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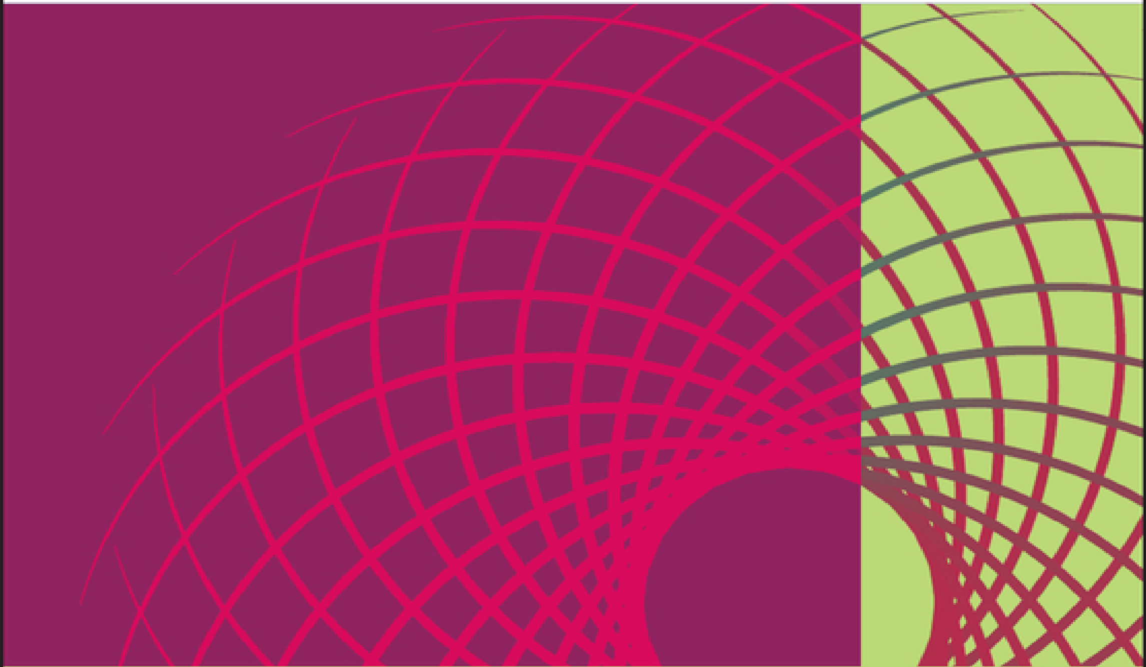
Volume 2

Fibonacci and Lucas Numbers with Applications

Thomas Koshy



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FIBONACCI AND LUCAS NUMBERS WITH APPLICATIONS

Volume Two

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Dedicated to
the loving memory of
Dr. Kolathu Mathew Alexander
(1930–2017)

CONTENTS

LIST OF SYMBOLS

Symbol	Meaning
$\boxed{\Leftarrow}$ or $\boxed{\Rightarrow}$	marginal symbol for alerting the change in notation
$\boxed{?}$	unsolved problem
■	end of a proof or solution; end of a lemma, theorem, or corollary when it does not end in a proof
$\mathbb{C}$	set of complex numbers
$(a_1, a_2, \dots, a_n)$	greatest common divisor (gcd) of the positive integers $a_1, a_2, \dots, a_n$
$[a_1, a_2, \dots, a_n]$	least common multiple (lcm) of the positive integers $a_1, a_2, \dots, a_n$
Δ	$\sqrt{x^2 + 4}$
$\alpha(x)$	$\dfrac{x + \Delta}{2}$
$\beta(x)$	$\dfrac{x - \Delta}{2}$
D	$\sqrt{x^2 + 1}$
$\gamma(x)$	$x + D$
$\delta(x)$	$x - D$
$a(x) \bmod b(x)$	remainder when $a(x)$ is divided by $b(x)$
$a(x) \equiv b(x) \pmod{c(x)}$	$a(x)$ is congruent to $b(x)$ modulo $c(x)$

