Serge Abrate
Editor

Impact Engineering of Composite Structures

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IMPACT ENGINEERING OF COMPOSITE STRUCTURES

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PREFACE

This book is a much improved version of a set of notes developed for a one-week course taught at the International Center for Mechanical Sciences (CISM) in Udine, Italy, July 21-25, 2005. The support and encouragement of the CISM staff and its Rector, Professor Giulio Maier are gratefully acknowledged for providing the opportunity to bring together different aspects of the problem of impact on composite structures. The opportunity to work with experts in different areas from different countries and to teach a very diverse group of students was also very much appreciated.

As the course coordinator, I want to express my thanks to the colleagues who participated in this long term venture: Professor Giovanni Belingardi from the Politecnico di Torino, Italy; Professor Wesley Cantwell from the University of Liverpool, England; Professor Uday Vaidya, University of Alabama-Birmingham, USA, and Professor Ramon Zaera from the University Carlos III, Madrid, Spain. It is also a pleasure to acknowledge the contributions of Professor Jorge Ambrosio from the University of Lisbon, Portugal who taught a portion of the course in Udine.

The book is intended for beginning graduate students and practitioners in industry who need an introduction with a strong technical background to this subject, one that enables them to pursue their own research or design activities and to pursue further studies on their own. We have attempted to present a broad range of topics. The two common threads throughout the book are that it deals with structures made of composite materials and that those structures are subjected to impacts. The structures can be part of aircrafts, motor vehicles, or armored military vehicles, for example. The impacts can be tool drops, ballistic projectiles or vehicle crashes. The book examines ways to model the impact event, to determine the size and severity of the damage and discusses general trends observed during experiments.

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CONTENTS

Introduction to the Mechanics of Composite Materials

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This chapter recalls the basic notions of stress and strain, the equations of motion of linear elasticity, and the constitutive equations for linear isotropic and orthotropic materials. Then, we introduce some of the most commonly used criteria for predicting failure inside a lamina and the delamination of the interface between adjacent layers. Finally, we discuss two types of approaches used to predict the propagation of delaminations.

1- Stress

This section introduces the concept of stress, the stress tensor as a quantity defining the intensity of the loading at a point, and the equations of motion of a solid in terms of these stress components. To define the intensity of the loading at a given point P in a solid, one can make an imaginary cut through that point. A small area ΔA surrounding P, will be subjected to a force $\overline{F}$ that has a component F_n in the normal direction and a component F_s in the tangential direction. Dividing a force by the area it acts on gives a measure of the intensity of the loading. Expecting this load to vary from point to point, we define the normal stress σ_n as the limit of the ratio $F_n / \Delta A$ as $\Delta A \to 0$. The shear stress σ_s is defined as the limit of the ratio $F_s / \Delta A$ as $\Delta A \to 0$. Since the shear stress can have an arbitrary orientation on the surface, it is usually split into two orthogonal components to account for its orientation. Therefore, on a surface passing through a point P there are three stress components: a normal stress and two shear stresses. The stresses have units of pressure and vary from point to point and also with the orientation of the surface. In the next section we discuss how to fully describe the state of stress at one point.

1.1- Stress Components Acting on a Small Element

Consider a small parallepiped with dimensions $\Delta x.\Delta y.\Delta z$. A face is called an x-face if the outside normal to that surface is oriented in the x direction. Similarly, for a y-face, the normal is oriented in the y direction and for a z-face the normal is in the z direction. A face of that element is called a positive face if the outside normal to that surface is pointing in the positive axis direction.

On a given face, three are 3 stress components: one normal stress and two shear stresses. On a positive x-face (Figure 1), the sign convention is that the stresses are positive in the positive x, y, and z direction. The stress components are designated by two indices. The first index denotes the surface that component is acting on. The second index indicates the direction in which that stress component is acting. For example, Figure 1 shows the three stress components acting on a positive x face. The first index is x for all three components because they are all acting on the x-face and then, σ_{xx} , σ_{xy} and σ_{xz} are acting in the positive x, y and z direction respectively. The same convention is shown to apply to the three stress components acting on a positive z-face in Figure 2 and Figure 3 shows the sign convention applied to a negative y-face.

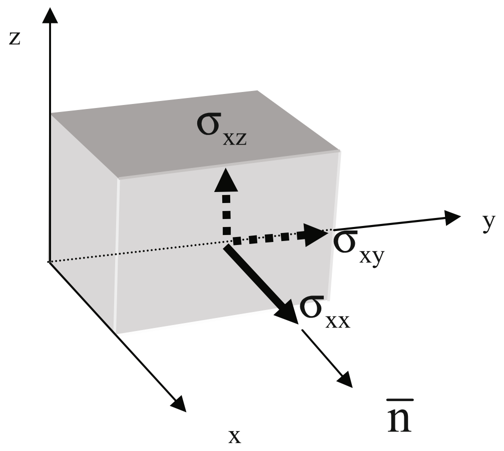

Figure 1. Stress components acting on a positive x face

These three figures define a total of nine components $\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$, σ_{yy}, σ_{yx}, $\sigma_{yz}, \sigma_{zz}, \sigma_{zx}, \sigma_{zy}$ and it can be shown that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, and $\sigma_{yz} = \sigma_{zy}$. Therefore, there are only six independent stress components that can be written as

$$[\sigma]=\begin{bmatrix}\sigma_{xx} & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz}\end{bmatrix} \quad (1)$$

$[\sigma]$ is called the stress tensor. A tensor is a quantity that follows transformation laws. One way to look at it is to think of a vector in space as a quantity with three components. If this vector joins two points in space or represents a force, it is a physical quantity with a given magnitude and orientation. If its components are known in one coordinate system, one can calculate the components of the vector in another coordinate system. Tensors are a generalization of vectors and follow some transformation laws when changing coordinate systems.

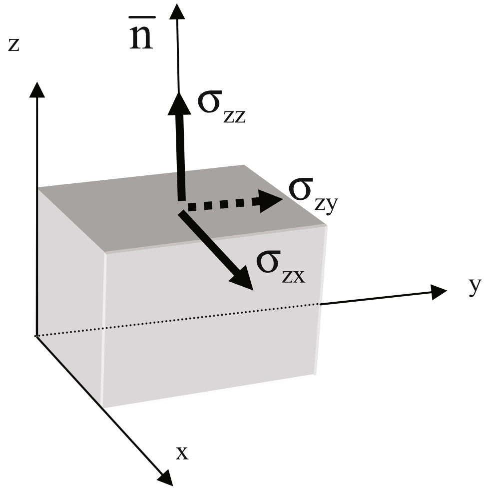

Figure 2. Stress components acting on a positive z face

1.2- Surface Tractions on an Arbitrary Surface

From the stress tensor (Equation 1), one can determine the loads acting on any surface passing through that point. Consider a wedge with three faces oriented in the x-, y- and z-directions and a fourth face oriented in the direction of an arbitrary vector $\bar{n}$ (Figure 4). The stresses acting on the first three faces are components of $[\sigma]$, the stress tensor in the xyz coordinate system. On the face oriented in the $\bar{n}$ direction, we define the surface tractions t_x, t_y and t_z which are forces per unit area acting in the x, y, and z directions. Stresses and surfaces tractions have the same dimension but stresses have components that are normal or in the place of the surface they are acting on while surface tractions are acting in the x, y, z directions.

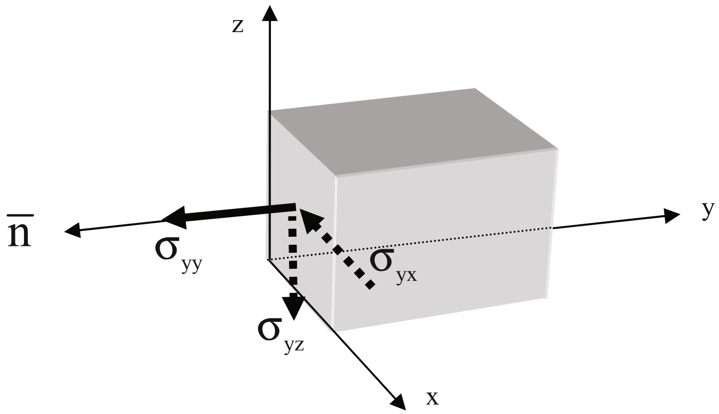

Figure 3. Stress components acting on a negative y face

To establish the relationship between the stress tensor $[\sigma]$ and the surface tensions $\{t\}$ we consider the equilibrium of the wedge (Figure 4) and find that

$$\begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \quad \text{or} \quad \{t\} = [\sigma]\{n\} \tag{2}$$

where i can be either x, y or z. Equation 2 shows that the stress tensor $[\sigma]$ completely defines the state of stress at that point because, given $[\sigma]$, one can calculate the surface tractions t_x, t_y and t_z acting on an arbitrary surface defined by its normal $\bar{n}$. The normal stress on an arbitrary surface is the dot product of the surface traction vector and the normal to the surface. That is

$$\sigma_N = \begin{bmatrix} t_x & t_y & t_z \end{bmatrix} \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} = t_x n_x + t_y n_y + t_z n_z \tag{3}$$

The shear stress acting on that surface is

$$\{\tau\} = \begin{Bmatrix} t_x \\ t_y \\ t_z \end{Bmatrix} - \sigma_N \begin{Bmatrix} n_x \\ n_y \\ n_z \end{Bmatrix} \tag{4}$$

Equation 4 gives the components of that shear stress in the xyz coordinate system. Eqs. (3, 4) show that, given $[\sigma]$ in one coordinate system, the stresses acting on any surface passing through that point can be determined. The stress tensor $[\sigma]$ completely defines the state of stress at that point.

