SVEN BODO WIRSING

SEPARABILITY WITHIN COMMUTATIVE AND SOLVABLE ASSOCIATIVE ALGEBRAS

Under consideration of non-unitary algebras. With 401 exercises

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disseminate knowledge

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Introduction

The truth is rarely pure and never simple. (Oscar Wilde)

Within the theory of associative algebras the nilradical and its factor algebra play an important role. The nilradical leads to the analysis of nilpotent and its factor algebra to the study of semisimple associative algebras. If the factor algebra by the nilradical of a finite-dimensional associative unitary algebra is separable, then the theorem of Wedderburn-Malcev ensures the existence of a subalgebra which is complementary to the nilradical. In other words, it is possible to lift the factor algebra by the nilradical into the algebra as a subalgebra. Furthermore, all such complements are conjugated under the action of the nilradical. An introduction to this topic is presented in chapter 1 of this work. In addition, we present the theorem of Taft for G-invariant radical complements (where G is a finite group acting on the algebra by auto- or anti-automorphism), include some examples of separable algebras, present the connection between separable algebras, derivations and factor sets, calculate the derivations of upper triangular matrices and present some examples and counterexamples within the context of the theorem of Wedderburn-Malcev

Within the standard literature the theorem of Wedderburn-Malcev is proven for unital algebras. In some papers and books it is stated afterwards that every algebra can be embedded into an unital algebra, and thus the theorem is valid also for non-unital algebras (see e.g. [8], first paragraph of page 3). This idea will be analyzed in details in this work within chapter 2.
We study the so-called adjunction of an unit and the embedding of an algebra into this adjunction. By determining its nilradical and the factor algebra by its nilradical we are able to transfer the existence part of the theorem of Wedderburn-Malcev to non-unitary algebras. For non-unitary algebras the question arise in what way two complements are conjugated. For this, we introduce the well-known star group (also called quasi regular or circle group). By analyzing the star group and the connection to the adjunction of an unit we prove that all complements are conjugated under the action of the star group.
We analyze this action further by determining its stabilizer: it is the central-

izer under the nilradical of a radical complement. This result is applied to the algebras of upper and lower triangular matrices over a field to determine the cardinality of the set of all radical complements.
The theorem of Taft for G-invariant radical complements is transferred to non-unitary algebras, too.
We proceed the chapter by analyzing compatibilities related to the theorem of Wedderburn-Malcev. The basic idea is to calculate radical complements of related structures (like subalgebras, ideals, factor algebras) based on radical complements of the entire algebra. For subalgebras, left and right ideals no meaningful compatibilities are provable in general (For the center one compatibility is proven within chapter 5.). For this, examples are presented. But we prove compatibilities for ideals and factor algebras by intersecting and factorizing radical complements in a natural way.
Chapter 2 is finalized by presenting algorithms for the determination of a radical complement. As a consequence we can calculate the decomposition for every element based on the nilradical and a radical complement. This decomposition can be used to calculate a decomposition based on every other radical complement by applying a transfer rule. The calculation of a decomposition for an element based on the decomposition of the entire algebra is called top-down calculation.

Two main topics are the guidelines of this work: the calculation of a radical complement and the presentation of an element as sum of a radical element and an element of a complement. In addition, the idea of compatibility is regarded as the third main topic within this work. For proving meaningful results related to this guidelines we will specialize the algebras to be analyzed.

Within chapter 3 we focus on so-called solvable associative algebras which are not or only little present in the basic literature. In the work [3] some deep insights in the theory of solvable associative algebras are proven. The analysis is motivated by Solomon's algebra and its connection to the representation theory of the symmetric groups. Solvable algebras are generalizing commutative algebras in a natural way. Thus, their analysis is also a basis for chapter 5 in this work.
Within the first section we prove that finite-dimensional associative solvable algebras are those algebras possessing a commutative factor algebra by its nilradical. Section two is dedicated to the result that solvable associative algebras (in uneven characteristic) are closely connected to the solvability of their associated Lie algebra. We use Cartan's criteria to characterize solvable associative algebras by a symmetric bilinear form. The connection to its associated Lie algebra motivated the question how the classes of solvabilities are connected. This topic is only analyzed for the algebras of upper triangular matrices over a field. We prove that both classes are identical

and also equal to class of solvability of its group of units (for uneven characteristic).
By the idea of compatability we calculate radical complements for subalgebras of solvable algebras and prove that all semisimple subalgebras are separable (if the radical factor algebra of the underlying algebra is separable).
We finalize the chapter by presenting an example summarizing the results proven so far and by analyzing the importance of the lower and upper triangular matrices for all solvable associative algebras.

