551 LECTURE NOTES IN ECONOMICS AND MATHEMATICAL SYSTEMS

Jacek Leskow
Martín Puchet · Lionello F. Punzo
Editors

New Tools of Economic Dynamics

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Dedicated to *Richard M. Goodwin* and *Michio Morishima*

Preface

Innovation and advances in the techniques of analysis of economic dynamics have been dramatic in recent years. Taken together, they have formed a sort of second wave following the wave that has revolutionised macro dynamics in the 80s. Impact has been relevant to both theoretical and applied work and it has involved also econometrics, of course.

On one side, we have witnessed the birth of families of what could be broadly defined new growth and development models. They are quite new in comparison to those of traditional approaches the endogenous and exogeneous types- and can be collectively characterised by the fact that their much richer internal structure is capable of producing a richer, and more interesting, variety of dynamics. On the other hand, econometrics and especially time series analysis began looking more closely at the finer structure of our economies, a greater number of variables being attributed to different agents and represented in the models.

In either case, the result was first that the models economists got used to work with, had (often, many) more dimensions than the traditional ones. Moreover, the main force driving the economy's dynamics began to be identified with the various rules and forms of interaction among many heterogeneous agents (industries, firms, individuals). The engine of dynamics was seen to be fundamentally endogenous, rather than the the mere response to the exogenous shocks of New Classical dynamics. Thus, the whole analytical framework based upon the impulse-response mechanism had to be entirely overhauled, changing their relative weights: more was put into the internal structure of the economy, less in the complications of the shock profiles.

The emerging new modelling framework obviously demanded new analytical tools, too. These had to be (and have been) imported from elsewhere, this ranging over a very broad field, from statistics to mathematics to physics. Among such tools, more and more important became e.g. numerical simulations as an exploratory device with a theoretical dignity of its own. Sometimes, using simulations was a choice; more often, however, it was a necessity given the size of the model at hand. Taken together, at any rate, all those new tools were employed, at times, to search out the capabilities and to explore the structure of a given model. On other occasions, they were to depict alternative scenarios for growth and/or for policy actions. Even when quantitative results were expected - as in all econometrics and time series studies still, some part of the added value of the research was in the qualitative nature of the information provided by its results. This calls for a comparison with the way the qualitative approach to dynamics entered into economics, and how it fared in the field since its discovery by the economists.

Qualitative analysis has been a key approach to dynamics since Poincar invented it at the end of 19th century and since its introduction into economics with the classical works of Frisch, Kaldor, Hicks and Goodwin, between the 30s and the 50s. It was born out of the incapability of handling certain non-linear dynamic models in a classical way, i.e. by explicitly finding their solutions. Going qualitative was a necessity, instead of a choice. It basically meant topological (hence, non numerical) analysis of individual models and the fundamentally topological theory of classes of models.

The qualitative approach that has been emerging recently is quite different, though it complements the classical one. Differences can be appreciated in many ways, but they all refer either to the intensive use of new, sometimes simulation and numerical techniques and the construction of models with greater dimensions than before, and/or to the deeper integration between theory and empirical evidence.

The New Tools project (and network) was born out of this challenge and it reflected the variety and heterogeneity of its aspects. Emphasis was however placed on the common ground, the exploration of tools rather than the construction of models around specific economic issues. The New Tools network now links researchers in various countries and universities of Europe and Latin America.

Most of the chapters collected in this volume are revised versions of research papers read in four workshops held consecutively at UNISI in Siena (December 2000), UDLA in Cholula (State of Puebla, Mexico, September 2001), CIMAT in Guanajuato (State of Guanajuato, October 2002) and in Nowy Sacz Graduate School of Business in Poland (September 2003). All papers were subjected to intense discussions during the network's meetings, with a varied public of researchers and students at different levels of their education. In fact, the purpose of the NT network is not only to promote research but also to enrich education, focusing on master and doctoral levels.

The broad areas in which the network's research activity fell so far, are reproduced in the volume's structure with the three sections: large interactive models of the economy; econometrics and time series; growth, development and structural change, Each section contains both theoretical and applied chapters as, in general, papers have been written with the need to look for such intersection in the authors'minds. In fact, it is a key hypothesis in the NT project that time is ripe for a reconciliation between the more theoretical and the more applied research lines in economic dynamics, ending thus a divorce and recomposing a unity that was at the birth of macro dynamics as envisioned by Ragnar Frisch and the founders of thee Econometric Society. We believe it is important to try in this direction by picking up the bits and pieces left from that divorce, in particular reconsidering the different tools that were developed then from the vantage point of the new ones. We now briefly go over the three sections, trying to highlight the novelty that is in the various applications of tools. Such novelty can often be appreciated more by the economists than by the pratictioner of those disciplines from which those tools have been imported. The main common ground can be identified with the study of various aspects of so called complex dynamics. As anticipated earlier, these aspects are hereafter investigated under the hypothesis that they spring from the endogenous mechanism more than from the characteristics of some exogenous forces. In other words, without denying the importance of the latter, often stochastic forces, it is the structure of the model economy, which is seen as the site of the basic explanation of its dynamics. Structure can be looked at in a variety of ways as shown in the various papers, and can also be seen in its evolution, dramatic or catastrophic as sometimes its discontinuous change is called (after the mathematical theory).

Thus, 6 out of 9 chapters in Section I are devoted to the analysis of the various effects and tools to analyse settings with heterogeneous agents, and to derive characteristics of the resulting (aggregate) dynamics. Thus, Aoki's Chapter 1 introduces the notion of classes or types of agents and deals with the issue of how to consider the uncertain appearance of new types along the economic trajectory. By looking at various schemes of local interaction between nearby firms, in Chpt. 2 Andergassen et als discuss the emergence of fluctuating growth and technological patterns shared by firms (trajectories in the evolutionary sense). On the other hand, through an explicit neural networks approach, Chapter 8 considers the emergence of firms and firms' networks as the result of processes of learning in an environments too complex to be handled efficiently by any individual, thus as the institutions adequate to solve the associated economic problems. Aoki's paper has implications for simulation techniques, which are heavily implemented in many of the other chapters. Through simulations, Chapter 2 tackles the problems of the emergence of different groupings of agents that are heterogeneous in their initial endowments and via bilateral exchange have to reach through an evolutionary process, equilibria implying different schemes of benefits sharing. Chapter 3 looks at a similar problem of social aggregation, there with a genetic algorithm approach, within a setting where the assumption of bounded rationality is central and a learning process is modelled. Chapter 4, on the other hand, innovates the conventional description of macroeconomic performance by studying (among other things) the effects of parameter perturbation over a system of equations tracing the time evolution of the first and second moments of the firms distributions (in terms of a chosen index of financial robustness). (Many of the implications of these analyses on growth and in particular on growth irregularity and fluctuations will come up again in Chapters of Section 3, while the statistical implications are practically dealt with in Section 2.) Finally, Chapter 9 reviews various easily available platforms for multi-agent simulations, thus providing a guide to the intriguing question of what to learn to do.