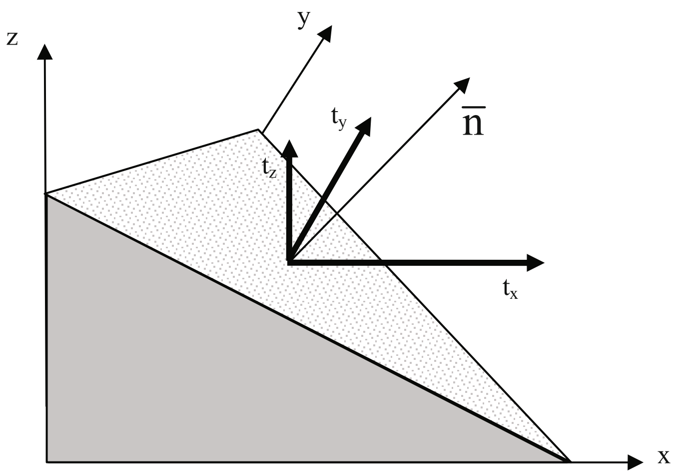

Figure 4. Surface tractions acting on an arbitrary surface

1.3- Equations of Motion

Consider a portion of a body with a volume V and a surface S. The surface is subjected to surface tractions $\bar{t}$, the body is also subjected to body forces $\bar{B}$, and the position of an arbitrary point inside the body is defined by the vector $\bar{r}$. Using Newton's law, the sum of the forces acting on the body gives

$$\int_S \bar{t}\, dS + \int_V \bar{B}\, dV = \int_V \rho \ddot{\bar{r}}\, dV \tag{5}$$

where ρ is the density of the material. In this expression we note that:

- $\int_S \bar{t}\, dS$ is the some of the forces acting on the external surface S
- $\int_V \bar{B}\, dV$ is the resultant of the body forces acting on the volume V
- $-\int_V \rho \ddot{\bar{r}}\, dV$ is the inertia force

Using Equation (2) and applying the divergence theorem, the first term becomes

$$\int_S \bar{t}\, dS = \int_S \bar{\bar{\sigma}}.\bar{n}\, dS = \int_V \bar{\nabla}.\bar{\bar{\sigma}}\, dV \tag{6}$$

Then, for this volume V, the motion is governed by

$$\int_V \left[\bar{\nabla}.\bar{\bar{\sigma}} + \bar{B} - \rho \ddot{\bar{r}} \right] dV = 0 \tag{7}$$

This equation holds for any volume inside the body so we must have

$$\bar{\nabla}.\bar{\bar{\sigma}} + \bar{B} - \rho \ddot{\bar{r}} = 0 \tag{8}$$

This vector equation (Equation 8) can be written as three scalar equations

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} + X = \rho \ddot{u}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} + Y = \rho \ddot{v} \tag{9}$$

and

$$\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + Z = \rho \ddot{w}$$

X, Y, and Z are the components of the body force vector $\bar{B}$. Equations 9 are the equations of motion of the linear theory of elasticity in terms of stresses and displacements. In the following, we will see how the stresses are related to the deformation of the body and how the three equations of motion can be expressed in terms of the three displacements u, v, and w.

2-Strain

Since stresses define the loading at a point, we also need to describe the deformation at that point and then relate those two measures knowing the behavior of the material. To describe the deformation at one point we consider three orthogonal line segments. After deformation, the length of each segment will be different and the angle between any two of those segments will be different from 90 degrees. We define three linear strains from the changes in length of these three line segments and three shear strains from the changes in the angles between them. These six strains will then describe the deformation at that point.

2.1- Linear Strains

Consider a small line segment of length L oriented in the x direction before deformation (Figure 5). After deformation, its length becomes $L+\Delta L$ and the linear strain in the x-direction is defined as $\varepsilon_{xx} = \Delta L / L$. In terms of u, the displacement in the x-direction, $\Delta L = \frac{\partial u}{\partial x}.L$ so the linear strain in the x-direction is $\varepsilon_{xx} = \frac{\partial u}{\partial x}$.

Similarly, considering line segments oriented in the y and z directions we can define the other two linear strain components: $\varepsilon_{yy} = \frac{\partial v}{\partial y}$ and $\varepsilon_{xx} = \frac{\partial w}{\partial z}$.

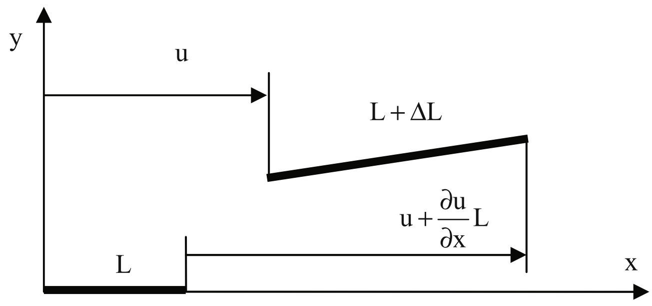

Figure 5. Deformation of a small line element in the x direction

2.2- Shear Strains

Shear strains provide another measure of the deformation at a point. Consider three small line segments initially oriented along the x, y and z directions respectively. These three segments are perpendicular to each other initially but, after deformation, the angle between them is no longer $\pi/2$. For example, the angle between the segments oriented in the x and y directions will be $\frac{\pi}{2} - \varepsilon_{xy}$ and ε_{xy} is defined as the shear strain in the xy plane. Figure 7 shows that $\varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$. Similarly, $\varepsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$ and $\varepsilon_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$.

The six strain components defined here can be shown to from a tensor

$$[\varepsilon] = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} \end{bmatrix} \tag{10}$$

called the strain tensor. It can be shown that the strain tensor defines the deformation at that point. That is, knowing $[\varepsilon]$, one can calculate the change in length of any short line segment passing through that point or the change of angle between any two line segments at that point.

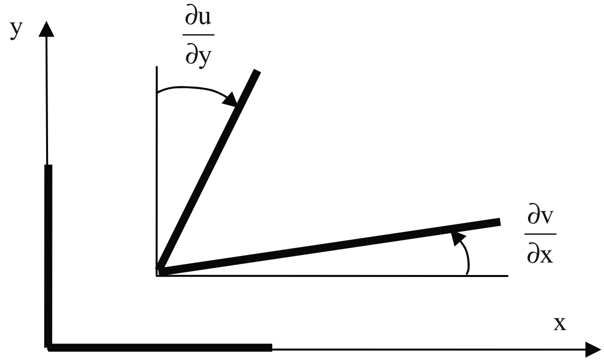

Figure 6. Deformations in the xy plane

3- Stress-Strain Behavior

In describing the mechanical behavior of materials, several definitions must be introduced. A material is isotropic if the behavior is the same in all directions. A material is homogeneous if its properties are the same as we move from point to point inside the material. In general, steel and cast iron are isotropic and are considered to be homogeneous on a macroscopic scale even if under a microscope one can observe different grains with different shapes and properties. Sometimes metals are not isotropic after rolling operations for example. Composite materials often consist of strong fibers embedded in a soft matrix. On a microscopic scale those materials are not homogeneous but on a macroscopic scale (many times the diameter of a fiber) these materials can be considered to be homogeneous. Composite materials are not isotropic either since they are usually stiffer and stronger in the fiber direction than in the transverse direction.

Another important concept in discussing the behavior of materials is whether or not the material is elastic. A material is elastic if it recovers it original length after being unloaded. Plotting the applied force versus the elongation or stress versus strain, a certain path is followed during the loading process and the same path is followed in reverse during the unloading. If for a material the unloading follows a different path, the material is said to be inelastic this can be due to strain rate effects (viscoelasticity), the introduction of permanent deformations (plasticity) or both. A material is said to be linear if the stress-strain curves are straight or, in other words, stress is proportional to strain. It should be pointed out that a material can be elastic without being linear which is the case for rubber.

In the following, we will first discuss the stress-strain behavior for an isotropic material and introduce different concepts such as Hooke's law and material properties such as the modulus of elasticity, Poisson's ratio and the shear modulus. The more general case of an orthotropic material is considered next.