Symbol	Meaning
$[a_0; \overline{a_1, \ldots, a_n}\,]$	infinite simple continued fraction
$w(\text{tile})$	weight of tile
$\mu(x)$	characteristic of the gibonacci family
F_n^*	$F_n F_{n-1} \cdots F_1$, where $F_0^* = 1$
$\begin{bmatrix} n \\ r \end{bmatrix}$	fibonomial coefficient $\dfrac{F_n^*}{F_r^* F_{n-r}^*}$
f_n^*	$f_n f_{n-1} \cdots f_1$, where $f_0^* = 1$
$\left[\begin{bmatrix} n \\ r \end{bmatrix}\right]$	gibonomial coefficient $\dfrac{f_n^*}{f_r^* f_{n-r}^*}$
$\begin{Bmatrix} n \\ r \end{Bmatrix}_q$	q-binomial coefficient $\dfrac{1-q^m}{1-q} \cdot \dfrac{1-q^{m-1}}{1-q^2} \cdots \dfrac{1-q^{m-r+1}}{1-q^r}$
$\Delta(x, y)$	$\sqrt{x^2 + 4y}$
$\boxed{!}$	switching variables

PREFACE

> Man has the faculty of becoming completely absorbed in one subject, no matter how trivial, and no subject is so trivial that it will not assume infinite proportions if one's entire attention is devoted to it.
>
> –Tolstoy, *War and Peace*

THE TWIN SHINING STARS REVISITED

The main focus of Volume One was to showcase the beauty, applications, and ubiquity of Fibonacci and Lucas numbers in many areas of human endeavor. Although these numbers have been investigated for centuries, they continue to charm both creative amateurs and mathematicians alike, and provide exciting new tools for expanding the frontiers of mathematical study. In addition to being great fun, they also stimulate our curiosity and sharpen mathematical skills such as pattern recognition, conjecturing, proof techniques, and problem-solving. The area is still so fertile that growth opportunities appear to be endless.

EXTENDED GIBONACCI FAMILY

The gibonacci numbers in Chapter 7 provide a unified approach to Fibonacci and Lucas numbers. In a similar way, we can extend these twin numeric families to twin polynomial families. For the first time, the present volume extends the gibonacci polynomial family even further. Besides Fibonacci and Lucas polynomials and their numeric counterparts, the extended gibonacci family includes Pell, Pell–Lucas, Jacobsthal, Jacobsthal–Lucas, Chebyshev, and

Vieta polynomials, and their numeric counterparts as subfamilies. This unified approach gives a comprehensive view of a very large family of polynomial functions, and the fascinating relationships among the subfamilies. The present volume provides the largest and most extensive study of this spectacular area of discrete mathematics to date.

Over the years, I have had the privilege of hearing from many Fibonacci enthusiasts around the world. Their interest gave me the strength and courage to embark on this massive task.

AUDIENCE

The present volume, which is a continuation of Volume One, is intended for a wide audience, including professional mathematicians, physicists, engineers, and creative amateurs. It provides numerous delightful opportunities for proposing and solving problems, as well as material for talks, seminars, group discussions, essays, applications, and extending known facts.

This volume is the result of extensive research using over 520 references, which are listed in the bibliography. It should serve as an invaluable resource for Fibonacci enthusiasts in many fields. It is my sincere hope that this volume will aid them in exploring this exciting field, and in advancing the boundaries of our current knowledge with great enthusiasm and satisfaction.

PREREQUISITES

A familiarity with the fundamental properties of Fibonacci and Lucas numbers, as in Volume One, is an indispensable prerequisite. So is a basic knowledge of combinatorics, generating functions, graph theory, linear algebra, number theory, recursion, techniques of solving recurrences, and trigonometry.