As said above, classical qualitative analysis basically meant topological methods applied to (classes of) models or of model predicted trajectories. This is very much the spirit in Chapter 6, where however the study of general equilibrium economies is carried on by means of much newer notions from Catastrophe theory. The point is the suggestion to focus upon the singular economies (that are structurally unstable) rather than on the structurally stables ones as is always the practice. (The theme of the importance of understanding instability comes up again in Section 3). But qualitative analysis can also be of a different type: an analysis where structural rather than functional dependence, and thus hierarchical and dominance relations are at the centre, as is in Chapter 7, where a pretopological approach is used to unveil the bare skeleton of an economy.

Virtually, all of the chapters mentioned above bear implications for observable dynamics, most of them do look also at empirical evidence. While this is a feature common with Section 3, empirical evidence and how to handle it is the very focus of Section II; as its title suggests. Here too, the common framework is one where effects of multidimensional economies and complex time evolution (including, uncertainty) are studied. This is the realm of econometrics, time series analysis and of course broadly defined simulation based-econometrics, a field fast growing specially in a version married with micro simulation.

The latter is basically the object of the two coordinated Chapters 13 and 14, and it appears in the topically related Chapter 15. In all three chapters, the study case of retirement choices is tackled for its own right, but also to demonstrate a variety of techniques to econometrically construct, handle and validate models with many agents, thus capable of exhibiting alternative outcomes through micro simulation experiments (in the former two chapters), or to endogenize choices (in this case, of retirement) as in the latter chapter. A critical review of outcomes of a bunch of econometric tools to evaluate monetary policy is presented in Chapter 16, with an application to a known difficult case, Mexico's highly volatility behaviour. The chapter makes a case for an informed policy decision-making process, whereby different scenarios produced by alternative techniques are systematically taken into account. This is again a link to themes in chapters of Section III, with their multiple illustrations of applications of complex dynamics tools to Latin America (and possibly to more recent events elsewhere). But before turning to that, we recall that the chapters opening this section, are all devoted to issues associated with detection of the driving force s behind seemingly irregular economic time series.

Thus, Chapter 12 reviews the recent advances of spectral analysis, a well-established technique in economics being associated with the still most favoured linear econometric framework, while in fact it has received major extensions through for instance the windowed filtering methods. The application to the well known Phillips' curve is a good link to Chapter 10, where time series are looked at as possibly embodying, next the more popular ingredients, also structural change. As a way to tackle such cases, the smooth transition formulation of an econometric model is exposed and some result shown. This is a rapidly expanding research in the filed of non linear econometrics, as much as is the modelling of financial markets, a sample to be found in Chapter 11 introducing an imported method of Value-at-Risk prediction (or VAR, not to be taken for the better known vector auto regression approach!), which promises to handle time series for which there are no multiple realizations, or it is safer not to assume it.

Section III deals with issues in what traditionally have been classified growth and development fields, until recently realm of well-defined theories with clearly understood predictions. The history of the last decades, and the theoretical reflections on it, has shown that the apparent consensus reached some time ago about their interpretation has definitely broken down. We are searching for an explanation, or more probably for various explanations for the series of events that have been happening in the various countries, explanations accounting of the variety of experiences and the evolution often dramatic shown by most of them. The so called convergence literature, enormously boosted by the growth debate and the availability of new statistical base in the mid of the 80s, has probably misled us by proposing the search for cross country uniform behaviours and long run stability towards some predicted equilibrium path. Neither prediction has proved to be reasonably tenable.

The critical implications of this failure are the common thread of the section, which opens with the revision of the notion of convergence in the light of its environmental implications. This leads to the unveiling of a double convergence hypothesis which is allegedly implicit in the growth re-interpretation of the so called environmental Kuznets curve, and to the rejection of the latter on the ground of the prevailing of different growth regimes across countries (the notion of regime recalls chapter 5 and 6 above). A re-examination of the growth findings in Chapter 18 focuses upon the much discussed issue of volatility in performance, an issue discussed by the authors resorting to distribution analysis with a Markov chain hypothesis. Chapter 19 re-examines in a detailed way a well known model of externally constrained growth, in the light of the Mexican and generally recent Latin American experiences, and it shows how it had to be to a large extent updated. The next two Chapters 20 and 21 dwell upon the uses of the notion of fractional brownian motion to explain, respectively, the time behaviour of indices of the Mexican stock market, where structural change is embedded as a result of the recent major organizational changes (including NAFTA), and the Argentinean high inflation before the parity with the US dollar as the evolution of self organized structures.

Optimisation is bread and butter for economists, and the intertemporal optimisation framework has become more so after the 80s. The book could not overlook this: Chapter 22 reviews the theory and various applications of a new field called semi-infinite programming, not as easy as standard programming, not so difficult as day to day, real life one.

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Part I

Large Interactive Economies

1

Modeling a Large Number of Agents by Types: Models as Large Random Decomposable Structures*

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Summary. This paper introduces methods, based on decomposable random combinatorial analysis, to model a large number of interacting agents. This paper also discusses a largely ignored possibility in the mainstream economic literature that hitherto unknown types of agents may enter the models at some future time. We apply the notion of holding times, and introduce the results of the one- and two-parameter inductive methods of Ewens, Pitman and Zabell to economic literature. More specifically, we use the notion of exchangeable random partitions of a finite set to produce a simple rule of sucession, that is, the expressions for the probabilties for entries by new or known types, conditional on the observed data. Then Ewens equilibrium distriution for the sizes of clusters is introduced, and its use to examine market behavior is sketched, especially when a few types of agents are dominant. We suggest that the approaches of this paper and the notion of holding times are relevant to agent-based simulations because holding times can be used to randomly select agents that ''act" first.