3.1- Hooke's Law

For isotropic materials, most of the information needed to characterize the mechanical behavior of the material can be obtained from a tensile test. For such a uniaxial loading, a normal stress produces an elongation in the direction of the applied stress and a contraction in the transverse direction but no shear deformation. The normal stress σ is directly proportional to ε, the normal strain in that direction

$$\sigma = E\varepsilon \tag{11}$$

for what are called linear elastic solids. Eqn. (11) is called Hooke's law and the proportionality constant E is the modulus of elasticity or Young's modulus of the material.

3.2- Poisson's Ratio

Under a uniaxial normal stress in the x direction for example, the material contracts in the transverse direction (y or z direction) and the ratio between the transverse and normal strains

$$\nu = -\varepsilon_{yy} / \varepsilon_{xx} \tag{12}$$

is a constant and a property of the material. This ratio is called Poisson's ratio. Note that when $\varepsilon_{xx} > 0$ in a tensile test, $\varepsilon_{yy} < 0$ and the negative sign in Equation 12 is introduced so that $\nu > 0$. Since the material is isotropic, $\nu = -\varepsilon_{zz} / \varepsilon_{xx}$ also.

3.3- Stress-Strain Relation in Shear

When the material is subjected to shear the shear stress is directly proportional to the shear strain so, for example,

$$\sigma_{xy} = G\varepsilon_{xy} \tag{13}$$

where G is the shear modulus. It can be shown that, for isotropic materials, the shear modulus is related to the modulus of elasticity and Poisson’s ratio by

$$G = \frac{E}{2(1+\nu)} \tag{14}$$

Therefore, only two independent constants are needed to characterize the elastic behavior of the material. Some authors give E and G while others provide E and ν.

3.4- Stress-Strain Relations for an Orthotropic Solid

Composite materials are heterogeneous on a microscopic scale which can be defined by the diameter of a fiber: typically $10\ \mu m$. However, the analysis of a structure takes place on a much larger scale where dimensions are measured in millimeters. On such a macroscopic scale it is possible to consider the heterogeneous mixture of fibers and matrix materials as a homogeneous material with different properties in different directions. Considering a rectangular block of fiber reinforced material with fibers oriented in the x_1 direction and subjected to a normal stress σ_{11} (Figure 7). The elongation of a line segment in the x_1 direction such as AB will define the strain ε_{11} and according to Hooke's law,

$$\sigma_{11} = E_1 \varepsilon_{11} \tag{15}$$

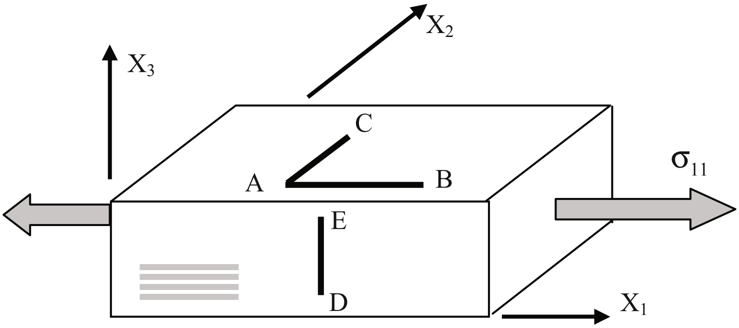

Figure 7. Orthotropic material subjected to tension in the fiber direction

The strains in the x_2 and x_3 directions determine the changes in length of the line segments AC and DE respectively. We define two Poisson's ratios

$$\nu_{12} = -\frac{\varepsilon_{22}}{\varepsilon_{11}} \qquad \nu_{13} = -\frac{\varepsilon_{33}}{\varepsilon_{11}} \tag{16}$$

so that

$$\varepsilon_{22} = -\nu_{12}\frac{\sigma_{11}}{E_1} \quad \text{and} \quad \varepsilon_{33} = -\nu_{13}\frac{\sigma_{11}}{E_1} \tag{17}$$

Note that ν_{12} was defined for the case when the load was applied in direction 1 and the transverse direction was direction 2. Similarly, for ν_{13} the load was applied in direction 1 and the transverse direction that was considered was direction 3. The first index is the direction in which the load is applied and the second index designates the transverse direction.

For the same orthotropic block, applying σ_{22} alone will produce strains

$$\varepsilon_{22} = \frac{\sigma_{22}}{E_2}, \quad \varepsilon_{11} = -\nu_{21}\frac{\sigma_{22}}{E_2}, \quad \varepsilon_{33} = -\nu_{23}\frac{\sigma_{22}}{E_2} \tag{18}$$

Similarly, applying σ_{33} alone will produce

$$\sigma_{33} = E_3\varepsilon_{33}, \quad \varepsilon_{11} = -\nu_{31}\frac{\sigma_{33}}{E_3}, \quad \varepsilon_{22} = -\nu_{32}\frac{\sigma_{33}}{E_3} \tag{19}$$

Combining the effects of these three normal stress components we have

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{Bmatrix} \tag{20}$$

In those equations, six Poisson's ratios have been introduced. It can be shown that the compliance matrix in Equation (20) must be symmetric so that

$$\frac{\nu_{21}}{E_2} = \frac{\nu_{12}}{E_1}, \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}, \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1} \tag{21}$$

and the number of independent Poisson's ratios is reduced to three. In shear, the stress-strain relations are

$$\sigma_{12} = G_{12}\varepsilon_{12}, \quad \sigma_{13} = G_{13}\varepsilon_{13}, \quad \sigma_{23} = G_{23}\varepsilon_{23} \tag{22}$$

where three independent shear moduli are introduced. The elastic behavior of orthotropic materials is characterized by 9 elastic constants: $E_1, E_2, E_3, \nu_{12}, \nu_{13}$, $\nu_{23}, G_{12}, G_{13}, G_{23}$. Combining Equations 20 and 22, the constitutive equations for an orthotropic material can be written as

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & \frac{-\nu_{31}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & \frac{-\nu_{32}}{E_3} & 0 & 0 & 0 \\ \frac{-\nu_{13}}{E_1} & \frac{-\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{13}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \tag{23}$$

and, for an isotropic material,

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\ \frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} \tag{24}$$

The usual convention for unidirectional composites is to take the x_1 axis in the fiber direction, the x_2 axis in the plane of the lamina, and x_3 to be perpendicular to that plane. This is called a material principal coordinate system and, in that system, normal stresses produce linear strains but no shear deformations. Similarly, shear stresses produce shear strains but no linear strains.

3.5- Stress-Strain Relations for a Lamina under Plane Stress

Consider a thin lamina of orthotropic material in material principal coordinates. If the top and bottom surfaces are stress free and the thickness is small, it may be assumed that the stress components $\sigma_{33}, \sigma_{31}, \sigma_{32}$ are negligible. The lamina is then in a state of plane stress and, for inplane loading, the constitutive equations are

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{21}}{E_2} & 0 \\ \frac{-\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} \tag{25}$$

Inverting this relationship gives

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{Bmatrix} \tag{26}$$

where the reduced stiffness constants are

$$Q_{11} = \frac{E_1}{1-\nu_{12}\,\nu_{21}},\ \ Q_{22} = \frac{E_2}{1-\nu_{12}\,\nu_{21}},\ \ Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\,\nu_{21}},\ \ Q_{66} = G_{12}$$

In this coordinate system, extension and shear deformations are uncoupled as indicated by the zeroes in the matrix [Q]. Applying any combination of stresses σ_{11} and σ_{22} will produce linear strains ε_{11} and ε_{22} but no shear strain. Similarly, a shear stress σ_{12} will induce a shear strain but no linear strain. The behavior of such a layer is governed by four independent constants $\left(E_1, E_2, \nu_{12}, G_{12}\right)$.

3.6- Coordinate Transformation

Composite structures often consist of a laminate with many layers oriented in different directions. For each layer, the constitutive equations can be written in a local coordinate system but then it is necessary to use one global coordinate system for all the layers. The stress components acting on a surface with any orientation can be found using a coordinate transformation law. If the three stress components in the xy coordinate system ($\sigma_{xx}.\sigma_{yy}$, and σ_{xy}) are known, the stresses in any coordinate system x'y' oriented at an angle θ (Figure 8) are given by

$$\begin{Bmatrix} \sigma_{x'x'} \\ \sigma_{y'y'} \\ \sigma_{x'y'} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \tag{27}$$

where $c = \cos\theta$, $s = \sin\theta$.