ORGANIZATION

The book is divided into 19 chapters of manageable size. Chapters 31 and 32 present an extensive study of Fibonacci and Lucas polynomials, including a continuing discussion of Pell and Pell–Lucas polynomials. They are followed by combinatorial and graph-theoretic models for them in Chapters 33 and 34. Chapters 35–39 offer additional properties of gibonacci polynomials, followed in Chapter 40 by a blend of trigonometry and gibonacci polynomials. Chapters 41 and 42 deal with a short introduction to Chebyshev polynomials and combinatorial models for them. Chapters 44 and 45 are two delightful studies of Jacobsthal and Jacobsthal–Lucas polynomials, and their numeric counterparts. Chapters 43, 46, and 48 contain a short discussion of bivariate gibonacci polynomials and their combinatorial models. Chapter 47 gives a brief discourse on Vieta polynomials, combinatorial models, and the relationships among the gibonacci subfamilies. Chapter 49 presents tribonacci numbers and polynomials; it also highlights their combinatorial and graph-theoretic models.

SALIENT FEATURES

This volume, like Volume One, emphasizes a user-friendly and historical approach; it includes a wealth of applications, examples, and exercises; numerous identities of varying degrees of sophistication; current applications and examples; combinatorial and graph-theoretic models; geometric interpretations; and links among and applications of gibonacci subfamilies.

HISTORICAL PERSPECTIVE

As in Volume One, I have made every attempt to present the material in a historical context, including the name and affiliation of every contributor, and the year of the contribution; indirectly, this puts a human face behind each discovery. I have also included photographs of some mathematicians who have made significant contributions to this ever-growing field.

Again, my apologies to those contributors whose names or affiliations are missing; I would be grateful to hear about any omissions.

EXERCISES AND SOLUTIONS

The book features over 1,230 exercises of varying degrees of difficulty. I encourage students and Fibonacci enthusiasts to have fun with them; they may open new avenues for further exploration. Abbreviated solutions to all odd-numbered exercises are given at the end of the book.

ABBREVIATIONS AND SYMBOLS INDEXES

An updated list of symbols, standard and nonstandard, appears in the front of the book. In addition, I have used a number of abbreviations in the interest of brevity; they are listed at the end of the book.

APPENDIX

The Appendix contains four tables: the first 100 Fibonacci and Lucas numbers; the first 100 Pell and Pell–Lucas numbers; the first 100 Jacobsthal and Jacobsthal–Lucas numbers; and a table of 100 tribonacci numbers. These should be useful for hand computations.

ACKNOWLEDGMENTS

A massive project such as this is not possible without constructive input from a number of sources. I am grateful to all those who played a significant role in enhancing the quality of the manuscript with their thoughts, suggestions, and comments.

My gratitude also goes to George E. Andrews, Marjorie Bicknell-Johnson, Ralph P. Grimaldi, R.S. Melham, and M.N.S. Swamy for sharing their brief biographies and photographs; to Margarite Landry for her superb editorial assistance; to Zhenguang Gao for preparing the tables in the Appendix; and to the staff at John Wiley & Sons, especially Susanne Steitz (former mathematics editor), Kathleen Pagliaro, and Jon Gurstelle for their enthusiasm and confidence in this huge endeavor.

Finally, I would be grateful to hear from readers about any inadvertent errors or typos, and especially delighted to hear from anyone who has discovered new properties or applications.

Thomas Koshy
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Framingham, Massachusetts
August, 2018

If I have been able to see farther, it was only
because I stood on the shoulders of giants.
–Sir Isaac Newton (1643–1727)

31

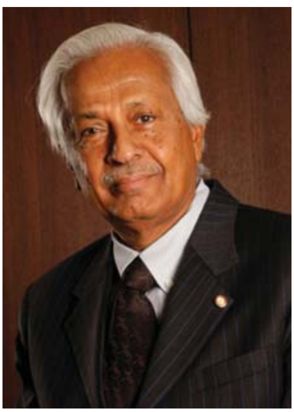
FIBONACCI AND LUCAS POLYNOMIALS I

A man may die,
nations may rise and fall,
but an idea lives on.
–John F. Kennedy (1917–1963)

The celebrated Fibonacci polynomials $f_n(x)$ were originally studied beginning in 1883 by the Belgian mathematician Eugene C. Catalan, and later by the German mathematician Ernst Jacobsthal (1882–1965). They were further investigated by M.N.S. Swamy at the University of Saskatchewan, Canada. The equally famous Lucas polynomials $l_n(x)$ were studied beginning in 1970 by Marjorie Bicknell of Santa Clara, California [37].