1.1 Introduction

Economists often face problems of modeling collective behavior of a large number of interacting agents, possibly of several different types. This paper discusses methods that are useful in this context. We indicate how the methods may explaine diverse phenomena such as equilibrium size distributions of clusters, that is subgroups formed by agents, market shares by different types of goods, changes in the adjustment speeds towards equilibria with model sizes, and emergence of macroeconomic regularities as the number of agents increases towards infinity, and so on.

We explicitly assume that there are several types of agents in our models, the number of which may not be known in advance, and that agents of new

* Prepared for the NT Book

types may enter the models at any time. We cannot assume in advance that we know all of them because new rules or new goods may be invented in the future. This is the so-called problem of unanticipated knowledge in the sense of Zabell, see Zabell (1992)[2]. In biology this problem is known as the sampling of species problem. In probability and statistics it is called laws of sucession, that is, how to specify the conditional probability that the next sample is a new type, given available sets of observation up to now. See Zabell (1982). In addition, agents may change their minds at any time about the decisions or behavioral rules they use. In other words, agents may change their types any time[3]. This paper presents some methods from the field of decomposable random combinatorial analysis. They are useful in modeling economic structures composed of a large number of possibly heterogeneous agents, components or basic units, and indicates some of their potential applications in macroeconomics and finance.

Large economic structures are regarded as decomposable random combinatorial structures. Of the many possible structures due to many possible configurations which a large number of components may assume, we wish to deal with "typical" structures of large sizes. By typical we mean structures which have high probabilities of being chosen, or observed from the set of all structures, when chosen at random in some sense from the set. This is the main reason for borrowing or adapting tools and concepts from combinatorial stochastic processes, and population genetics literature.

We interpret the word "types" broadly. This word may refer to some characteristics or rules that are used to partition the set of agents. Or, it may refer to some rules or behavioral patterns adopted by economic agents, or it may refer to some other characteristics to distinguish one subgroup of agents from other groups. We assume that the number of types are at most countable.

The methods described in this paper are not in the tool kit of traditionally trained economists or econometricians, but we have found them to be useful for understanding macroeconomic or financial phenomena from our new perspectives. Stochastic combinatorial tools are used to show how agents form clusters, and jump Markov processes are used to model how the clusters evolve over time through interaction among agents of several types. To describe dynamic phenomena, master equations (backward Chapman- Kolmogorov equations) are used to describe probability distributions over states. We use a new notion of states, called partition vectors, which is more appropriate in dealing with exchangeable or delabelled agents and category, i.e., type indices.

Some of the questions we examine are: How do we describe the process by which agents form clusters, that is, subgroups in modeling a collection of

[2] Zabell describes the problem faced by statisticians in classifying samples of insects collected in unexplored regions, since they may contain new species of insects, say. The naive Bayesian approach is not applicable. See, however, Antoniak (1969) on non-parametric Bayesian approach. He obtained the same distribution as the Ewens sampling formula, Ewens (1972).

[3] There is no lock-step behavior by agents.

interacting agents? What are the stationary distributions of sizes of fractions of agents of different types? What are the market shares of a typical largest cluster, two largest clusters, and so on? Distributions of cluster sizes matter, because a few of the larger clusters, if and when formed, approximately determine the market excess demands for whatever goods in the markets and the nature and magnitudes of fluctuations.

The methods mentioned in this paper have diverse origins. To discuss clusters and entries by agents of new types, such as new goods, new business models, new (sub)optimization procedures, and so on, we borrow from the literature of population genetics such as Ewens (1972, 1990), Watterson (1976), and Watterson and Guess (1977), and from statistics and stochastic processes such as Kingman (1978a, b), Arratia and Tavaré (1992), and Pitman (2002). See also Aoki (2002).

The Ewens sampling formula is an example of one-parameter inductive model. It is specified by a single parameter θ, which controls the rate of entries of new types, and correlations among agents of different types. We also describe its two-parameter extension by Pitman (1992), which is specified by two parameters, α and θ discussed later.

More concretely, we introduce the notion of partition vector as state vectors, which is different from the empirical distributions, and use the asssumption of exchangeable partitions induced by agents of different types in the models, in the technical sense of exchangeable random variables in the probability literature. We utilize the notion of holding times from the literature on continous-time Markov chain (also called jump Markov process) to decide randomly which agent acts first. We apply the equilibrium distribution to discuss the question of market shares, behavior of rates of returns, and volatilities of returns.

To conclude, we list some financial and economic applications: In finance we mention the work on power laws and volatility switching in Aoki (2002b, c). In economics, we briefly compare the approach of the traditional economists in allocating capital stock between two sectors as formulated by Dixit (1989), and our modeling procedure in terms of continuous-time Markov chains with a large but finite number of interacting agents in Aoki (2002a, Chapt.8). A new approach to the Diamond search model from our perspective is in Aoki and Shirai (2000). A new approach to growth model is in Aoki and Yoshikawa (2002).

1.2 New Concepts

Partition vectors

When it is known that there are n agents, and K distinct types of agents, a common choice of state vector is $\mathbf{n} = (n_1, n_2, \ldots, n_K)$ where n_i is the number of agents of type i, i = 1,2, ..., n. This choice of state vector is natural, and appears to be satisfactory. There is, however, another choice of state which suits our needs better when identities of agents of various types are not the issue. In some cases, only the numbers of agents of different types may matter. Labels we assign to agents may be merely for convenience of reference. Permuting these labels often leaves nothing of substance changed. For example, agents may be labelled in the order we sample or examine them, or in the order they enter the market, but there is no essential meaning or substance to the labels. They are for mere convenience of referring to them. Permuting labels assigned to agents should not cause any essential changes in our conclusions about the models in such cases. When this holds true, agents are called exchangeable in the technical sense defined in probability literature[4]. To indicate this, exchangeable agents are sometimes called delabelled agents. For a collection of exchangeable agents their joint probability is invariant to permutations of indices assigned to agents in order to refer to them.

We regard collections of agents as exchangeable agents, and assume also that types are exchangeable. That is, agents are partitioned into distinct clusters. Labels of the clusters may again be for convenience. Category indices may again be for mere convenience of reference with no substance. If categories are delabelled, then the probability is also invariant with respect to permutations of category indicies. This is the notion of exchangeable partitions.