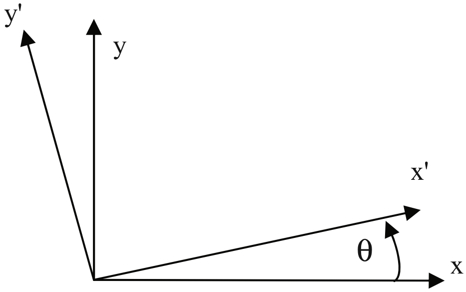

Figure 8. Coordinate transformation in the plane of a lamina

The stresses in the new coordinate system are related to the stresses in the old coordinate system by

$$\{\sigma'\} = [T]\{\sigma\} \tag{28}$$

where [T] is the coordinate transformation matrix. The displacements in the new coordinate system are related to those in the old system by

$$u' = u \cos\theta + v \sin\theta, \quad v' = -u \sin\theta + v\cos\theta \tag{29}$$

The strains in the new coordinate system are

$$\varepsilon_{x'x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'}$$

$$\varepsilon_{y'y'} = \frac{\partial v'}{\partial y'} = \frac{\partial v'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial y'} \tag{30}$$

$$\varepsilon_{x'y'} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial y'} + \frac{\partial v'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial v'}{\partial y}\frac{\partial y}{\partial x'}$$

The coordinates in the new and old systems are related by

$$x = x' \cos\theta - y' \sin\theta \ , \quad y = x' \sin\theta + y' \cos\ \theta \tag{31}$$

Then, the relationships between the strain components in the new coordinate system and the strain components in the old system are

$$\begin{gathered} \varepsilon_{x'x'} = \varepsilon_{xx} \cdot \cos^2\theta + \varepsilon_{yy} \cdot \sin^2\theta + \sin\theta\ \cos\theta\ \varepsilon_{xy} \\ \varepsilon_{y'y'} = \varepsilon_{xx} \cdot \sin^2\theta + \varepsilon_{yy} \cdot \cos^2\theta + \sin\theta \cos\theta\ \varepsilon_{xy} \\ \varepsilon_{x'y'} = -\ \varepsilon_{xx} . 2\sin\theta\ \cos\theta + \varepsilon_{yy} . 2\ \cos\theta\ \sin\theta + \left(\cos^2\theta\ -\ \sin^2\ \theta\right)\ \varepsilon_{xy} \end{gathered} \tag{32}$$

These strain transformation equations (Eqs. 32) can be written in matrix form as

$$\begin{Bmatrix} \varepsilon_{x'x'} \\ \varepsilon_{y'y'} \\ \varepsilon_{x'y'}/2 \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & 2\,sc \\ s^2 & c^2 & 2\,sc \\ -\,sc & sc & (c^2 - s^2) \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy}/2 \end{Bmatrix} \tag{33}$$

or, in short hand notation,

$$\{\varepsilon'\} = [T]\,\{\varepsilon\} \tag{34}$$

where the transformation matrix T is the same matrix used for stress transformations (Equation 28). Eqns. (33) show that, in two-dimensions, three strain components are sufficient to define the state of deformation at one point. The strain in any other direction can be found using the transformation equations.

In a laminate, the material principal directions for a given layer are oriented at an angle θ from the global coordinate system xy (Figure 9). The stress-strain relations in the material principal coordinates (Equation 26) can be written as

$$\begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix}$$

or,

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -2cs & 2cs & c^2 - s^2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \quad (35)$$

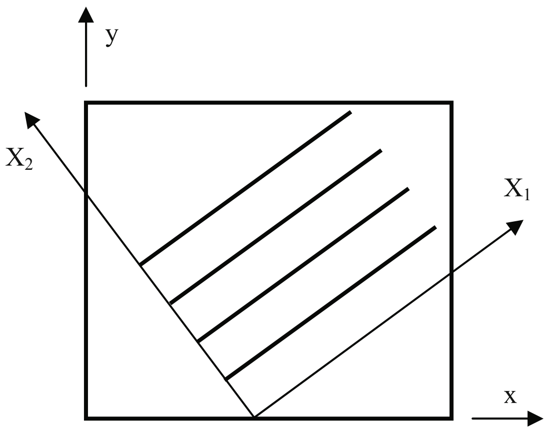

Figure 9. Material principal coordinates for an orthotropic lamina with fibers oriented at an angle θ from an arbitrary coordinate system xy.

Finally, the constitutive equations for a single orthotropic layer are oriented at an angle θ from the x-axis can be written as

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} \quad (36)$$

where

$$\overline{Q}_{11} = Q_{11}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{22}s^4,$$

$$\overline{Q}_{12} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{12}(c^4 + s^4)$$

$$\overline{Q}_{22} = Q_{22}c^4 + 2(Q_{12} + 2Q_{66})c^2s^2 + Q_{11}s^4,$$

$$\overline{Q}_{16} = Q_{11}c^3s - (Q_{12} + 2Q_{66})(c^2 - s^2)cs - Q_{22}s^3c$$

$$\overline{Q}_{26} = -Q_{22}c^3s + (Q_{12} + 2Q_{66})(c^2 - s^2)cs + Q_{11}s^3c,$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 4Q_{66})c^2s^2 + Q_{66}(c^2 - s^2)^2$$

with $c = \cos(\theta)$ and $s = \sin(\theta)$. When the coordinate system is not oriented in the material principal direction $(\theta \neq 0 \text{ or } 90^o)$, the $\overline{Q}_{16}$ and $\overline{Q}_{26}$ terms are not zero and there will be coupling between the extension and shear deformations. A normal stress will produce linear strains and a shear strain. Similarly a shear stress will produce both extensions and shear deformations.

4- Failure Criteria for Composite Materials

Under some given loads, the analysis of a composite structure determines stresses and strains that satisfy the equations of motion and the constitutive equations. It is also important to determine whether or not the structure will fail. In a composite structure, failure can occur inside a ply or at interfaces between plies. Many failure criteria have been proposed for predicting failure of composites and this section will describe some of the most commonly used criteria. A common feature to all the criteria described here is that there are based on the composite stresses (or strains). That is on the stresses calculated assuming that each ply is a homogeneous orthotropic solid as opposed to detailed stress distribution inside a fiber and the surrounding matrix.

First we examine criteria for predicting failure of the fibers, failure of the matrix, or delamination failure. It is often thought that it is easier to develop criteria for individual failure modes and that, when successful, they bring insight into the behavior of the composite. Then, we discuss criteria capable of predicting failure under any combination of stresses. The use of a single criterion for predicting failure of a lamina is easier to implement but the drawback is that, usually, they cannot predict the mode of failure. In each case we introduce the various failure criteria in order of increasing complexity.

4.1- Criteria for Fiber Failure

The simplest approach is to assume that fiber failure depends only on the normal stress in the fiber direction and that failure occurs when the magnitude of that stress exceeds a critical value. Following Hashin (1980), tensile fiber failure is predicted using a maximum stress criterion

$$\sigma_{11} / X_T = 1 \qquad \text{when} \qquad \sigma_{11} > 0 \tag{37.a}$$

This criterion is used by Green et al (2000), Luo et al (1999, 2001), Foo et al (2008) and Cesari et al (2007). Foo et al (2008) assumed that fiber compressive failure occurs when

$$|\sigma_{11}| / X_C = 1 \qquad \text{when} \qquad \sigma_{11} < 0 \tag{37.b}$$

X_T is the strength of the composite in the fiber direction and X_C is the strength in compression and in general $X_C < X_T$. Other researchers have proposed ways to account for the effects of other stress components. Lee (1982) and Hatami-Marbini and Pietruszczak (2007) use Equations 37 to predict tensile and compressive fiber failure. In addition, fiber failure due to shear is predicted using

$$\left(\frac{\sigma_{12}}{S_1}\right)^2 + \left(\frac{\sigma_{13}}{S_1}\right)^2 = 1 \tag{38}$$

where S_1 is the shear strength in the plane perpendicular to the material principal axis 1. This implies that the material is transversely isotropic and that the effect of normal and shear stresses are independent.

Yamada and Sun (1978) and Sun and Yamada (1978) considered that the shear stress σ_{12} also has an effect on fiber failure and used the quadratic criterion

$$\left(\frac{\sigma_1}{X}\right)^2 + \left(\frac{\sigma_{12}}{S_{12}}\right)^2 = 1 \tag{39}$$

where X =X_t for $\sigma_{11} > 0$ and X=X_c for $\sigma_{11} < 0$. S_{12} is the shear strength in the plane of the lamina. In the $\sigma_{12} - \sigma_1$ plane, the failure curve consists of two half ellipses: one for $\sigma_{11} > 0$ and the other for $\sigma_{11} < 0$. The strength of a ply is not the same when it is tested by itself compared with that of a ply inside a laminate. Sun and Yamada (1978) noted that the in-situ shear strength in a laminate is often two or three time higher than that measured on a single layer. Hou et al

(2000) and Zhang (2002) used Equation 39 for $\sigma_{22} > 0$. Hou et al (2000) also included the effect of the transverse shear stress σ_{13} on the tensile fiber failure

$$\left(\frac{\sigma_1}{X_T}\right)^2 + \left(\frac{\sigma_{12}^2 + \sigma_{13}^2}{S_f^2}\right) \geq 1 \tag{40}$$

Gomez del Rio et al (2005) used a slightly different version of Equation 40

$$\left(\frac{\sigma_1}{X_T}\right)^2 + \left(\frac{\sigma_{12} + \sigma_{13}}{S_f}\right)^2 \geq 1 \tag{41}$$

Hashin's fiber failure criteria where

$$\left(\frac{\sigma_{11}}{X_T}\right)^2 + \left(\frac{\sigma_{12}}{S_{12}}\right)^2 + \left(\frac{\sigma_{13}}{S_{13}}\right)^2 = 1 \quad \text{when } \sigma_{11} > 0 \tag{42}$$

and

$$|\sigma_{11}| = X_C \text{ when } \sigma_{11} < 0 \tag{43}$$

was used by Zhang et al (2002) and Li et al (2002). Zhang et al (2002) also introduced a modified Hashin criterion by replacing stresses by strains in Equations (42, 43).