Fibonacci and Lucas Numbers with Applications, Volume Two. Thomas Koshy.

Eugène Charles Catalan (1814–1894) was born in Bruges, Belgium, and received his Doctor of Science from the École Polytechnique in Paris. After working briefly at the Department of Bridges and Highways, he became professor of mathematics at Collège de Chalons-sur-Marne, and then at Collège Charlemagne. Catalan went on to teach at Lycée Saint Louis. In 1865, he became professor of analysis at the University of Liège. He published *Éléments de Géométrie* (1843) and *Notions d'astronomie* (1860), as well as many articles on multiple integrals, the theory of surfaces, mathematical analysis, calculus of probability, and geometry. Catalan is well known for extensive research on spherical harmonics, analysis of differential equations, transformation of variables in multiple integrals, continued fractions, series, and infinite products.



M.N.S. Swamy was born in Karnataka, India. He received his B.Sc. (Hons) in Mathematics from Mysore University in 1954; Diploma in Electrical Engineering from the Indian Institute of Science, Bangalore, in 1957; and M.Sc. (1960) and Ph.D. (1963) in Electrical Engineering from the University of Saskatchewan, Canada.

A former Chair of the Department of Electrical Engineering and Dean of Engineering and Computer Science at Concordia University, Canada, Swamy is currently a Research Professor and the Director of the Center for Signal Processing and Communications. He has also taught at the Technical University of Nova Scotia, and the Universities of Calgary and Saskatchewan.

Swamy is a prolific problem-proposer and problem-solver well known to the Fibonacci audience. He has published extensively in number theory, circuits, systems, and signal processing and has written three books. He is the editor-in-chief of *Circuits, Systems, and Signal Processing*, and an associate editor of *The Fibonacci Quarterly*, and a sustaining member of the Fibonacci Association.

Swamy received the Commemorative Medal for the 125th Anniversary of the Confederation of Canada in 1993 in recognition of his significant contributions to Canada. In 2001, he was awarded D.Sc. in Engineering by Ansted University, British Virgin Islands, "in recognition of his exemplary contributions to the research in Electrical and Computer Engineering and to Engineering Education, as well as his dedication to the promotion of Signal Processing and Communications Applications."

Marjorie Bicknell-Johnson was born in Santa Rosa, California. She received her B.S. (1962) and M.A. (1964) in Mathematics from San Jose State University, California, where she wrote her Master's thesis, *The Lambda Number of a Matrix*, under the guidance of V.E. Hoggatt, Jr.

The concept of the lambda number of a matrix first appears in the unpublished notes of Fenton S. Stancliff (1895–1962) of Meadville, Pennsylvania. (He died in Springfield, Ohio in 1962.) His extensive notes are pages of numerical examples without proofs or coherent definitions, that provided material for further study. Bicknell developed the mathematics of the lambda function in her thesis [40].

A charter member of the Fibonacci Association, Bicknell-Johnson has been a member of its Board of Directors since 1967, as well as Secretary (1965–2010) and Treasurer (1981–1999). In 2012, she wrote a history of the first 50 years of the Association [39].

Bicknell-Johnson has been a passionate and enthusiastic contributor to the world of Fibonacci and Lucas numbers, as author or co-author of F_{11} research papers, 32 of them written with Hoggatt. Her 1980 obituary of Hoggatt remains a fine testimonial to their productive association [38].

31.1 FIBONACCI AND LUCAS POLYNOMIALS

As we might expect, they satisfy the same polynomial recurrence $g_n(x) = xg_{n-1}(x) + g_{n-2}(x)$, where $n \geq 2$. When $g_0(x) = 0$ and $g_1(x) = 1$, $g_n(x) = f_n(x)$; and when $g_0(x) = 2$ and $g_1(x) = x$, $g_n(x) = l_n(x)$. Table 31.1 gives the first ten Fibonacci and Lucas polynomials in x. Clearly, $f_n(1) = F_n$ and $l_n(1) = L_n$.