State of a population is described by the (unordered) set of type-frequencies i.e., fractions or proportions of different types without stating which frequency belongs to which type. In the context of economic modeling, this way of description does not require model builders to know in advance how many or what types of agents are in the population. It is merely necessary to recognize that there are K_n distinct types in his sample of size n, and that there are a_j types with j agents or goods in the sample. The vector $\mathbf{a}$ with these components is called partition vector by Zabell (1992), and we adopt this name. Note that $\sum_j a_j = K_n$, and $\sum j a_j = n$. The first equation counts the number of occupied boxes, and the second total number of agents. Partition vector is just the right notion to discuss models with delabelled agents and delabelled categories. The problem is the same as the occupancy problem of allocating unlabelled or indistinguishable balls and unlabelled or indistinguishable boxes. The same concept is known under different names in Kingman (1980), and Sachkov (1996).

Exchangeable random partitions

A partition of a finite set F into K blocks is an unordered collection of non-empty, disjoint sets $\{A_1, \ldots, A_K\}$ whose union is F. To be definite, we may use a convention that the blocks of partitions are listed in the order of appearance, that is, by the least elements of the blocks. Let $[n]$ denote a set of n elements, $\{1, 2, \ldots, n\}$. Let X_i, $i = 1, 2, \ldots n$ be random variables with values on $[K]$.

[4] See Feller (1968).

These X_s are grouped into subsets of K or less and induce partition of $[n]$. Any partition of $[n]$ defines a composition of n, which is a sequence of positive numbers with sum n. By using the size of the sets A_i, $n_i = \mid A_i \mid$, the set defines the composition $n = n_1 + n_2 + \ldots + n_K$. A partition Π_n of $[n]$ is exchangeable if its distribution is invariant with respect to permutations, i.e.,

$$Pr(\Pi_n = \{A_1, A_2, \ldots A_K) = p(\mid A_1 \mid, \ldots, \mid A_K \mid);$$

where $p(\cdots)$ is some symmetric function of the components.

Exchangeable random partition is such that

$$Pr(N^{ex}_{n,1}, \cdots, N^{ex}_{n,k}) = \frac{n!}{n_1! n_2! \cdots n_k!} \frac{1}{k!} \times p(n_1, n_2, \cdots, n_k)$$

Two partitions with the same vector **a** are equiprobable when the partitions are exchangeable. Sequences associated with exchangeable random partitons are exchangeable sequences.

Exchangeable agents

This newer representation has roots in the exchangeable random partitions of a set of agents into clusters, which arises in examining clusters or subsets of agents of the same types. Stirling numbers of the first and second kind also appear in counting the configurations of clusters of agents of various types. These have roots in random combinatorial analysis. Probability distributions such as the Poisson-Dirichlet distributions and the multivariate Ewens distribution are not in the tool kit of conventionally trained economists or econometricians, but are important in examing the distributions of sizes of clusters of agents. We therefore present these as well as some others, as needs arise, to advance and support our views expressed in this book.

Let X_i, $i = 1, 2, \ldots, n$ be a sequence of random variables whose values are in the set of type indices. When an original sequence of random variables and that of permuted random variables, that is, $X_1, X_2, \ldots, X_n$ and $X_{\sigma(1)}, \ldots X_{\sigma(n)}$, where σ denotes permutation of the subscript, have the same probability depending on the empirical distribution $n_j = \mid \{X_i = typej\} \mid, j = 1, 2, \cdots, K$, that is $p(n_1, n_2, \cdots, n_K)$, where $p(\cdots)$ is symmetric of its arguments, then we call the sequence exchangeable. Two sequences with the same empirical distributions are equiprobable for exchangeable sequences of random variables.

Limit behavior of large fractions: Poisson-Dirichlet distributions

Suppose that a large number of agents interact in a market where each agent uses one of K available trading rules, where K is large. Then the set of agents is partitioned into at most K clusters. The number of clusters depends crucially on the correlations among agents which affects the probability that two randomly chosen agents in the market are using the same strategy. The sizes of clusters are arranged in non-increasing order as $n_{(1)} \geq n_{(2)} \geq \cdots$. With high correlations, a small number of large clusters tend to form. When correlations among agents are small, many smaller clusters are likely to emerge, as we later mention. We will also mention later that the sum of the sizes of the first two largest clusters, $n_{(1)} + n_{(2)}$, alone in some cases account for the majority, 70 per cent say, of the total number of participants. This observation is useful in characterizing aggregate behavior when it happens. See Aoki (2002b). Order statistics of the fractions have a well-defined limit distribution, called the Poisson-Dirichlet distribution, as the number of agents go to infinity. The probability density of the first few of the orders sizes of fractions can be used in our discussion of approximations of market excess demands or return dynamics of some traded assets.

1.3 Dynamics of Clustering Processes

Agents and goods are classified into clusters or subsets by associating types with strategies or choices of agents. Here we interpret the word types broadly.

As agents interact, new clusters form or some existing clusters break up into smaller ones. The transitions of these processes are captured by specifying how the partition vector $\mathbf{a}$ is transformed over a small time intervals. For example, transition rate specification

$$w(\mathbf{a}, , \mathbf{a} + e_1) = \lambda(n)$$

where $\mathbf{e_i}$ is a vector with the only non-zero component 1 at the ith component, refers to an event that a single agent enters the market without joining any existing cluster, while

$$w(\mathbf{a}, \mathbf{a} + \mathbf{e}_{j+1} - \mathbf{e}_j) = j a_j \lambda(n)$$

specifies one agent joints a cluster of size j, thereby increasing the number of clusters of size $j+1$ by one, and reducing that of size j by one. The right-hand side specifies that the rates are proportional to some constant $\lambda(n)$, and ja_j which counts the total number of agents in the clusters of size j. There are many other possibilities, of course. At this stage of our exposition, let us pause to take stock of what we have done. Our model building procedures may be summarized as follows: We start with a collection of a large, but a finite number of microeconomic agents in some economic or financial context. We first select state space and specify a set of transition rates on it to model agent interactions stochastically. Agents may be households, firms, or countries depending on the context of models. Unlike examples in textbooks in simple probability, our transition rates are usually state-dependent, that is functions of states, to model effects of endogenously generated aggregate effects, called field effects in Aoki (1996), such as effects of aggregate behavior such as total outputs, crowding, fashion, group pressures on individual agents, and so on, as well as evaluations of consequences of specific choices subject to uncertainty or imperfect information that go into evaluations of value function maximization associated with alternative choices.

Then we describe by master equation the dynamics of the joint probability of the components of a state vector for the model which incorporates specified transition rates. Stationary or nonstationary solutions of the master equations are then examined to deduce model aggregate dynamic behavior.

