \max_{\mu_2 \in \mathcal{C}_2} \mu_2(A)$. We give applications to maxmin expected utility and to the core of a lower probability.

Keywords: Lyapunov theorem, maximin expected utility, lower probability

1 Introduction

In the theory of decision making under uncertainty as well as in the theory of cooperative games several questions can be reduced to the problem of whether or not two distinct sets of measures disagree on a set. For instance, if two maxmin expected utility preferences have the same utility on the prize space and the same willingness to bet, are they necessarily the same? Under which conditions, does the core of a lower probability coincide with the weak* closed and convex hull of any set of measures defining it? Both questions are answered in the affirmative if and only if one knows that there exists a set A such that $\min_{\mu_1 \in \mathcal{C}_1} \mu_1(A) > \max_{\mu_2 \in \mathcal{C}_2} \mu_2(A)$, whenever $\mathcal{C}_1$ and $\mathcal{C}_2$ are two (convex, weak* closed) disjoint sets of measures. This is our main result, which we prove in the next section under the conditions stated therein. In Sect. 3, we provide a quick sample of the usefulness of Theorem 1, by answering the two questions stated above. We do not discuss, however, the full range of applications of Theorem 1. For another less immediate application, we refer the reader to [2], where our Theorem 1 turns out to be a key tool to characterize those events which are unambiguous either in the sense of [14] or of [5]. In general, we expect Theorem 1 to be widely applicable in areas different from the ones we consider such as in Quasi-Bayesian Statistics (due to the central role played by upper probabilities; see, for instance [13]) or in social choice theory.