In the interest of brevity and clarity, we drop the argument in the functional notation, when such deletions do *not* cause any confusion. Thus g_n will mean $g_n(x)$, although g_n is technically a functional name and *not* an output value. ⇐

TABLE 31.1. First 10 Fibonacci and Lucas Polynomials

n	$f_n(x)$	$l_n(x)$
1	1	x
2	x	x^2+2
3	x^2+1	x^3+3x
4	x^3+2x	x^4+4x^2+2
5	x^4+3x^2+1	x^5+5x^3+5x
6	x^5+4x^3+3x	$x^6+6x^4+9x^2+2$
7	$x^6+5x^4+6x^2+1$	$x^7+7x^5+14x^3+7x$
8	$x^7+6x^5+10x^3+4x$	$x^8+8x^6+20x^4+16x^2+2$
9	$x^8+7x^6+15x^4+10x^2+1$	$x^9+9x^7+27x^5+30x^3+9x$
10	$x^9+8x^7+21x^5+20x^3+5x$	$x^{10}+10x^8+35x^6+50x^4+25x^2+2$

For the curious-minded, we add that f_n is an even function when n is odd, and an odd function when n is even; and l_n is an odd function when n is odd, and even when n is even.

TABLE 31.2. Triangular Array *A*

n \ k	0	1	2		Row Sums
1	1				1
2	1				1
3	1	1			2
4	1	2			3
5	1	3	1		5
6	1	4	3		8
7	1	5	6	1	13
8	1	6	(10)	4	21
			↑ t_n		↑ F_n

Table 31.1 contains some hidden treasures. To see them, we arrange the nonzero coefficients of the Fibonacci polynomials in a left-justified array A; see Table 31.2. Column 2 of the array consists of the triangular numbers $t_n = n(n+1)/2$, and the nth row sum is F_n.

Let $a_{n,k}$ denote the element in row n and column k of the array. Clearly, $a_{n,k}$ is the coefficient of x^{n-2k-1} in f_n; so $a_{n,k} = \binom{n-k-1}{k}$. Recall that $\sum_{k\geq 0}\binom{n-k-1}{k} = F_n$ [287].

Consequently, it can be defined recursively:

$$a_{1,0} = 1 = a_{2,0}$$
$$a_{n,k} = a_{n-1,k} + a_{n-2,k-1},$$

where $n \geq 3$ and $k \geq 1$; see the arrows in Table 31.2. This can be confirmed; see Exercise 31.1.

Let d_n denote the nth rising diagonal sum. The sequence $\{d_n\}$ shows an interesting pattern: 1, 1, 1, 2, 3, 4, ⑥, 9, 13, …; see Figure 31.1. We can also define d_n recursively:

$$d_1 = d_2 = d_3 = 1$$
$$d_n = d_{n-1} + d_{n-3},$$

where $n \geq 4$.

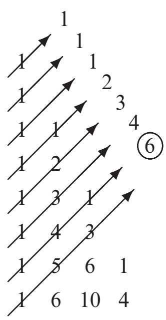


Figure 31.1.

Since $a_{n,k} = \binom{n-k-1}{k}$, it follows that

$$d_n = \sum_{k=0}^{\lfloor (n-1)/3 \rfloor} a_{n-k,k}$$
$$= \sum_{k=0}^{\lfloor (n-1)/3 \rfloor} \binom{n-2k-1}{k}.$$

For example, $d_8 = \sum_{k=0}^{2} \binom{7-2k}{k} = \binom{7}{0} + \binom{5}{1} + \binom{3}{2} = 9.$

The falling diagonal sums also exhibit an interesting pattern: 1, 2, 4, 8, 16, …; see Figure 31.2. This is so, since the nth such sum is given by

$$\sum_{k=0}^{n-1} a_{n+k,k} = \sum_{k=0}^{n-1} \binom{n-1}{k}$$
$$= 2^{n-1},$$

where $n \geq 1$.