In models which focus on the decomposable random combinatorial aspects, distributions of a few of the largest order statistics of the cluster size distributions are examined to draw economic consequences.

Ewens Distribution

Here we follow Aoki (1996, 1998, 2000a,b, 2002a, b,c) and sketch the basic ingredients for our modeling procedure without too much detail. The reader is asked to consult the cited references for detail.

Continuous-time Markov chains, also known as jump Markov processes, are completely specified by transition rates, when state spaces are at most countable.

For ease of explanation we use vector **n** some of the times and **a** at other times. Define a state vector X_t which takes on the value $\mathbf{n} := (n_1, n_2, \cdots, n_K)$, called frequency or occupancy vector, where n_i is the number of agents of type i, $i = 1, 2, \cdots, K$, $N = n_1 + n_2 + \cdots + n_K$.

In our model we need to specify entry rates, exit rates and rates of type changes. Over a small time interval Δ, rates are multiplied by the length of interval to approximate the conditional probabilities up to $O(\Delta)$. For example, entry rates by an agent of type j may be specified by

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_j) = \phi_j(n_j, athbfn);$$

where $\mathbf{e}_j$ is a vector with the only nonzero element of one at component j.[5] Exit rates of an agent of type k is specified by

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_k) = \psi_k(n_k, \mathbf{n});$$

and transition rates of type i agent changing into type j agent by

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_i + \mathbf{e}_j) = \lambda_{i,j}\nu(n_i, n_j, \mathbf{n});$$

With transition rates between states specified, the dynamics for the probability is given by the following equation, where s, s', and s'' refer to some states

[5] For example, $w(\mathbf{n}; \mathbf{n} + \mathbf{e}_j)\Delta \approx Pr(X_{t+\Delta} = \mathbf{n} + e_j \mid X_t = n)$.

$$dP(\mathbf{s},t)/dt = \sum_{s'} w(s',s)P(s',t) - \sum_{s''} w(s,s'')P(s,t)$$

This is called the master equation in phyics, ecology and chemistry, and we follow their usage of the name. A specific example of interest has the transition rates:

$$w(\mathbf{n}, \mathbf{n} + \mathbf{e}_k) = c_k(n_k + h_k)$$

for $n_k \geq 0$,

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) = d_j n_j$$

$n_j \geq 1$, and

$$w(\mathbf{n}, \mathbf{n} - \mathbf{e}_j + \mathbf{e}_k) = \lambda_{jk} d_j n_j c_k(n_k + h_k)$$

with $\lambda_{jk} = \lambda_{kj}$, and where $j, k = 1, 2, \cdots K$. We assume that $d_j \geq c_j > 0$, and $h_j > 0$, and $\lambda_{jk} = \lambda_{kj}$ for all j, k pairs.

The first transition rate specifies entry rate of type k agents, and the second that of the exit or departure rate by type j agents and the last specifies the transition intensity of changing types by agents from type j to type k. In the entry transition rate specification $c_k n_k$ stands for attractiveness or disadvantage of larger group, such as network externality which makes it easier for others to join the cluster or group, or congestion which may induce avoidance of larger groups, as the case may be. The term $c_k h_k$ stands for the innovation effects which is independent of the group size. These transition rates for type changes are in Kelly (1979), for example. We need interactions or correlations among agents. It turns out that parameter θ, to be introduced in connection with (2) below, plays this role. See Aoki (2000a, 2002b). The jump Markov process thus specified has the steady state or stationary distribution

$$\pi(\mathbf{n}) = \prod_{j=1}^{K} \pi_j(n_j),$$

where

$$\pi_j(n_j) = (1 - g_j)^{h_j} \binom{-h_j}{n_j} (-g_j)^{n_j}$$

where $g_j = c_j/d_j$

These expressions are derived straightforwardly by applying the detailed balance conditions to the transition rates. See Kelly (1979, Chapt.1), or Aoki (2002, p. 148) for example.

To provide simpler explanation, suppose that $g_j = g$ for all j. Then, noting that $\prod_j (1-g)^{h_j} = (1-g)^{\sum_j h_j}$, the joint probability distribution is expressible as

$$\pi(\mathbf{n}) = \binom{-\sum h_k}{n}^{-1} \prod_{j=1}^{K} \binom{-h_j}{n_j} \tag{1.1}$$

By a suitable limiting process this distribution goes to the Ewens distribution. To see this suppose that K becomes very large and h very small, while the product Kh approaches a positive constant θ . We note that the negative binomial expression

$$\binom{-h}{j}^{\alpha_j}$$

approaches $(h/j)^{\alpha j}(-1)^{ja_j}$ as h becomes smaller. Suppose $K_n = k \le K$. Then, there are

$$\frac{K!}{a_1!a_2!\cdots a_n!(K-k)!}$$

many ways of realizing a vector. Hence

$$\pi(\mathbf{n}) = \binom{-\theta}{n}(-1)^n \frac{K!}{a_1!a_2!\cdots a_n!(K-k)!} \prod_j (\frac{h}{j})^{\alpha_j} \tag{1.2}$$

Noting that $K! = (K-k)! \times h^k$ approaches θ^k in the limit of K becoming infinite and h approaching 0 while keeping Kh at θ, we arrive, in the limit, at the probability distribution, known as the Ewens distribution, or Ewens sampling formula very well known in the genetics literature, Ewens (1972).

$$\pi(\mathbf{n}) = \frac{n!}{\theta^{[n]}} \prod_{j=1}^{n} (\frac{\theta}{j})^{a_j} \frac{1}{a_j!},$$

where $\theta^{[n]} := \theta(\theta+1)\cdots(\theta+n-1)$. This distribution has been investigated in several ways. See Arratia and Tavaré (1992), or Hoppe (1987). Kingman (1980) states that this distribution arise in many applications. There are other ways of deriving this distribution. We next examine some of its properties.