4.2- Criteria for Matrix Failure

Like when the material is loaded in the fiber direction, failure under normal stress in the transverse direction can be thought to depend only on the magnitude of that stress σ_{22}. To predict the onset of matrix cracking, Li et al (2006) used the maximum stress criterion

$$\sigma_{22} / Y_T = 1 \quad \text{when} \quad \sigma_{22} > 0 \quad \text{or} \quad |\sigma_{22}| / Y_c = 1 \text{ when} \quad \sigma_{22} < 0 \tag{44}$$

which accounts for the different strength in tension and compression when loaded in the transverse direction but does not account for the effect of shear stresses. In addition to Equations 44, Lee (1982) and Hatami-Marbini and Pietruszczak (2007) predicted matrix failure due to shear using

$$\left(\frac{\sigma_{12}}{S_2}\right)^2+\left(\frac{\sigma_{13}}{S_2}\right)^2=1 \tag{45}$$

where S_2 is the shear strength in the plane perpendicular to the material principal axis 2. Equation 45 predicts failure due to shear stresses independently of the normal component σ_{22}.

In a laminate, stresses inside a ply vary through the thickness. The criterion proposed by Choi and Chang (1992)

$$\left(\frac{{}^n\overline{\sigma}_{22}}{{}^nY}\right)^2+\left(\frac{{}^n\overline{\sigma}_{23}}{{}^nS_i}\right)^2=e_M^2 \tag{46}$$

where ${}^nY = {}^nY_t$ if $\sigma_{yy} \geq 0$ and ${}^nY = {}^nY_c$ if $\sigma_{22} < 0$ was used in several publications (Her and Linag (2004), Krishnamurthy et al (2001), Mahanta et al (2004), Pradhan and Kumar (2000), Rahul et al (2005, 2006), Zheng et al (2006)). Overbars in Equation 46 indicate that the stresses are averaged through the thickness of layer n. When $\overline{\sigma}_{23}$ is plotted versus $\overline{\sigma}_{22}$ this failure criterion is represented by half of an ellipse when $\overline{\sigma}_{22} > 0$ and another half ellipse when $\overline{\sigma}_{22} < 0$. This criterion accounts for interactions between the normal and shear stress components in the yz plane.

Green et al [2] and Luo et al [3,4] assumed that matrix failure depends on the three stress components σ_{22}, σ_{12}, and σ_{23}

$$\left(\frac{\sigma_{22}}{Y_T}\right)^2+\left(\frac{\sigma_{12}}{S_{12}}\right)^2+\left(\frac{\sigma_{23}}{S_{23}}\right)^2=1 \quad \text{when} \quad \sigma_{22} \geq 0 \tag{47}$$

Green et al (2002) and Luo et al (2001) further assumed that $S_{12}=S_{23}$. Cesari et al (2007) used this criterion to predict failure in tension. They also used it for compression but with different strengths.

Green et al (2002), Luo et al (1999, 2001), Hou et al (2000), Zhang (2002), Li et al (2002) and Gómez-del Río et al (2005) assume that matrix cracking occurs when

$$\left(\frac{\sigma_{22}}{Y_T}\right)^2+\left(\frac{\sigma_{12}}{S_{12}}\right)^2=1 \qquad \text{when} \qquad \sigma_{22} \geq 0 \tag{48}$$

or

$$\frac{1}{4}\left(\frac{\sigma_{22}}{S_{12}}\right)^2+\frac{Y_c\sigma_{22}}{4S_{12}^2}-\frac{\sigma_{22}}{Y_c}+\left(\frac{\sigma_{12}}{S_{12}}\right)^2\geq 1 \quad \text{when} \quad \sigma_{22}<0 \tag{49}$$

Sun et al (1996) proposed a criterion that accounts for the reduction in shear strength when $\sigma_{22} < 0$

$$\left(\frac{\sigma_{22}}{Y}\right)^2+\left(\frac{\sigma_{12}}{S-\mu\sigma_{22}}\right)^2=1 \text{ with } \quad \mu=\begin{cases}\mu_o & \text{when} \quad \sigma_{22}<0\\ 0 & \text{when} \quad \sigma_{22}>0\end{cases} \tag{50}$$

where Y=Y_t for $\sigma_{22} > 0$ and Y=Y_c for $\sigma_{22} < 0$.

Zhang et al (2002) and Kim et al (1997, 2007), Foo et al (2008) used Hashin's criteria for predicting matrix failure. Matrix tensile failure occurs when $(\sigma_{22}+\sigma_{33})>0$ and

$$\left(\frac{\sigma_{22}+\sigma_{33}}{Y_T}\right)^2+\frac{\sigma_{23}^2-\sigma_{22}\sigma_{33}}{S_{23}^2}+\frac{\sigma_{12}^2}{S_{12}^2}+\frac{\sigma_{13}^2}{S_{13}^2}=1 \tag{51}$$

Similarly, compressive matrix failure is expected to occur when $(\sigma_{22}+\sigma_{33})<0$ and

$$\left(\frac{\sigma_{22}+\sigma_{33}}{2\sigma_{12}}\right)^2+\frac{(\sigma_{22}+\sigma_{33})}{Y_C}\left[\left(\frac{Y_C}{2S_T}\right)^2-1\right]+\frac{\sigma_{23}^2-\sigma_{22}\sigma_{33}}{S_{23}^2}+\frac{\sigma_{12}^2}{S_{12}^2}+\frac{\sigma_{13}^2}{S_{13}^2}\geq 1 \tag{52}$$

Kim et al (1997, 2007) averaged the stress components though the thickness of the layer in order to predict matrix failure (Equations 51, 52). They also assumed that $S_{12} = S_{23}$.

4.3- Delamination Failure Criteria

Debonding between adjacent layers depends on the stresses acting on that interface: the normal component σ_{33} and the two shear stresses σ_{13} and σ_{23}. Lee (1982), Zhang (2002), and Hatami-Marbini and Pietruszczak (2007) predicted delamination using a maximum stress criterion for the normal stress and a quadratic criterion for the two shear components

$$\sigma_{33} / Y_T = 1 \qquad \text{or} \qquad \left(\frac{\sigma_{23}}{S_3}\right)^2 + \left(\frac{\sigma_{13}}{S_3}\right)^2 = 1 \tag{53}$$

where S_3 is the shear strength in the plane perpendicular to the material principal axis 3. It is assumed that no failure occurs when $\sigma_{33} < 0$. The criterion proposed by Christensen and DeTeresa (2004) and mentioned by Cesari et al (2007)

$$\left(\frac{\sigma_{13}}{S_{13}}\right)^2 + \left(\frac{\sigma_{23}}{S_{23}}\right)^2 \geq 1 \tag{54}$$

allows for different strengths for σ_{13} and σ_{23} but does not account for interactions between the normal and shear stresses acting at the interface.

Several criteria accounting for the interaction between the three stress components acting at the interface have been introduced. Green et al (2000), Luo et al (1999, 2001), Cesari et al (2007), Zhao and Cho (2004, 2007), Wagner et al (2001), and Hou et al (2000, 2001), postulated that the onset of delamination is governed by

$$\left(\frac{\sigma_{33}}{Z_T}\right)^2 + \left(\frac{\sigma_{13}}{S_{13}}\right)^2 + \left(\frac{\sigma_{23}}{S_{23}}\right)^2 \geq 1 \qquad \text{when } \sigma_{33} > 0 \tag{55}$$

Again, Green et al (2000) and Luo et al (2001), Wagner et al (2001), Hashin (1982) assumed that $S_{12} = S_{23}$. Cesari et al (2007) used the same criterion when $\sigma_{33} < 0$ but with Z_c instead of Z_T in that case. The ellipsoid defined by Equation 55 accounts for the interaction between the three stress components acting at the interface.

Her and Liang (2004), Kim et al [27, 28], Zhang et al (2002), Li (2002), Gómez-del Río et al (2005), Huang and Lee (2003), Lee and Huang (2003), Fuoss et al (1998) use the delamination criterion proposed by Choi and Chang (1992)

$$D_a \left[\left(\frac{{}^{n}\bar{\sigma}_{yz}}{{}^{n}S_i}\right)^2 + \left(\frac{{}^{n+1}\bar{\sigma}_{xz}}{{}^{n+1}S_i}\right)^2 + \left(\frac{{}^{n+1}\bar{\sigma}_{yy}}{{}^{n+1}Y}\right)^2 \right] = e_D^2 \tag{56}$$

where ${}^{n+1}Y = {}^{n+1}Y_t$ if $\sigma_{yy} \geq 0$ and ${}^{n+1}Y = {}^{n+1}Y_c$ if $\sigma_{yy} < 0$. D_a is a scaling parameter and failure occurs when $e_D \geq 1$. Note the similarity between Equations

55 and 56. In the latter, stresses are averaged over the thickness of the layer above the interface (layer n+1) and the strengths can be those of either the layer above or the layer below (n).