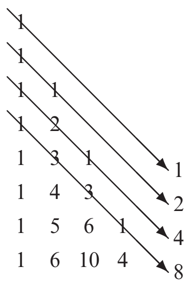


Figure 31.2.

The nonzero elements of Lucas polynomials also manifest interesting properties; see array B in Table 31.3.

TABLE 31.3. Triangular Array *B*

n \ k	0	1	2	3	4	Row Sums
1	1					1
2	1	2				3
3	1	3				4
4	1	4	2			7
5	1	5	5			11
6	1	6	9	2		18
7	1	7	14	7		29
8	1	8	20	16	2	47

↑ L_n

Let $b_{n,k}$ denote the element in row n and column k, where $n \geq 1$ and $k \geq 0$. Then

1) $\sum_{k=0}^{\lfloor n/2 \rfloor} b_{n,k} = L_n$.
2) $b_{n,k} = b_{n-1,k} + b_{n-2,k-1}$, where $b_{1,0} = 1 = b_{2,0}$, $b_{2,1} = 2$, $n \geq 3$, and $k \geq 0$.
3) Let x_n denote the nth rising diagonal sum. Then $x_1 = 1 = x_2$, $x_3 = 3$, and $x_n = x_{n-1} + x_{n-3}$, where $n \geq 4$.
4) $x_n = \sum_{k=0}^{\lfloor (n-1)/3 \rfloor} \frac{n-k}{n-2k}\binom{n-2k}{k}$.

For example, $x_7 = \sum_{k=0}^{2} \frac{7-k}{7-2k}\binom{7-2k}{k} = \frac{7}{7}\binom{7}{0} + \frac{6}{5}\binom{5}{1} + \frac{5}{3}\binom{3}{2} = 12.$

In the interest of brevity, we omit their proofs; see Exercises 31.2–31.5.

Next we construct a graph-theoretic model for Fibonacci polynomials.

Weighted Fibonacci Trees

Recall from Chapter 4 that the nth *Fibonacci tree* T_n is a (rooted) binary tree [287] such that

1) both T_1 and T_2 consist of exactly one vertex; and
2) T_n is a binary tree whose left subtree is T_{n-1} and right subtree is T_{n-2}, where $n \geq 3$. It has $2F_n - 1$ vertices, F_n leaves, $F_n - 1$ internal vertices, and $2F_n - 2$ edges.

Figure 31.3 shows the first five Fibonacci trees.

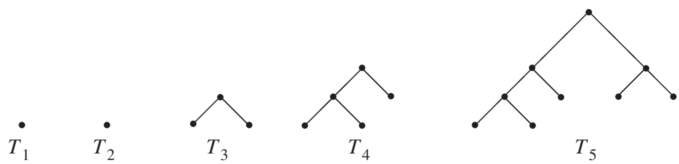


Figure 31.3.

We now assign a *weight* to T_n recursively. The weight of T_1 is 1 and that of T_2 is x. Then the weight $w(T_n)$ of T_n is defined by $w(T_n) = x \cdot w(T_{n-1}) + w(T_{n-2})$, where $n \geq 3$.

For example, $w(T_3) = x \cdot w(T_2) + w(T_1) = x^2 + 1$; and $w(T_4) = x \cdot w(T_3) + w(T_2) = x^3 + 2x$.

Since $w(T_1) = f_1$, and $w(T_2) = f_2$, it follows by the recursive definition that $w(T_n) = f_n$, where $n \geq 1$. Clearly, $w(T_n)$ gives the number of leaves of T_n when $x = 1$.

Binet-like Formulas

Using the recurrence $g_n = xg_{n-1} + g_{n-2}$ and the initial conditions, we can derive explicit formulas for both f_n and l_n; see Exercises 31.6 and 31.7:

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad l_n = \alpha^n + \beta^n,$$

where $\alpha = \alpha(x) = \dfrac{x + \Delta}{2}$ and $\beta = \beta(x) = \dfrac{x - \Delta}{2}$ are the solutions of the equation $t^2 - xt - 1 = 0$ and $\Delta = \Delta(x) = \sqrt{x^2 + 4}$. Notice that $\alpha + \beta = x$, $\alpha - \beta = \Delta$, and $\alpha\beta = -1$.