The number of clusters and value of θ

Ewens sampling formula has a single parameter θ. Its value influences the number of clusters formed by the agents. Smaller values of θ tends to produce a few large clusters, while larger values produce a large number of smaller clusters. To obtain some quick feels for the influences of the value of θ, take $n = 2$ and $a_2 = 1$. All other as are zero. Then

$$\pi_2(a_1 = 0, a_2 = 1) = \frac{1}{1+\theta}$$

This shows that two randomly chosen agents are of the same type with high probability when θ is small, and with small probability when θ is large. In fact, θ controls correlation between agents' types or classification. Furthermore, the next two expreme situations may convey the relation between the value of θ and the number of clusters. We note that the probability of n agents forming a single cluster is given by

$$\pi_n(a_j = 0, 1 \le j \le (n-1), a_n = 1) = \frac{(n-1)!\theta}{\theta^{[n]}}$$

while the probability that n agents form n singleton is given by

$$\pi_n(a_1 = n a_j = 0, j \ne 1) = \frac{\theta^{n-1}}{(\theta+1)(\theta+2)\cdots(\theta+n-1)}$$

With θ much smaller than one, the former probability is approximately equal to 1, while the latter is approximately equal to zero. When θ is much larger than n the opposite is approximately true.

We can show that

$$P_n(K_n = k) = \frac{1}{\theta^{[n]}} c(n,k)\theta^k$$

where $c(n,k)$ is known as the signless Stirling numbers of the first kind, and is defined by

$$\theta^{[n]} = \sum_1^n c(n,k)\theta^k.$$

See Hoppe (1987) for the derivation. Stirling numbers are discussed in van Lint and Wilson (1992, p.104) for example. Another class of interesting transition rates arise by applying what is called the Johnson's sufficientness postulate[6] in the statistical literature. In modeling industrial sector with ni being the number of agents of type i, the word type may refer to the kinds of goods being produced by firm i or n_i may refer to the size of the "production line", that is, a measure of capacity utilization by firm producing typ i good. Zabell (1982) proved that under the assumption of exchangeable partitions the functional form of f is specified by

$$f(n_i, n) = \frac{n_i}{n+\theta}$$

with some positive scalar parameter θ . Therefore, the entry rate of a new type is given by $\theta/(n+\theta)$. More generally, with K types, it is of the form

$$w(\mathbf{n}, \mathbf{n} + e_k) = \frac{\alpha + n_k}{K\alpha + n}$$

[6] Johnson's sufficientness postulate stipulates that the conditional probability that the next agent which enters is of type i, given the current state vector, is $f(n_i, n)$, that is, a function of the existing number of agents of type i and that of the total number of agents in the model. See Zabell (1982).

which reduces to (2) in the limit of α going to zero, and K to infinity while their product approaches θ, and

$$w(\mathbf{n}, \mathbf{n} - e_j) = \frac{n_j}{n}$$

See Costantini (1979, 2000), and Zabell (1982) for circumstances under which these transition rates arise. See Aoki and Yoshikawa (2001) and Aoki (2002a, Sec.8.6) for an application of this type of transition rates in models of economy or sectors of economy.

Densities of the large fractions

Aoki (2002a, Sec.10.6) gives expressions for the densities of the r largest fractions of clusters. In the case where the largest fraction x is greater than 1/2, its density is given by

$$p(x) = \frac{\theta}{x}(1 - x)^{\theta - 1}$$

For the largest two fractions x and y such that $y \geq (1 - x)/2$, the joint density is

$$f(x, y) = \frac{\theta^2}{xy}(1 - x - y)^{\theta - 1}$$

These are used in Aoki (2002b) to discuss asset returns in an asset market in which there are two dominant groups of agents, that is two largest clusters such that $x + y$ is about 0.7 or larger.

1.4 Gibbs Partitions

We can construct more complex combinatorial structures by introducing the notion of compound or internal states of a particular combinatorial structures. They could be called colors of structures, and may correspond to internal energies in the case of physical components.

The multiset $\{\mid A_1 \mid, \cdots \mid A_k \mid\}$, related to the sizes of k blocks of a partiton of $[n]$, of unordered sizes of blocks defines a partition of n. Earlier we have introduced two ways of representing or encoding these: non-increasing order of sizes, and partiton vectors.

Let V be some kind of combinatorial structures (called species in Pitman (2002)). As before $[n]$ is the set of n agents, objects or elements. Denote by $V([n])$ a set of V-structures such that the number of V-structures on $[n]$ is $\mid V([n]) \mid = v_n$.

Let W be another species of combinatorial structures. Let w_j be the number of W-structures on a set of j agents.

We construct the composite structure on $[n]$, $(VoW)([n])$, which is the set of all ways of partitioning $[n]$ into blocks $\{A_1, \cdots A_k\}$ for some $k = 1, 2, \cdots, n$, and assigning to each block A_i a W-structure. The number of composite structure is then

$$\mid (VoW)([n]) \mid := B_n(\mathbf{v}, \mathbf{w}) := \sum_{k=1}^{n} v_k B_{n,k}(\mathbf{w})$$

where

$$B_{n,k}(\mathbf{w}) := \sum \prod_{i=1}^{k} w_{|A_i|}$$

and where the sum is over all possible partitions of n agents into k clusters. Using the composition $n = n_1 + n_2 + \cdots + n_k$, we may write this as

$$B_{n,k}(\mathbf{w}) = \frac{n!}{k!} \sum \prod_i \frac{w_{n_i}}{n_i!}$$

where the sum is over $(n_1, n_2, \cdots, n_k)$. Since there are a_j of blocks of size j, and j agents can be arranged in $j!$ ways, and a_j blocks in $a_j!$ ways, we have

$$\left[\prod_j w_j^{a_j} \right] B_{n,k}(\mathbf{w} = \frac{n!}{\prod_j (j!)^{a_j} a_j!}$$

with $\sum_j = k \sum_j j a_j = n$. Here the notation $\left[x^j\right] f(x) = c_j$ means that the polynomial $f(x)$ has c_j as the coefficient of power x^j. The symbol $B_{n,k}()$ is known as the Bell polynomial in the combinatorics literature. We use the generating function

$$w(x) := \sum_{j=1}^{\infty} w_j \frac{x^j}{j!}$$

to write

$$B_{n,k}(\mathbf{w}) = \left[\frac{x^n}{n!} \right] \frac{w(x)^k}{k!}$$

For each partition of $[n]$

$$Pr(\prod_n = \{A_1, A_2, \cdots, A_k\}) = p(\mid A_1 \mid, \cdots (\mid A_k \mid\mid; \mathbf{v}, \mathbf{w}),$$

where

$$p(n_1, \cdots, n_k; \mathbf{v}, \mathbf{w}) = \frac{v_k \prod_i w_{n_i}}{B_n(\mathbf{v}, \mathbf{w})}$$

Here

$$B_n(\mathbf{v}, \mathbf{w}) = \sum_{k=1}^{n} v_k B_{n,k}((w).$$

We call $\prod_n$ a Gibbs ($\mathbf{v}$,$\mathbf{w}$) partition if the distribution of $\prod_n$ on the set of all partitions of $[n]$ is as given above. A random partition of n induced by a random partiton n of $[n]$ is represented by partition vector $\mathbf{a}$ with

$$Pr(\mid \pi_n \mid_j = a_j, j = 1, \cdots, n) = \frac{n! v_k}{B(\mathbf{v}, \mathbf{w})} \prod (\frac{w_j}{j!})^{a_j} \frac{1}{a_j!}$$

where $\mid \prod_n \mid_j$ is the number of blocks of size j.