The quadratic delamination criterion of Brewer and Lagace (1988) is similar to Equations (55, 56) and can be written as

$$\left(\frac{\sigma_{33}}{Z}\right)^2 + \left(\frac{\sigma_{13}}{S_{13}}\right)^2 + \left(\frac{\sigma_{23}}{S_{23}}\right)^2 = 1 \tag{57}$$

where $Z=Z_t$ when $\sigma_{33} > 0$ and $Z=Z_c$ when $\sigma_{33} < 0$. Naik et al (2000, 2001) used Equation 57 and averaged the values of the stresses through the thickness of the ply. Li et al (2008) used the Brewer-Lagace criterion as defined when $\sigma_{33} > 0$ and omitted the effect of the transverse normal stress when $\sigma_{33} < 0$. In that case, we recover the criterion proposed by Yeh and Kim (2004) which predicts that tensile delaminations occurs when $\sigma_{33} > 0$ and

$$\left(\frac{\sigma_{33}}{Z_T}\right)^2 + \left(\frac{\sigma_{13}}{S_{13}}\right)^2 + \left(\frac{\sigma_{23}}{S_{23}}\right)^2 \geq 1 \tag{58}$$

and shear delaminations occurs when $\sigma_{33} < 0$ and

$$\left(\frac{\sigma_{13}}{S_{13}}\right)^2 + \left(\frac{\sigma_{23}}{S_{23}}\right)^2 \geq 1 \tag{59}$$

Equation 59 is identical to Equation 54, the criterion proposed by Christensen and DeTeresa (2004). Huang and Lee (2003), Liu and Wang (2007) used Yeh's criterion (Equations 58, 59). Zhao and Cho (2007) used only the tensile part of the criterion (Equation 58). The modified Yeh criteria (2004) was used by Huang and Lee (2003) and Lee and Wang (2003). This criterion predicts a tensile delamination mode when

$$\left(\frac{\varepsilon_{33}}{Z_T^{\varepsilon}}\right)^2 + \left(\frac{\varepsilon_{13}}{S_{13}^{\varepsilon}}\right)^2 + \left(\frac{\varepsilon_{23}}{S_{23}^{\varepsilon}}\right)^2 \geq 1 \qquad (\varepsilon_{33} > 0) \tag{60}$$

and a shear delamination mode when

$$\left(\frac{\varepsilon_{13}}{S_{13}^{\varepsilon}}\right)^2+\left(\frac{\varepsilon_{23}}{S_{23}^{\varepsilon}}\right)^2\geq 1 \qquad (\varepsilon_{33}<0) \tag{61}$$

Chen (2004) included the effect of the in-plane transverse stress σ_{22}

$$\left(\frac{{}^{n}\sigma_{33}}{Z_C}\right)^2+D_1\left(\frac{{}^{n}\sigma_{23}^2+{}^{n+1}\sigma_{13}^2}{S^2}\right)+D_2\left(\frac{{}^{n+1}\sigma_{22}}{Y}\right)^2=e_D \tag{62}$$

Hou et al (2000, 2001) assumed that delamination occurs when

$$\left(\frac{\sigma_{33}}{Z_T}\right)^2+\frac{\sigma_{23}^2+\sigma_{13}^2}{S_{13}^2\left(d_{ms}\,d_{fs}+\delta\right)}=1 \qquad \text{when} \qquad \sigma_{33}\geq 0 \tag{63}$$

or

$$\frac{\sigma_{23}^2+\sigma_{13}^2-8\sigma_{33}^2}{S_{13}^2\left(d_{ms}d_{fs}+\delta\right)}=1 \;\;\text{when}\;\; -\sqrt{\left(\sigma_{23}^2+\sigma_{13}^2\right)/8}\leq\sigma_{33}<0 \tag{64}$$

They also assumed that no delamination occurs when

$$\sigma_{33}<-\sqrt{\left(\sigma_{23}^2+\sigma_{13}^2\right)/8} \tag{65}$$

In Eqs. (63, 64), d_{ms} is a damage coefficient of matrix cracking and d_{fs} is a damage coefficient of fibre failure and δ is the ratio between interlaminar stresses before and after matrix or fiber failure.

Zou et al (2002-a,b) proposed a single criterion that accounts for different strength for tension and compression in the transverse direction and for the effect of the two transverse shear stresses

$$\frac{\sigma_{33}^2}{Z_t Z_c}+\frac{\sigma_{13}^2+\sigma_{23}^2}{S^2}+\left(\frac{1}{Z_t}-\frac{1}{Z_c}\right)\sigma_{33}=1 \tag{66}$$

Fenske and Vizzini (2001) extended the Brewer-Lagace criterion by including the effects of inplane stresses.

These various criteria attempt to predict the onset of delamination at the interface between two adjacent plies in terms of the stresses acting at that interface. It is important to remember that the behavior is very different depending on the sign of the transverse normal stress.

4.4- Stress Invariants

Many yield or failure criteria are expressed in terms of invariants of the stress tensor, invariants of the deviatoric stress tensor, or in terms of mean stress and equivalent stress. These quantities are recalled here for future reference. The state of stress at one point is defined be the stress tensor $[\sigma]$ and its six components. The surface tractions acting on a surface oriented by a vector $\{n\}$ are given by $\{t\}=[\sigma]\{n\}$. The principal stress directions are such that only normal stresses are acting on the surfaces oriented in those directions. In other words, the vector $\{t\}$ acts in the normal direction or

$$\{t\}=\lambda\{n\} \tag{67}$$

where the scalar quantity λ is called the principal stress. Equation 67 can be written as

$$[\sigma]\{n\}=\lambda\{n\} \quad \text{or} \quad ([\sigma]-\lambda[I])\{n\}=0 \tag{68}$$

where [I] is the 3x3 identity matrix. Solving this eigenvalue problem (Equation 68) means finding values of the principal stresses λ and the corresponding vectors $\{n\}$ that satisfy Equation 68. Non-trivial solutions occur when $\left| [\sigma_{ij}]-\lambda[I] \right|=0$ which leads to the cubic algebraic equation

$$-\lambda^3+I_1\lambda^2+I_2\lambda+I_3=0 \tag{69}$$

where

$$I_1=\sigma_{xx}+\sigma_{yy}+\sigma_{zz}\,,$$

$$I_2=\sigma_{xy}^2+\sigma_{xz}^2+\sigma_{yz}^2-\sigma_{xx}\sigma_{yy}-\sigma_{yy}\sigma_{zz}-\sigma_{yy}\sigma_{zz} \tag{70}$$

and

$$I_3=\sigma_{xx}\sigma_{yy}\sigma_{zz}+2\sigma_{xy}\sigma_{yz}\sigma_{zx}-\sigma_{zz}\sigma_{xy}^2-\sigma_{yy}\sigma_{xz}^2-\sigma_{xx}\sigma_{yz}^2$$

The state of stress at a point being independent of the coordinate system used to describe it, the principal stresses must be independent of the coordinate system. That means that the coefficients I_1-I_3 in Equation 69 must remain constant under coordinate transformation. For this reason, I_1-I_3 are called the invariants of the stress tensor. Solving Equation 69 gives the values of the three principal stresses and substituting into Equation 68, one can find the principal directions. These principal directions and principal stresses are important because they give the lowest and the highest normal stresses acting at that point and they are often used to define failure criteria. The mean stress or hydrostatic pressure is usually defined as

$$\sigma_m = I_1/3 = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})/3 \tag{71}$$

Both I_1 and σ_m are used in the development of many yield or failure criteria. The deviatoric stress tensor is defined as

$$[s_{ij}] = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} (\sigma_{11} - \sigma_m) & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & (\sigma_{22} - \sigma_m) & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & (\sigma_{33} - \sigma_m) \end{bmatrix} \tag{72}$$

The eigenvalue problem $\left| [s_{ij}] - \lambda [I] \right| = 0$ leads to

$$-\lambda^3 + J_1\lambda^2 + J_2\lambda + J_3 = 0 \tag{73}$$

where

$$J_1 = s_{xx} + s_{yy} + s_{zz},$$

$$J_2 = s_{xy}^2 + s_{xz}^2 + s_{yz}^2 - s_{xx}s_{yy} - s_{yy}s_{zz} - s_{yy}s_{zz} \tag{74}$$

and

$$J_3 = s_{xx}s_{yy}s_{zz} + 2s_{xy}s_{yz}s_{zx} - s_{zz}s_{xy}^2 - s_{yy}s_{xz}^2 - s_{xx}s_{yz}^2$$