Chapter 4 is dedicated to algebras related to generalized quaternion algebras. We derive some examples for commutative algebras which are used within chapter 5. In addition, an algebra is presented possessing two non-conjugated radical complements. In the literature only very few examples related to this topic are existing. Finally, we classify the derived algebras up to isomorphism.

Our main questions are answered in details for commutative algebras within chapter 5 of this work. Commutative algebras possess exactly one radical complement. Standard examples of commutative algebras are centers of associative algebras. By using the idea of compatibility we describe the radical complement of the center based on the entire algebra: the intersection of every radical complement of the entire algebra with the center is exactly the radical complement of the center. As a consequence, we prove for solvable algebras that the intersection of all radical complements is exactly the radical complement of the center.
Afterwards, we turn our focus to the inner structure of commutative algebras. The unique radical complement is identified by the set of elements possessing a minimal polynomial which is squarefree and separable. These elements are called fully separable within this work. The decomposition topic is also solved by generalizing the Jordan decomposition. The latter decomposition is to be calculated by solving congruences within the algebra of polynomials. In the generalized version - applying not only to splitting endomorphism but also to separable ones - special divisions are to be done additionally. The set of diagonalizable and splitting elements are the connection to the well-known Jordan decompositions. Both sets are analyzed, and it is proven that they are subalgebras, too. The connection to the nilradical and the radical complement is presented for them.
The results are illustrated by using commutative group algebras and the algebras analyzed within chapter 4. In addition, we transfer all results to non-unitary associative algebras.
Finally, we answer our main topics for solvable associative algebras within chapter 5. The radical complements can be determined by using the set of fully separable elements. The generalized Jordan decomposition is used to

answer the decomposition question for the elements. We are able to generalize this calculation to solvable and to basic algebras. Thus, we are able to calculate a decomposition for an element without knowing a decomposition of the entire algebra. This approach is called bottom-up calculation. We can use the bottom-up calculation for describing a radical complement of the underlying algebra, too.

The appendix starts by analyzing a theorem within the work [3] of T. Bauer. The proof that solvable associative algebras can be characterized by algebras possessing solvable group of units is analyzed in details. The result is transferred to non-unitary algebras using again the star group and the adjunction of an unit. Thus, solvable associative algebras are characterized by possessing solvable groups of units and solvable associated Lie algebras. Afterwards we present proofs for the theorem of Wedderburn-Malcev and for Taft's theorem about G-invariant radical complements for unitary algebras using cohomology of algebras, groups and direct calculations.

Some applications are also transferred to the exercises at the end of each chapter. Some exercises are included enhancing the theory presented so far. In addition, at the beginning of each exercise series some open-ended topics are included which can be used by the reader – and also by the author – to do additional researches within this theory. The author has included some graphics – mostly so called Hasse diagrams – to visualize the main results of this work.

Notation

Numbers and sets

$\mathbb{P}$	the set of prime numbers
$\mathbb{N}$	the set of natural numbers without 0
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{H}$	the set of real quaternions
$\underset{\sim}{n}$	$\{a \mid a \in \mathbb{N}, a \leq n\}$
$[x]$	maximal integer less than or equal to x (Gauss bracket)
$A \times B$	the set of all pairs $(a;b)$, $a \in A$ and $b \in B$
M^n	the set of all n-tuples over M

Fields and polynomial rings

$(K;L)$	field extension with extension field L and basic field K
$K[t_1,...,t_n]$	polynomial algebra in commutating variables $t_1,...,t_n$ over K
$K(t_1,...,t_n)$	field of fractions of $K[t_1,...,t_n]$
$grad(f)$	degree of the polynomial $f \in K[t]$
$char(K)$	characteristic of the field K
(f)	notation for the K-ideal $fK[t]$ of $K[t]$
$K[a]$	smallest unitary K-subalgebra of a K-algebra containing a
$GF(p), p \in \mathbb{P}$	notation for the field $\mathbb{Z}/p\mathbb{Z}$
$halb(f)$	product of the pairwise irreducible divisors of the polynomial f
$max(f)$	greatest multiplicity of the irreducible divisors of the decomposition into irreducible polynomials of the polynomial f

Groups and magmas

G/U	the set of right cosets of a subgroup U of a group G
$G \times H$	the direct product of the groups G and H
$Stab_G(m)$	the stabilizer of an element m of a G-set
mG	the orbit of an element m of a G-set