Since $\alpha = \alpha f_1 + f_0$ and $\alpha^2 = \alpha x + 1$, it follows by the principle of mathematical induction (PMI) that $\alpha^n = \alpha f_n + f_{n-1}$, where $n \geq 1$; see Exercise 31.8. Similarly $\beta^n = \beta f_n + f_{n-1}$.

Using the Binet-like formulas, we can extend the definitions of Fibonacci and Lucas polynomials to negative subscripts: $f_{-n} = (-1)^{n-1} f_n$ and $l_{-n} = (-1)^n l_n$.

Using the Binet-like formulas, we can also extract a plethora of properties of Fibonacci and Lucas polynomials; see Exercises 31.14–31.97. For example, it is fairly easy to establish that

$$f_n l_n = f_{2n};$$

$$f_{n+1} + f_{n-1} = l_n; \tag{31.1}$$

$$x f_{n-1} + l_{n-1} = 2 f_n; \tag{31.2}$$

$$l_{2n} + 2(-1)^n = l_n^2;$$

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n;$$

$$l_{n+1} l_{n-1} - l_n^2 = (-1)^{n-1}(x^2 + 4).$$

The last two identities are Cassini-like formulas. It follows from the Cassini-like formula for f_n that every two consecutive Fibonacci polynomials are relatively prime; that is, $(f_n, f_{n-1}) = 1$, where (a, b) denotes the greatest common divisor (gcd) of the polynomials $a = a(x)$ and $b = b(x)$.

Cassini-like Formulas Revisited

Since $l_n(2i) = 2i^n$, it follows that $(x \pm 2i) \nmid l_n$, where $i = \sqrt{-1}$. Consequently, by the Cassini-like formula for l_n, every two consecutive Lucas polynomials are relatively prime, that is, $(l_n, l_{n+1}) = 1$.

The Cassini-like formulas have added dividends. For instance, $(f_{n+4k} + f_n, f_{n+4k-1} + f_{n-1}) = l_{2k}$. To see this, we have

$$\begin{aligned} \Delta(f_{n+4k} + f_n) &= \left(\alpha^{n+4k} - \beta^{n+4k}\right) + (\alpha^n - \beta^n) \\ &= \left(\alpha^{n+2k} - \beta^{n+2k}\right)(\alpha^{2k} + \beta^{2k}) \\ f_{n+4k} + f_n &= f_{n+2k} l_{2k}. \end{aligned}$$

Replacing n with $n-1$, this implies $f_{n+4k-1} + f_{n-1} = f_{n+2k-1} l_{2k}$. Thus

$$\begin{aligned}(f_{n+4k} + f_n, f_{n+4k-1} + f_{n-1}) &= l_{2k} \cdot (f_{n+2k}, f_{n+2k-1}) \\ &= l_{2k} \cdot 1 \\ &= l_{2k}. \end{aligned} \tag{31.3}$$

Similarly,

$$(l_{n+4k} + l_n, l_{n+4k-1} + l_{n-1}) = l_{2k}; \tag{31.4}$$

see Exercise 31.102.

It follows from properties (31.3) and (31.4) that

$$(F_{n+4k} + F_n, F_{n+4k-1} + F_{n-1}) = L_{2k};$$

$$(L_{n+4k} + L_n, L_{n+4k-1} + L_{n-1}) = L_{2k}.$$

For example, $(L_{23} + L_7, L_{22} + L_6) = (64079 + 29, 39603 + 18) = 47 = L_8$.

Pythagorean Triples

The identities $l_{n+1} + l_{n-1} = \Delta^2 f_n$ and $l_{2n} = \Delta^2 f_n^2 + 2(-1)^n$ (see Exercises 31.32 and 31.49) can be employed to construct Pythagorean triples (a, b, c). To see this, let $c = \Delta^2 f_{2n+3}$ and $a = x l_{2n+3} - 4(-1)^n$. We now find b such that (a, b, c) is a Pythagorean triple.