Using the definitions of s_{11}, s_{22} and s_{33} (Equation 72), it is easy to show that the first invariant of the deviatoric stress tensor is zero (J_1=0). J_2 can be rewritten as

$$J_2 = s_{xx}^2 + s_{yy}^2 + s_{zz}^2 + s_{xx}s_{yy} + s_{yy}s_{zz} + s_{yy}s_{zz} + s_{xy}^2 + s_{xz}^2 + s_{yz}^2 \tag{75}$$

or in tensor notation, $J_2 = s_{ij}s_{ij}$. In terms of stress components, the second invariant is written as

$$J_2 = \frac{1}{3}\left\{\sigma_{xx}^2 + \sigma_{yy}^2 + \sigma_{zz}^2 - \sigma_{xx}\sigma_{yy} - \sigma_{yy}\sigma_{zz} - \sigma_{zz}\sigma_{xx}\right\} + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 \tag{76}$$

Alternatively, the second invariant J_2 can be expressed as

$$J_2 = \frac{1}{6}\left\{(\sigma_{xx} - \sigma_{yy})^2 + (\sigma_{yy} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{xx})^2\right\} + \sigma_{xy}^2 + \sigma_{yz}^2 + \sigma_{zx}^2 \qquad (77)$$

or, in terms of principal stresses,

$$J_2 = \frac{1}{6}\left\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right\} \qquad (78)$$

The equivalent stress, or von Mises stress, is defined as

$$\sigma_e = \sqrt{3J_2} = \sqrt{\frac{1}{2}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]} \qquad (79)$$

and the octahedral shear stress is defined as

$$\tau_{oct} = \sqrt{\frac{2}{3}J_2} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} \qquad (80)$$

The second invariant of the deviatoric stress tensor J_2, the equivalent stress σ_e and the octahedral stress τ_{oct} are used in the development of many yield or failure criteria.

4.5- Strain Energy Density for Isotropic Solids

This section recalls that, for isotropic solids, the strain energy density depends exclusively on I_1, the first invariant of the stress tensor and J_2, the second invariant of the deviatoric stress tensor. For isotropic solids, the stress-strain relations (Equations 24) can be used to write the strain energy density $U = \frac{1}{2}\sigma_{ij}e_{ij}$ as

$$U = \frac{J_2}{G} + \frac{1-2\nu}{6E} I_i^2 \qquad \text{or} \qquad U = \frac{1}{2\overline{E}}\left[\sigma_e^2 + \beta^2\sigma_m^2\right] \qquad (81)$$

where $\overline{E}=\dfrac{3E}{2(1+\nu)}$ and $\beta^2=\dfrac{9(1-2\nu)}{2(1+\nu)}$. The first term in Equation (81) represents the distortional strain energy and the second term is the strain energy corresponding to the hydrostatic loading. This result explains why, as we shall see, many yield criteria for isotropic materials are written in terms the two invariants I_1 and I_2 or, equivalently, in terms of the equivalent stress and the mean stress σ_e and the mean stress σ_m.

4.6- Von Mises Criterion

The fact that the strain energy density depends on I_1 and J_2 lead to the development of a number of failure or yield criteria based on stress invariants. Experiments conducted by Bridgman (1947) showed that, for metals, yielding is independent of the hydrostatic pressure. Therefore, a number of criteria written in terms of stress invariants are independent of I_1. The von Mises yield criterion in terms of the second invariant of the deviatoric stress tensor is

$$J_2=\frac{1}{3}\sigma_{YT}^2=k^2 \qquad (82)$$

where σ_{YT} is the yield strength in tension and k is the yield in pure shear. Therefore, yielding occurs when J_2 reaches a critical value. Recalling Equation 79, the definition of the equivalent stress σ_e, the Von Mises criterion states that yielding occurs when σ_e reaches the value of the yield strength in tension. With this criterion the strength in tension and compression are equal. For the biaxial loading case, in terms of the principal stresses,

$$\sigma_1^2-\sigma_1\sigma_2+\sigma_2^2=\sigma_{YT}^2 \qquad (83)$$

The yield curve in the $\sigma_1-\sigma_2$ plane is an ellipse centered at the origin with its major axis orientted at 45^o from the σ_2 axis and the lengths of the semi-axes are $a=\sigma_{YT}\sqrt{2}$ and $b=\sigma_{YT}\sqrt{2/3}$. In the $\sigma_m-\sigma_e$ plane, the Von Mises criterion represented by the horizontal lines $\sigma_e=\pm\sigma_{YT}/3$. The Von Mises criterion can also be written in terms of principal stresses as

$$\left\{\left[(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2\right]/2\right\}^{1/2}=\sigma_Y \qquad (84)$$

or, in terms of the regular stress components, as

$$\frac{1}{6}\left\{ \left(\sigma_{xx}-\sigma_{yy}\right)^2+\left(\sigma_{yy}-\sigma_{zz}\right)^2+\left(\sigma_{zz}-\sigma_{xx}\right)^2 \right\}+\sigma_{xy}^2+\sigma_{yz}^2+\sigma_{zx}^2=\frac{\sigma_{YT}^2}{3} \quad (85)$$

These forms of von Mises' criterion (Eqs. 84, 85) are given for comparison with criteria used for anisotropic materials.

4.7- Hill's anisotropic criterion and extensions

In 1948, Hill proposed a criterion capable of accounting for anisotropic behavior that can be seen as a generalization of the von Mises criterion to account for the effect of orthotropy. This criterion was developed for metals such as aluminum that were subjected to rolling operations and became orthotropic. We will show that it is inadequate for modeling foams and other materials for which the effects of hydrostatic pressure are significant and that have different behavior in tension and compression. An extension to this criterion that addresses these deficiencies is discussed next. The well-known Hill criterion (1948, 1950) is written as

$$H\left(\sigma_{11}-\sigma_{22}\right)^2+F\left(\sigma_{22}-\sigma_{33}\right)^2+G\left(\sigma_{33}-\sigma_{22}\right)^2+2L\sigma_{23}^2+2M\sigma_{31}^2+2N\sigma_{12}^2=1 \quad (86)$$

The similarity with von Mises' criterion is obvious when comparing with Equation 85. The coefficients H-N are introduced to account for the orthotropy of the material. If σ_{1T}, σ_{2T} and σ_{3T} are the yield strengths in the principal directions of anisotropy, the strength parameters in Equation 86 are

$$F=\frac{1}{2}\left\{\frac{1}{\sigma_{2T}^2}+\frac{1}{\sigma_{3T}^2}-\frac{1}{\sigma_{1T}^2}\right\}, \quad G=\frac{1}{2}\left\{\frac{1}{\sigma_{3T}^2}+\frac{1}{\sigma_{1T}^2}-\frac{1}{\sigma_{2T}^2}\right\} \quad (87)$$

$$H=\frac{1}{2}\left\{\frac{1}{\sigma_{1T}^2}+\frac{1}{\sigma_{2T}^2}-\frac{1}{\sigma_{3T}^2}\right\}, \quad L=\frac{1}{2\left(\tau_{23}^S\right)^2}, \quad M=\frac{1}{2\left(\tau_{31}^S\right)^2}, \quad N=\frac{1}{2\left(\tau_{12}^S\right)^2}$$

where τ_{12}^S, τ_{23}^S, and τ_{31}^S are the yield stresses with respect to the axes of anisotropy. Hill's anisotropic criterion (Equation 86) does not include the effect of hydrostatic pressure. This can easily be seen by adding a pressure p to σ_{11}, σ_{22} and σ_{33} and substituting into Equation (86) where the term p will then drop out thus proving that point. The strength parameters F, G and H are based on tensile strengths only and Equation 87 predicts equal strengths in tension and compression. Therefore, while capable of accounting for orthotropy, Hill's criterion (Equation 86) is inadequate for modeling foams because it does not address the other two complicating factors discussed here.

The von Mises criterion is inspired by the form of the strain energy density of isotropic solids. Similarly, Hill's criterion bears a similarity with the strain energy density in orthotropic solids. With stress strain relations given by Equations 24, the strain energy density can be written as

$$2U = \frac{\sigma_{11}^2}{E_1} + \frac{\sigma_{22}^2}{E_2} + \frac{\sigma_{33}^2}{E_3} - 2\frac{\nu_{12}}{E_1}\sigma_{11}\sigma_{22} - 2\frac{\nu_{13}}{E_1}\sigma_{11}\sigma_{33} - 2\frac{\nu_{23}}{E_2}\sigma_{22}\sigma_{33} + \frac{2}{G_{23}}\sigma_{23}^2 + \frac{2}{G_{31}}\sigma_{31}^2 + \frac{2}{G_{12}}\sigma_{12}^2 \tag{88}$$

The similarity between the strain energy density (Equation 88) and Hill's criterion (Equation 86) is obvious. As we shall see later, a number of authors use this analogy between the strain energy density, written as $U = \frac{1}{2}\{\sigma\}^T[S]\{\sigma\}$, to define an equivalent stress of the form $\sigma_e = \{\sigma\}^T[P]\{\sigma\}$. [P] being a matrix of the same form as the compliance matrix [S].