$st(G)$	the solvable class of a solvable group G
$[g, h]$	the commutator of elements g, h of a group
$G^{'}$	the commutator subgroup of a group G
$G^{(n)}$	the n-th derived subgroup of a group G
S_n	the symmetric group over $\underline{n}$
A_n	the alternating group over $\underline{n}$
D_{2n}	the dihedral group of order $2n$
Q_{4n}	the quaternion group of order $4n$
$O_p(G), p \in \mathbb{P}$	the intersection of all p-Sylow subgroups of a group G
$Aut(M)$	the set of all automorphism of a magma M

Spaces and matrices

$\langle T \rangle_K$	the K-linear span of a set T of vectors
Kv	the set $\langle v \rangle_K$
$End_K(V)$	the set of all K-endomorphism of a K-space V
$f(k)$	the inserting of k into the polynomial f
$dim_K(V)$	the K-dimension of a K-space V
$U \oplus_K W$	the inner direct sum of the K-subspaces U and W of a K-space
$V \otimes_K W$	the tensor product of the K-spaces V and W
$v \otimes w$	tensors of a tensor product
$ker(\alpha)$	kernel of the endomorphism α
$Im(\alpha)$	image of the endomorphism α
$\alpha \otimes \beta$	the tensor product of the K-linear functions α and β
tr	the trace function
$M_B(\alpha)$	the representing matrix of a K-linear function α based on a basis B
A_{ij}	the $(i;j)$-value of a matrix A
a_{ij}	the $(i;j)$-value of the matrix $A = (a_{ij})$
$K^{n \times m}$	$n \times m$-matrix space over K
$GL(n, K)$	the group of units of $K^{n \times n}$
$rad(f)$	the radical of a symmetric bilinear form f
$QA(K)$	the set of squares of a field K
$Pot(n, K)$	the set of n-th powers of a field K
τ	the transpose function on $K^{n \times n}$
$Aut(A)$	group of algebra automorphism of A
$Ant(A)$	set of anti-automorphism of A
$Der(A, M)$	set of derivations of a (A, A)-bimodule M
$Z^1(A, M)$	set of 1-cocycles of a (A, A)-bimodule M
$Inder(A, M)$	set of inner derivations of a (A, A)-bimodule M
$B^1(A, M)$	set of 1-coboundaries of a (A, A)-bimodule M
$H^1(A, M)$	first Hochschild cohomology group of a (A, A)-bimodule M
$Z^2(A, M)$	set of 2-cocycles of a (A, A)-bimodule M
$B^2(A, M)$	set of 2-coboundaries of a (A, A)-bimodule M
$H^2(A, M)$	second Hochschild cohomology group of a (A, A)-bimodule M

Algebras

$(K,A), A^K$	the adjunction of an unit
φ	the embedding-function of A into (K,A)
$\bigoplus_{i=1}^{r} A_i$	the direct sum of K-algebras A_i
A^-, A^{op}	the opposite or inverse algebra of A
$\cdot_{op}$	$a \cdot_{op} b := ba$
A_L	scalar extension of A, $A \otimes_K L$
$Aut_K(A)$	the set of all K-algebra automorphism of a K-algebra A
$Z(A)$	the center of a K-algebra A
$C_A(T)$	the centralizer of the set T of an algebra A
$N_A(T)$	the normalizer of the set T of an algebra A
$\langle T \rangle_{\trianglelefteq \mathcal{A}}$	the ideal-span of the subset T within an algebra
α_a	the shift of a within an algebra
$N(A)$	the set of all divisors of zero within an algebra A
$J(A)$	the Jacobson radical of an algebra A