Since $c = l_{2n+4} + l_{2n+2}$, we have

$$\begin{aligned} c + a &= l_{2n+4} + (x l_{2n+3} + l_{2n+2}) - 4(-1)^n \\ &= 2[l_{2n+4} - 2(-1)^{n+2}] \\ &= 2\Delta^2 f_{n+2}^2; \\ c - a &= (l_{2n+4} - x l_{2n+3}) + l_{2n+2} + 4(-1)^n \\ &= 2[l_{2n+2} - 2(-1)^{n+1}] \\ &= 2\Delta^2 f_{n+1}^2. \end{aligned}$$

Therefore, $b^2 = c^2 - a^2 = (2\Delta^2 f_{n+2}^2)(2\Delta^2 f_{n+1}^2) = 4\Delta^4 f_{n+2}^2 f_{n+1}^2$; so we obtain $b = 2\Delta^2 f_{n+2} f_{n+1}$.

Thus $(a, b, c) = (x l_{2n+3} - 4(-1)^n, 2\Delta^2 f_{n+2} f_{n+1}, \Delta^2 f_{2n+3})$ is a Pythagorean triple.

Clearly, $\Delta^2 | b$ and $\Delta^2 | c$; so $\Delta^4 | (c^2 - b^2)$. Consequently, $\Delta^4 | a^2$ and hence $\Delta^2 | a$. Thus (a, b, c) is *not* a primitive Pythagorean triple.

H.T. Freitag (1908–2005) of Roanoke, Virginia, studied the Pythagorean triple for the special case $x = 1$ in 1991 [168].

Recall from Chapter 16 that $\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \alpha$. So what can we say about $\lim_{n\to\infty} \frac{f_{n+1}}{f_n}$? Next we investigate this.

Suppose $x > 0$. Then $0 < x/\Delta < 1$. Since $\frac{\beta}{\alpha} = \frac{x-\Delta}{x+\Delta} = -\frac{1-x/\Delta}{1+x/\Delta}$, $|\beta/\alpha| < 1$. Consequently,

$$\begin{aligned}\frac{f_{n+1}}{f_n} &= \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha^n - \beta^n}\\ &= \frac{\alpha^{n+1}}{\alpha^n} \cdot \frac{1-(\beta/\alpha)^{n+1}}{1-(\beta/\alpha)^n}\\ \lim_{n\to\infty} \frac{f_{n+1}}{f_n} &= \alpha \cdot \frac{1-0}{1-0}\\ &= \alpha.\end{aligned}$$

Similarly, $\lim_{n\to\infty} \frac{l_{n+1}}{l_n} = \alpha$. Thus

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \alpha = \lim_{n\to\infty} \frac{l_{n+1}}{l_n}, \tag{31.5}$$

where $x > 0$.

For the curious-minded, we add that

$$\frac{f_{n+1}(0)}{f_n(0)} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \text{undefined} & \text{otherwise;} \end{cases}$$

$$\frac{l_{n+1}(0)}{l_n(0)} = \begin{cases} \text{undefined} & \text{if } n \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

It follows by the recursive definition that $\deg(f_n) = n - 1$ and $\deg(l_n) = n$, where $\deg(h_n)$ denotes the degree of the polynomial $h_n(x)$ and $n \geq 1$. Suppose $a, b \geq 2$. Then $(a-1)(b-1) \geq 1$; consequently, $ab > a + b - 1$. Suppose also that $x \geq 1$. Since $\deg(f_a f_b) = \deg(f_a) + \deg(f_b) = a + b - 2$, it follows that $f_{ab} > f_a f_b$. Likewise, $l_{ab} > l_a l_b$.

The facts that $2\alpha = x + \Delta$, $2\beta = x - \Delta$, and $\Delta = \sqrt{x^2+4}$ can be used to develop two interesting identities, one involving Fibonacci polynomials and the other involving Lucas polynomials.