Caddell, Raghava and Atkins (1973) proposed a modified Hill criterion for polymers where

$$H(\sigma_{11} - \sigma_{22})^2 + F(\sigma_{22} - \sigma_{33})^2 + G(\sigma_{33} - \sigma_{11})^2 + 2L\sigma_{23}^2 + 2M\sigma_{31}^2 + 2N\sigma_{12}^2 + K_1\sigma_{11} + K_2\sigma_{22} + K_3\sigma_{33} = 1 \tag{89}$$

The last three terms on the left hand side of Equation (89) have been added in order to account for differences in yield strengths in tension and compression. Both Raghava and Caddell (1974) and Caddell and Kim (1981) used Equation 89 to model the yield strength of anisotropic polycarbonate.

4.8- Development of Quadratic Criteria

To better understand the similarities between the quadratic failure criteria that have been proposed by various authors and that are widely used, the following presents a step by step development of such a criterion. In a case where the loading consists of two stress components σ_1 and σ_2, for example, we start with the simple expression

$$A\sigma_1^2 + B\sigma_2^2 = 1 \tag{90}$$

which is the equation for an ellipse in the $\sigma_1 - \sigma_2$ plane with major axes in the σ_1 and σ_2 directions. With this expression the strength in tension and compression are the same and $A = 1/X^2$ and $B = 1/Y^2$ where X and Y are the strengths in directions 1 and 2. Often the failure locus is an ellipse with major axes at an angle from the σ_1 and σ_2 directions. Adding an additional term to Equation 90, the criterion

$$A\sigma_1^2 + B\sigma_2^2 + C\sigma_1\sigma_2 = 1 \tag{91}$$

represents an ellipse with a major axis at an angle θ from the σ_1 axis. That angle is given by

$$\tan 2\theta = \frac{C}{A - B} \tag{92}$$

When A=B, $\theta = 45^{\circ}$ and the ratio C/A controls the ratio of the major and minor axes. For uniaxial tests in direction 1, Equation 91 predicts the same strength X in tension and compression. Similarly, in direction 2, the strength Y is the same in tension and compression. Then, $A=1/X^2$ and $B=1/Y^2$. With composite materials, tensile and compressive strength are very different. Therefore, the criterion has to allow for cases where X_t, the strength in tension, is different from X_c, the strength in compression. Adding linear terms to the previous equation gives

$$A\sigma_1^2 + B\sigma_2^2 + C\sigma_1\sigma_2 + D\sigma_1 + E\sigma_2 = 1 \tag{93}$$

Under uniaxial tension and compression in direction 1, this criterion gives

$$AX_t^2 + DX_t = 1 \quad \text{and} \quad AX_c^2 - DX_c = 1 \tag{94}$$

respectively. Solving these two equations, we find the constants A and D

$$A = \frac{1}{X_t X_c}, \qquad D = \frac{1}{X_t} - \frac{1}{X_c} \tag{95}$$

Similarly, from the uniaxial strengths in direction 2, we find the constants B and E

$$B = \frac{1}{Y_t Y_c}, \qquad E = \frac{1}{X_t} - \frac{1}{X_c} \tag{96}$$

The constant C must be determined from biaxial tests. When introducing a shear stress component σ_{12}, simply add a σ_{12}^2 term. No σ_{12} term is required since the shear strength S is the same for positive and negative shear stresses. Then, the criterion becomes

$$\frac{\sigma_1^2}{X_t X_c} + \frac{\sigma_2^2}{Y_t Y_c} + C\sigma_1\sigma_2 + \left(\frac{1}{X_t} - \frac{1}{X_c}\right)\sigma_1 + \left(\frac{1}{Y_t} - \frac{1}{Y_c}\right)\sigma_2 + \frac{\sigma_{12}^2}{S^2} = 1 \tag{97}$$

Extending this approach to three dimensions, we can add σ_3^2, $\sigma_1\sigma_3$, $\sigma_2\sigma_3$ and σ_3 terms to account for the third normal stress components and its interaction with the first two. σ_{13}^2 and σ_{23}^2 must also be added to account for the additional shear stress components. This leads to the three- dimensional Tsai-Wu criterion discussed below.

4.9- Three Dimensional Tsai-Wu Failure Criterion

The approach discussed in the previous section can be generalized to three dimensional loading of a unidirectional composite. In that case, we have six stress components: three normal stresses σ_{11}, σ_{22}, and σ_{33}, and three shear stresses σ_{23}, σ_{13}, and σ_{12}. For the normal stresses, the criterion should include three linear terms $(\sigma_{11}, \sigma_{22}, \text{and } \sigma_{33})$, three square terms $(\sigma_{11}^2, \sigma_{22}^2, \text{and } \sigma_{33}^2)$, and three interaction terms $(\sigma_{11}\sigma_{22}, \sigma_{11}\sigma_{33}, \text{and } \sigma_{22}\sigma_{33})$. For the shear stresses, only the square terms are needed. Then, we obtain what is called the Tsai-Wu failure criterion

$$\begin{aligned} &F_1\sigma_{11} + F_2\sigma_{22} + F_3\sigma_{33} + F_{11}\sigma_{11}^2 + F_{22}\sigma_{22}^2 + F_{33}\sigma_{33}^2 + F_{44}\sigma_{23}^2 + F_{55}\sigma_{13}^2 \\ &+ F_{66}\sigma_{12}^2 + 2F_{12}\sigma_{11}\sigma_{22} + 2F_{13}\sigma_{11}\sigma_{33} + 2F_{23}\sigma_{22}\sigma_{33} = 1 \end{aligned} \tag{98}$$

that contains 12 coefficients. Nine of the coefficients can be determined from unidirectional tests

$$F_1 = \frac{1}{X_t} - \frac{1}{X_c}, \qquad F_2 = \frac{1}{Y_t} - \frac{1}{Y_c}, \qquad F_3 = \frac{1}{Z_t} - \frac{1}{Z_c},$$

$$F_{11} = \frac{1}{X_t X_c}, \qquad F_{22} = \frac{1}{Y_t Y_c}, \qquad F_{33} = \frac{1}{Z_t Z_c}, \tag{99}$$

$$F_{44} = \frac{1}{S_{23}^2}, \quad F_{55} = \frac{1}{S_{13}^2}, \quad F_{44} = \frac{1}{S_{12}^2}$$

The three coefficients F_{12}, F_{13}, and F_{23} can have a significant effect on the shape of the failure surface and they are best determined through biaxial tests that are difficult to perform. When this information is not available, one can use the fact the three terms $\frac{1}{2}F_{11}\,\sigma_{11}^2 + \frac{1}{2}F_{22}\,\sigma_{22}^2 + 2F_{12}\,\sigma_{11}\sigma_{22}$ form the perfect square $\frac{1}{2}\left(\sqrt{F_{11}}\,\sigma_{11} - \sqrt{F_{22}}\,\sigma_{22}\right)^2$ if

$$F_{12} = -\frac{1}{2}\sqrt{F_{11}F_{22}} \tag{100}$$

In addition, if we also take

$$F_{13} = -\frac{1}{2}\sqrt{F_{11}F_{33}} \quad \text{and} \quad F_{23} = -\frac{1}{2}\sqrt{F_{33}F_{22}} \tag{101,102}$$

then, the criterion can be written as

$$\frac{1}{2}\left(\sqrt{F_{11}}\,\sigma_{11} - \sqrt{F_{22}}\,\sigma_{22}\right)^2 + \frac{1}{2}\left(\sqrt{F_{33}}\,\sigma_{33} - \sqrt{F_{22}}\,\sigma_{22}\right)^2 + \frac{1}{2}\left(\sqrt{F_{11}}\,\sigma_{11} - \sqrt{F_{33}}\,\sigma_{33}\right)^2 + F_1\sigma_{11} + F_2\sigma_{22} + F_3\sigma_{33} + F_{44}\,\sigma_{23}^2 + F_{55}\,\sigma_{13}^2 + F_{66}\,\sigma_{12}^2 = 1 \tag{103}$$

a form similar to the criterion proposed by Caddell et al (1973) (Equation 89). Combining Equations (99-103) gives expressions that are used by many authors

$$F_{12} = -\frac{1}{2\sqrt{X_t X_c Y_t Y_c}}, \quad F_{23} = -\frac{1}{2\sqrt{Y_t Y_c Z_t Z_c}}, \quad F_{13} = -\frac{1}{2\sqrt{X_t X_c Z_t Z_c}} \tag{104}$$

In the case of a transversely isotropic material where the 23 plane is the plane of symmetry, the number of constants are reduced (1971) since F_2=F_3, F_{22}=F_{33},