Associative algebras

D_n	the Solomon algebra
$\Delta u, n$	the set of lower triangular matrices of $K^{n\times n}$
$\Delta o, n$	the set of upper triangular matrices of $K^{n\times n}$
$s\Delta u, n$	the set of strict lower triangular matrices of $K^{n\times n}$
$s\Delta o, n$	the set of strict upper triangular matrices of $K^{n\times n}$
$D(n,K)$	the set of diagonal matrices of $K^{n\times n}$
$E(A)$	the group of units of an associative unitary algebra A
κ_e	the conjugation by an unit e within an associative algebra
a^e resp. T^e	$a\kappa_e$ resp. $T\kappa_e$
$*$	the star or circle or quasi regular composition
$Q(A)$	the quasi regular or star or circle group of an associative algebra A
$A^\star$	another notation for $Q(A)$
$e^{(-1)}$, e'	the inverse of a quasi regular element e
$\kappa_{(e)}$	the conjugation with a quasi regular element e
$a^{(e)}$ resp. $T^{(e)}$	$a\kappa_{(e)}$ resp. $T\kappa_{(e)}$
$rad(A)$	the nilradical of an associative algebra A
$Nil(A)$	the set of nilpotent elements of an associative algebra A
KG	the group algebra based on a group G and a field K
$Aug(KG)$	the augmentation ideal of KG
$cl(A)$	the class of nilpotency of an associative algebra A
$cl(a)$	the class of nilpotency of an element a of an associative algebra
ρ	the right regular representation of an associative algebra
λ	the left regular representation of an associative algebra

$\mathcal{R}_1$	the class of unitary rings
$\mathcal{A}$	the class of associative algebras
$\mathcal{A}_1$	the class of associative unitary algebras
$\mathcal{A}$-isomorphic, $\cong_{\mathcal{A}}$	isomorphism within the class $\mathcal{A}$
$\mathcal{A}_1$-isomorphic, $\cong_{\mathcal{A}_1}$	isomorphism within the class $\mathcal{A}_1$
$\langle T\rangle_{\mathcal{A}_1}$	the algebra span of T within $\mathcal{A}_1$
$\langle T\rangle_{\mathcal{A}}$	the algebra span of T within $\mathcal{A}$
$A^{<n>}$	the n-th power of an associative algebra A
$<,>_\rho$	the standard trace form of ρ
$<,>_\lambda$	the standard trace form of λ
$< a,b >_{\lambda,\rho}$	$= tr(a\lambda\, b\rho + a\rho\, b\lambda)$
$A(a,b,K), A(a,b)$	the generalized quaternion algebra
$S(A_i,n)$	see image 8
Z_A	see image 8
$A^{n\times m}$	the set of $n\times m$-matrices over A

Lie algebras

A°	the associated Lie algebra
$\circ$	$a\circ b := ab - ba$
$L^{(n)}$	the n-th derived K-subalgebra of a K-Lie algebra L
$cl(L)$	the nilpotency class of a nilpotent K-Lie algebra L
$S\circ T$	the K-linear span of the set $\{s\circ t \mid s\in S, t\in T\}$
ad	the adjoint representation
$st(L)$	the solvable class of a solvable K-Lie algebra L

Solvable algebras

$AUF(A)$	the solvable radical of an associative K-algebra A
$auf(A)$	the solvable residuum of an associative K-algebra A
A'	the derived K-subalgebra of a K-algebra A
$A^{(n)}$	the n-th derived K-subalgebra K-algebra A
$st(A)$	the solvable class of a solvable K-algebra A

Commutative algebras

$H(A)$	the set of semisimple elements of an algebra A
$D(A)$	the set of diagonalizable elements of an algebra A
$Sep(A)$	the set of separable elements of an algebra A
$VSep(A)$	the set of fully separable elements of an algebra A
$ZF(A)$	the set of splitting elements of an algebra A
$min_{a,K}, \widetilde{min}_{a,K}$	the minimum polynomial of a over a field K
F_a	isomorphism between $K[a]$ and $K[t]/(min_{a,K})$

$\widetilde{F}_a$	the inserting of a into $tK[t]$
χ	the isomorphisms within the Chinese Remainder theorem
$char_{a,K}$	the characteristical polynomial of a over a field K
aug	the augmentation function of KG onto K
χ_i	an irreducible character of a group
e_i	an idempotent related to χ_i
ω_d	a primitive d-th root of unity
ϕ	the Phi-function
$Gal(L;K)$	the Galois group of a field extension $(K;L)$
$h(G)$	the class number of a group G
$K(\omega_d)$	the adjunction of a primitive d-th root of unity to the field K
$Irr_K(G)$	the set of all irreducible characters of a group G.

Chapter 1

Separable algebras and the theorem of Wedderburn-Malcev

1.1 Separable algebras

1.1.1 Characterizations, properties and examples

Within this section we provide a short introduction to separable algebras. The reader may also read and study the corresponding chapters within the text books of Richard Pierce [35] and Yurij Drozd [8].