Economic interpretation

Our economic interpretation is as follows: Suppose that n agents are partitioned into clusters such that each agent belongs to a unique cluster. The collection of clusters is represented by a partition of $[n]$.

Assume that each cluster of size j can be in any one of w_j different internal "states", or "color" $\mathbf{w} := (w_1, w_2, \cdots)$, where w_i is a non-negative integer.

Configuration of the system of n agents is a partition of the set $[n]$ into clusters, plus the assignment of an internal state to each cluster. For each partition of $[n]$ into k blocks of sizes $n_1, n_2, \cdots, n_k$, there are $\prod_i w_{n_i}$ different configurations.

When $v_k = 1$ for some k and zero elsewhere, the Gibbs partition corresponds to those in which all configurations with k clusters are equiprobable. This is the microcanonical configurations in physics. A general weight sequence $\mathbf{v}$ randomizes k to allow all probabilistic mixture over k of these microcanonical states.

Ewens distributions are derived as a special case of the above in Pitman (2002). Because of our interest in underlying dynamics generating the distributions, we have earlierr provided a jump Markov process derivation for them. Whittle used reversible equilibrium distribution of a Markov process to construct particualr cases of Gibbs partitions, Whittle (1986).

Entry of new types

Our view on economic growth is that growth is sustained by continual introduction of goods of new types which stimulate demands for these new goods, not by R & D activities which re ne existing goods, Aoki and Yoshikawa (2002).

New entries could be newly invented or improved goods, new business models, new behavioral patterns and so on. Law of succession in the statistical literature address these questions as conditional probabilities of agents entering models from outside being new or one of existing types in the model. Here we rely on recent works by Kingman and Pitman. Their models can be approximated as birth-immigration models in the context of continous time branching processes and we introduce their resuls into our models. See Feng and Hoppe (1998)for the mathematical set-up.

Let $X_1, \ldots X_n \ldots$ be an infinite sequence of random variables taking on any of a finite numbe of values, say 1, 2, $\cdots$ k. The subscripts on X are thought of time index, or the order in which samples are taken or agents enter the system.

The sequence is said to be *exchangeable* if for every n, the cylinder set probabilities

$$Pr(X_1 = j_1, \ldots X_n = j_n) = Pr(j_1, j_2, \cdots j_n)$$

are invariant under all possible permutations of the time index. Two sequences have the same probability if one is a rearrangement of the other, or the probability is the function of the frequency vector, $\mathbf{n} = (n_1, n_2, \ldots n_k)$. The observed frequency counts, $n_j = n_j(X_1, X_2, , \cdots X_n)$ are sufficient statistics for the sequence in the sense that probabilities conditional on the frequence counts depend only on the frequency vector

$$Pr(X_1, X_2, \ldots, X_n \mid \mathbf{n}) = \frac{n_1! n_2! \ldots n_k}{n!}$$

de Finetti theorem says that

$$Pr(X_1 = j_1, X_2 = j_2, \ldots, X_n = j_n) = \int p_1^{n_1} p_2^{n_2} \cdots p_1^{n_k} d\mu(p_1, p_2, \cdots, p_k),$$

over the simplex Δ_k of p_js which sum to one. Once the prior $d\mu$ is implicitly or explicitly specified, it is immediate that

$$Pr(X_{n+1} = j_1 \mid X_1, X_2, \ldots, X_n) = Pr(X_{n+1} = j_i \mid \mathbf{n}).$$

Such a conditional probability is sometimes called a rule of succession. Johnson's sufficientness postulate[7] is

$$Pr(X_{n+1} = i \mid \mathbf{n}) = f(n_i, N) :$$

If $X_1, X_2, \ldots$ is an exchangeable sequence satisfying the sufficientness postulate, and $k \geq 3$, then assuming that the relevant conditional probabilities exist

$$Pr(X_{n+1} = i \mid \mathbf{n}) = \frac{n_i + \alpha}{n + k\alpha} :$$

See Zabell (1982).

How different are the estimates of probability of new types with the Ewens distribution and multinomial distribution? Ewens (1996) has some numerical examples which show that they are quite different. With the multinomial approach, a_1 (the number of singleton) is critical. With the Ewens formula, the total number of types, $\sum a_j$ is only relevant.

[7] so called by I. J. Good to avoid confusion with the notion of sufficient statistics

The Pitman two-parameter model

Pitman (1992) generalized the Ewens' distribution by using the transition rates

$$w(n, n+e_j) = \frac{n_j - \alpha}{n+\theta}$$

where $\theta + \alpha > 0$. In terms of the rule of sucession it becomes

$$Pr(X_{n+1} \mid \mathbf{n}) = \frac{n_i - \alpha}{n + \theta - K_n\alpha} :$$

for α between 0 and 1, and θ positive, the conditional probability for a new type is

$$\Pr(X_{n+1} = \text{new}) = \frac{\theta}{n+\theta-K_n\alpha} :$$

Pitman (1995). With this, the conditional probability that a new type enters in the next Δ time interval is approximately given by $\frac{K_n\alpha+\theta}{n+\theta}\Delta$. Pitman also derived the equilibrium distribution for this two-parameter version.