Definitions 1 For all $n \in \mathbb{N}$ we define $\underline{n} := \mathbb{N}_{\leq n}$. The symbols $\mathcal{R}_1$, $\mathcal{A}$ resp. $\mathcal{A}_1$ are used for the classes of unitary rings, associative algebras resp. associative unitary algebras. If $\mathcal{K}$ is one the classes $\mathcal{R}_1$, $\mathcal{A}$ or $\mathcal{A}_1$, then $\langle ... \rangle_{\mathcal{K}}$ resp. $\cong_{\mathcal{K}}$ denotes the span resp. the isomorphism within the class $\mathcal{K}$. In addition, we use the word $\mathcal{K}$-isomorphism or say that two objects of the class $\mathcal{K}$ are $\mathcal{K}$-isomorphic. By A^- resp. A^{op} we denote the opposite algebra of an algebra A with respect to the multiplication $a \cdot_{op} b := ba$ for all $a, b \in A$. The center of an algebra A is symbolized by $Z(A)$.$\diamond$

Within this work we use definitions used within module theory like module, algebra module, (A, B)-bimodule, semisimple module, projective module, irreducible module etc. An (A, A)-bimodule is also called A-bimodule within this work. The reader may also read and study the relevant chapters within the text books of Richard Pierce [35] and Yurij Drozd [8]. From algebra and linear algebra theory we use terms K-algebra, K-subalgebra, K-ideal, K-right ideal, K-left ideal, K-space, K-subspace etc. If the connection to the field K is unambiguous, then we omit it and use the terms algebra, subalgebra etc.

Definition 1 *(separable algebra)* An associative unitary K-algebra A is called separable if and only if A is projective as $A^- \otimes_K A$-algebra module (see [35], section 10.2, definition). The next characterization shows us how to detect this property inside the algebra itself. In addition, separable algebras are closely connected to separable field extension.$\diamond$

Theorem 1 *(characterizations of separable algebras) Let K be a field and A an associative unitary K-algebra. The following statements are equivalent:*

(i) *A is separable.*

(ii) *$A \otimes A^-$ possesses a so-called separating idempotent: an element $t \in A \otimes A^-$ exists such that $\mu(t) = 1_A$ and $at = ta$ for every $a \in A$ are valid. Here μ is related to the multiplication of A (μ is defined by $\mu(a \otimes b) := ab$) and the expression $at = ta$ is noted within the A-bimodule structure of $A \otimes A^-$.*

(iii) *$A^- \otimes_K A$ is semisimple and finite-dimensional.*

(iv) *For every field extension $(K; L)$ the L-algebra $A_L := A \otimes_K L$ (basic field or scalar extension) is semisimple.*

(v) *A natural number $r \in \mathbb{N}$ and associative finite-dimensional unitary simple K-algebras $A_1, ..., A_r$ exist such that $A \cong_{A_1} \bigoplus_{i=1}^{r} A_i$ is valid and for every $i \in \underline{r}$ the pair $(K1_{A_i}; Z(A_i))$ is a separable field extension.*

<u>Proof</u>. see theorem 6.1.2 in [8] and the proposition on page 182 in [35].$\diamond$

Based on this theorem we can deduce some properties and also present some examples of separable algebras.

Corollary 1 *(properties and examples of separable algebras) Let K be a field and A an associative unitary K-algebra.*

(i) *If A is separable, then A is semisimple and finite-dimensional.*

(ii) *If A is separable, then $Z(A)$ is separable.*

(iii) *Direct products of separable algebras are separable.*

(iv) *For every $n \in \mathbb{N}$ the K-algebra K^n is separable.*

(v) *Direct products of full matrix algebras over K are separable.*

(vi) *Let K be algebraical closed. A is separable if and only if A is finite-dimensional and semisimple.*

(vii) Let K be perfect. A is separable if and only if A is finite-dimensional and semisimple.

(viii) Let $(K; L)$ be a finite-dimensional field extension. The following statements are valid:

(a) $(K; L)$ is a separable field extension.

(b) L is separable as K-algebra.

(ix) A direct product of separable field extensions of K is a separable K-algebra.

Proof. The proof is a direct consequence of part (iv) of theorem 1.$\diamond$

Examples 1 (i) $\mathbb{C}$ is – based on part (ii) of corollary 1 – a separable $\mathbb{R}$-algebra.

(ii) $\mathbb{R}$ is not finite-dimensional as $\mathbb{Q}$-algebra, and hence – based on part (i) of corollary 1 – not separable as $\mathbb{Q}$-algebra.

(iii) Let K be a field and A an associative n-dimensional unitary central-simple K-algebra. A is – based on part (iv) of corollary 1 – separable. In particular, the quaternion algebra $\mathbb{H}$ is separable as $\mathbb{R}$-algebra and for all $n \in \mathbb{N}$ the K-algebra $K^{n\times n}$ is separable.$\diamond$

1.1.2 Group algebras and separability

Within this section we analyze on what terms the group algebra is separable. Let K be a field and G a finite group. By $char(K)$ we denote the characteristic of the field K. The group algebra is symbolized by KG.

Remark 1 *Let K be a field and G a finite group. The following statements are valid:*

(i) $KG \cong_{\mathcal{A}_1} (KG)^-$

(ii) For every finite group H the statement $K(G \times H) \cong_{\mathcal{A}_1} KG \otimes_K KH$ is valid.

Proof. The K-linear extension of the function

$$G \longrightarrow KG, g \longmapsto g^{-1}$$

is a $\mathcal{A}_1$-isomorphism between KG and $(KG)^-$. In addition, the K-algebras $K(G \times H)$ and $KG \otimes_K KH$ are $\mathcal{A}_1$-isomorphic based on the K-linear extension of the map

$$G \times H \longrightarrow KG \otimes_K KH, (g; h) \longmapsto g \otimes h.$$

Theorem 2 *(separability of group algebras) Let K be a field and G a finite group. The following statements are equivalent:*

(i) KG is separable.

(ii) KG is semisimple.

(iii) $char(K)$ is not a divisor of the order of G.

Proof. The equivalence of (ii) and (iii) is the content of the theorem of Maschke. The implication (i) to (ii) can be proven based on part (i) of corollary 1. We need to prove the implication (ii) to (i). By using remark 1 we deduce $KG \otimes_K (KG)^- \cong_{\mathcal{A}_1} K(G \times G)$. $char(K)$ is zero or a prime number. Thus, based on the semisimplicity and the theorem of Maschke $K(G \times G)$ is semisimple, too. By using part (ii) of corollary 1 we finish the proof.$\diamond$

1.1.3 Matrix algebras of separable algebras

Within this section we prove that matrix algebras of separable algebras are separable, too.

Remark 2 *(isomorphism of matrix algebras) Let K be a field, $n, m \in \mathbb{N}$ and A an associative unitary finite-dimensional K-algebra. The following statements are valid:*

(i) $(A^{n\times n})^-$ and $(A^-)^{n\times n}$ are isomorphic.

(ii) $A^{n\times n}$ is isomorphic to $K^{n\times n} \otimes_K A$.

(iii) $K^{n\times n} \otimes_K K^{m\times m}$ and $K^{(nm)\times(nm)}$ are isomorphic.

Proof. The reader may execute the proof within the exercises.$\diamond$

Within the Morita-theory of associative algebras the nilradical is determined for matrix algebras:

Remark 3 *(nilradical of matrix algebras) Let $n \in \mathbb{N}$ and A an associative right artian K-algebra. The nilradical of the matrix algebra $A^{n\times n}$ is exactly $rad(A)^{n\times n}$ (which is the matrix algebra of the nilradical). In particular, A is semisimple if and only if $A^{n\times n}$ is semisimple:*

$$rad(A^{n\times n}) = rad(A)^{n\times n}.$$

Proof. The reader may execute the proof as an exercise.$\diamond$

Theorem 3 *(separability of matrix algebras) Let K be a field, $n \in \mathbb{N}$ and A an associative separable K-algebra. The matrix algebra $A^{n\times n}$ is separable.*

Proof. Based on theorem 1 we have to prove that $(A^{n\times n}) \otimes_K (A^{n\times n})^-$ is semisimple. Based on this statement and remark 2 as well as the commutativity and associativity of the tensor product we deduce:

$$\begin{aligned}(A^{n\times n}) \otimes_K (A^{n\times n})^- &\cong_{\mathcal{A}} \\ K^{n\times n} \otimes_K A \otimes_K K^{n\times n} \otimes_K A^- &\cong_{\mathcal{A}} \\ K^{n^2\times n^2} \otimes_K (A \otimes_K A^-) &\cong_{\mathcal{A}} \\ (A \otimes_K A^-)^{n^2\times n^2}.\end{aligned}$$

A is separable, and thus $A \otimes A^-$ is semisimple based on theorem 1. By using remark 3 the matrix algebra $(A \otimes A^-)^{n^2\times n^2}$ is semisimple and the proof is finished$\diamond$

1.1.4 Separable algebras, derivations and factor sets

1.1.4.1 Derivations

We begin this section by defining derivations within the context of bimodules.

Definition and remark 1 *(derivation, inner derivation, first Hochschild cohomology group)* Let A be an associative K-algebra and M an A-bimodule. A K-linear map $d : A \longrightarrow M$ is referred to as a derivation or 1-cocycle from A into M if

$$d(ab) = a.d(b) + d(a).b$$

is valid for all $a, b \in A$. By $Der(A, M) = Z^1(A, M)$ we denote the set of all derivations from A in to M. Given $m \in M$, the linear map

$$ad(m) : A \longrightarrow M, a \mapsto a.m - m.a$$

is the inner derivation or 1-coboundary effected by m. By $Inder(A, M) = B^1(A, M)$ we denote the set of all inner derivations from A into M. If we consider A as A-bimodule the set $Der(A) := Der(A, A)$ resp. $Inder(A) := Der(A, A)$ is the collection of all derivations resp. inner derivations of A. Using the bar resolution one sees that the first Hochschild[1] cohomology group $H^1(A, M)$ is the factor space of derivations by inner derivations from

[1]Gerhard Paul Hochschild (born April 29, 1915 in Berlin, died July 8, 2010 in El Cerrito, California) was a German-born American mathematician who worked on Lie groups, algebraic groups, homological algebra and algebraic number theory. Hochschild wrote his thesis in 1941 at Princeton University with Claude Chevalley on Semisimple Algebras and Generalized Derivations. From 1956 up to 1957 he was at the Institute for

A into M. For more details see e.g. [35], [9], [10], [14], [15] and [16]. By $Aut(A)$ or $Aut_K(A)$ resp. $Ant(A)$ or $Ant_K(A)$ we denote the set of all algebra automorphism resp. anti-automorphism of A.$\diamond$

Examples 2 *(examples of derivations) Let A be an associative K-algebra, M an A-bimodule, $d \in Der(A)$ and $m \in M$.*

(i) d_m is a derivation.

(ii) $\mathbb{R}[t]$ possesses no inner derivation different from zero. The formal derivation of polynomials is a derivation.

(iii) Let K be a field and $A := K[t]/(t^2)$. If $char(K) \neq 2$, then $Der(A)$ is of dimension 1 and $Inder(A)$ is of dimension 0.

(iv) Let $\mu : A \otimes_K A \to A$ be the multiplication morphism of the K-algebra A. Consider the A-A-bimodule $A \otimes_K A$. $\ker\mu$ is a sub-bimodule of $A \otimes_K A$ (since μ is an A-A-bimodule map). Consider the map

$$\delta : A \to \ker\mu, \;\; a \mapsto a \otimes 1 - 1 \otimes a$$

and prove that δ is a derivation.

(v) $Der(A, M)$ is a K-space.

(vi) The kernel of a derivation is a (unital) subalgebra of A.

(vii) Let $d \in Der(A, M)$ and $g \in Aut(A)$. The map $d^g := g^{-1}dg$ is a derivation of A into M.

(viii) Let $d \in Der(A, M)$ and $g \in Ant(A)$. The map $d^g := g^{-1}dg$ is a derivation of A into M.

Proof. ad(i): Let $a, b \in A$. We calculate

$$d_m(ab) = (ab)m - m(ab)$$

and

$$\begin{aligned} d_m(a)b + ad_m(b) &= \\ (am - ma)b + a(bm - mb) &= \\ (am)b - (ma)b + a(bm) - a(mb) &= \\ d_m(ab). \end{aligned}$$

Advanced Study. He was professor at the University of Illinois at Urbana-Champaign and from the end of the 1950s at the University of California, Berkeley. Hochschild introduced Hochschild cohomology, a cohomology theory for algebras, which classifies deformations of algebras. Hochschild and Nakayama introduced cohomology into class field theory. Along with Bertram Kostant and Alex F. T. W. Rosenberg, the Hochschild-Kostant-Rosenberg theorem is named after him. Among his students were Andrzej Bialynicki-Birula and James Ax. In 1955 he was a Guggenheim Fellow. In 1979 he was elected to the National Academy of Sciences, and in 1980 he was awarded the Leroy P. Steele Prize of the AMS.

ad(ii): The algebra is commutative and possesses therefor no inner derivation different from zero. The other statement is valid because of the following well-know rules:

$$\begin{aligned}(fg)^{'} &= f^{'} + g^{'}\\ (kf)^{'} &= kf^{'}\\ (fg)^{'} &= f^{'}g + fg^{'}.\end{aligned}$$

These are the rules for being a derivation.

ad(iii): Let K be a field. Consider the commutative, unitary, associative and 2-dimensional algebra $A := K[t]/(t^2)$. This algebra possesses a basis $\{1, r\}$ such that $r^2 = 0$ is valid. A is commutative and therefore $Inder(A)$ is the zero space. Let d be a derivation of A. We calculate

$$d(1) = d(1 \cdot 1) = 1 \cdot d(1) + d(1) \cdot 1 = 2 \cdot d(1).$$

Hence, $d(1) = 0$ is true. Let $char(K) \neq 2$. Because of $r^2 = 0$ we derive

$$0 = d(r^2) = 2rd(r).$$

Let $k, l \in K$ such that $d(r) = k1 + lr$ is valid. We calculate

$$0 = 2rd(r) = 2kr.$$

Thus, $k = 0$ is valid. The derivation is defined by $d(1) = 0$ and $d(r) = lr$. If we define a linear function by these rules, then we can prove that this linear function is indeed a derivation. Let $x_1, x_2 \in A$ and $r, s, t, u \in K$ such that $x_1 = n1 + sr$ and $x_2 = t1 + ur$ are valid. A straightforward calculation shows

$$d(x_1x_2) = l(nu + st)r = d(x_1)x_2 + x_1d(x_2).$$

Thus, the set of derivations is of dimension 1 possessing no inner derivations different from zero.

ad(iv): Let $a, b \in A$. We calculate

$$(ab)\delta = (ab) \otimes 1 - 1 \otimes (ab).$$

In addition, the following calculation is valid:

$$(a\delta).b + a.(b\delta) = (a \otimes 1 - 1 \otimes a).b + a.(b \otimes 1 - 1 \otimes b).$$

From left, A acts on the left component, and from right A acts on the right components of the tensors within this bimodule structure. Thus, we derive

$$(a\delta).b + a.(b\delta) = (a \otimes b) - (1 \otimes (ab)) + ((ab) \otimes 1) - (a \otimes b) = (ab) \otimes 1 - 1 \otimes (ab).$$

ad(v): Let $d, e \in Der(A, M)$, $a, b \in A$ and $k \in K$. We calculate

$$(ab)(d+e) = (ab)d + (ab)e = a(bd) + (ad)b + a(be) + (ae)b = a(b(d+e)) + (a(d+e))b$$

and

$$(ab)(ke) = k(ab)e = k((ae)b + a(be)) = k(ae)b + ka(be) = a(ke)b + a(b(ke)).$$

ad(vi): The kernel of a K-linear function is a K-space. Let $d \in Der(A, M)$ and $a, b \in ker(d)$. We calculate

$$d(ab) = ad(b) + d(a)b = 0 + 0 = 0.$$

Furthermore, we derive:

$$d(1) = d(1 \cdot 1) = 1 \cdot d(1) + d(1) \cdot 1 = 2 \cdot d(1).$$

Thus, we conclude $d(1) = 0$.

ad(vii): Let $a, b \in A$. We calculate

$$\begin{aligned} (ab)d^g &= \\ (ab)g^{-1}dg &= \\ (ag^{-1})(bg^{-1})dg &= \\ ((ag^{-1})d(bg^{-1}) + (ag^{-1})(bg^{-1})d)g &= \\ (ag^{-1})dg \cdot b + a(ag^{-1})dg. \end{aligned}$$

ad(viii): This statement can be derived from part (vii) by using the opposite algebra A^{op} or by direct calculation as done within part (vii).$\diamond$

Bi-module derivations are closely connected to separable algebras. This is the content of the next theorem. This characterization is used within the appendix for proving the theorems of Wedderburn-Malcev and Taft.

Theorem 4 *(characterization of separable algebras by inner derivations) Let R be a field and A be an unital R-algebra. Then, A is a separable R-algebra if and only if every derivation from A to an A-A-bimodule is inner.*

Proof. (see [63]) $\Longrightarrow$: Assume that A is a separable R-algebra. Then, based on theorem 1, there exists an element $t \in A \otimes_R A$ (where all tensor products are over R) satisfying $\mu(t) = 1$ (where $\mu : A \otimes_R A \to A$ is the multiplication morphism of the R-algebra A) and $at = ta$ for every $a \in A$ (where we are using the standard A-A-bimodule structure on $A \otimes_R A$). Now, let M be an A-A-bimodule, and $d : A \to M$ be a derivation. Since $t \in A \otimes_R A$