The two-parameter Poisson-Dirichlet distribution, $PD(\alpha, \theta)$, for some α between 0 and 1 and $\theta > -\alpha$ is a probability distribution on the sequence of fractions V_n, with $V_1 > V_2 > \cdots$, and $\sum_n V_n = 1$. Let X_n, $n = 1, 2, \ldots$ be independent random variables with $Beta(1-\alpha, \theta + n\alpha$ distribution. Let U_n be the random variables with residual allocation, that is, $U_1 = X_1$; $U_2 = (1 - X_1)X_2$, ... Let V_n be the decreasing order statistics. This is Pitman's $PD(\alpha, \theta)$ distribution. When α is zero, it reduces to the Kingman $PD(\theta)$. This corresponds to the conditional probability

$$\Pr\ (\text{new type} \mid X_1, X_2, \ldots X_n) = \frac{\alpha+\theta}{n+\theta},$$

and

$$\Pr\ (\text{existing type} \mid X_1, X_2, \ldots X_n) = \frac{n-\alpha}{n+\theta},$$

In other words, when there are k clusters of types in the data, the probability of a new type appearing as the next observation is increased from the $\theta/(\theta+n)$ to $(\theta+k\alpha)/(\theta+n)$, and correspondingly the probability of observing type i next is reduced to $(n_i - \alpha)/(\theta + n)$. With the partition vector $\mathbf{a}$, the probability is

$$Pr(\mathbf{a} : \alpha, \theta) = \frac{1}{\theta^{[n]}} \prod_1^{K_n} (\theta + (i-1)\alpha) \prod_1^{n} \left[(1-\alpha)^{[j-1]}\right]^{a_j}$$

where $K_n = \sum a_j$, and $\sum ja_j = n$. There are interpretations in terms of what is called size-biased sampling of these, which is briefly mentioned below. See Kingman (1992), and Pitman (1992, 1995)

Urn models

It is most instructive to use urn models to describe the ways conditional probabilities of new types are constructed. Hoppe (1984) generalizes the Polya urn by introducing a special ball, the black ball say, of weight . All other balls of various colors have weight one. Initially, the urn contains just the black ball. So, at the first drawing, the black ball is drawn, and is returned to the urn, together with a ball of color 1, say. When a non-black ball is drawn in subsequent drawings it is returned together with another ball of the same color. When the black ball is drawn, it is returned togehter with a ball of color so far not seen. That is, a ball of new color signifies a new type of balls, that is agents. After n drawings, the urn contains balls of total weight $\theta + n$. Thus, the probability of a new type next is $\theta/(\theta + n)$, while that of drawing a ball of existing color is $n/(\theta + n)$. In (3), as α goes to zero, and $K_n\alpha$ approaches θ, Ewens (1996) discusses how the conditional probability assigned by the Ewnes distribution differs from the multinomial distributions.

Let a process $\{X_n\}$ generated by sampling from the urn. Let K_n be the number of colors in the urn.

Fix **a** and consider a possible sample path $\{X_1 = j_1, X_2 = j_2, \ldots, X_n = j_n\}$

$$Pr(X_1 = j_1, X_2 = j_2, \ldots, X_n = j_n) = \frac{\theta^{K_n} \prod_{j=1}^{n}(n_j - 1)!}{\theta^{[n]}}$$

Here $\theta^{[n]} := \theta(\theta + 1) \cdots (\theta + n - 1)$. Recall that the black ball has been selected K_n times, and that each ball of new color is followed $n_j - 1$ times by balls of the same color.

To count the number of samples, there are two constraints:

1. we observe that the first ball of color 1 precedes the first ball of color 2, which precedes the first ball of color 3, and so on.
2. We merely know that there are n_1 balls of some color, n_2 of another and so on.

To count the sample path subject to the above two constraints, arrange of occupancy numbers in decreasing order, $n_{(1)} \geq n_{(2)} \cdots n_{(K_n)}$. Suppose that there are p distinct integers in the set of the ordered frequencies. Define α_1 to be such that $n_{(i)} = n_{(1)}$, α_2 be such that $n_{(i)} = n_{(\alpha_1+1)}$, and so on. Finally, α_p is such that $n_{(i)} = n_{(\underbrace{K}_{n})}$.

There are $K_n!/\prod_{i=1}^{p} \alpha_i!$ ways of distributing $\{n_1, n_2, \ldots n_{K_n}\}$ among K_n types. For each such distribution of the occupancy numbers there are $n!/n_1! \cdots n_{K_n}$ permutations of labels agreeing with the occupancy numbers. In total, there are

$$\frac{K_n!n!}{\prod \alpha_i! \prod n_j!}$$

permutations which meets with constraint (2). Not all among these meet the constraint (1). Separate the permuta- tions into disjoint classes by the order of the first appearance of the digits $\{1, 2, \cdots K_n\}$. There are $K_n!$ such disjoint classes, all of the same cardinality by symmetry. Only one of them satis es (1). Thus dividing the above by $K_n!$, and multiplying by the probability of one sample, we have the probability of the random partition being given by $\mathbf{a}$. This is the Ewens sampling formula.

When we let k go to infinity, and α to zero in such a way that $k\alpha$ goes to a positive limit, we have

$$Pr(X_{n+1} = i \mid \mathbf{n}) = \frac{n_i}{n + \theta}$$

Then

$$Pr(X_{n+1} = new \mid \mathbf{n}) = \frac{\theta}{n + \theta}.$$

This leads us to the Ewens sampling formula, well known in the population genetics literature.

The notion of partition exchangeability generalizes the notion of exchangeble sequence by imposing exchangeability with respect to categories, as well as time indices. That is, a probability function P is partition exchangeable if the cylinder set probabilities $Pr(X_1 = j_1, \ldots X_n = j_n)$ are invariant under permutation of the time index and the type index. The role of frequency vector is now played by the partition vector $\mathbf{a}$ where a_i denotes the number of types with i entries, that is, the number of n_j that are equal to i.

The predictive probabilities for partition exchangeable probabiities will have the form

$$Pr(X_{n+1} = i \mid X_1, X_2, \ldots, X_n) = f(n_i \mid \mathbf{a}).$$

Sampling, residual allocation models, and distributions of order statistics

Given K categories or types, and the associated random probability vector

$$\mathbf{p} = (p_1, p_2, \ldots, p_K);$$

let ν be a random variable having values $1, 2, \ldots K$ such that

$$Pr(\nu = r) = p_r;$$

$r = 1, 2, \ldots, K$. This probability p_μ is said to be obtained from $\mathbf{p}$ by sizebiased sampling. This process is repeated by renormalizing the remaining probability by dividing it by $1 - p$ and proceeding as before.

The components of p is rearranged as a vector q, the components of which are expressible by

$$q_1 = v_1, q_2 = (1 - v_1)v_2, q_3 = (1 - v_1)(1 - v_2)v_3, \ldots$$

The random variables $v_1, v_2, \ldots$ are independent. For example, vs may have beta distributions. This last construction is an example of a residual allocation model, also known as a broken stick model. With the one-parameter size-biased samples, when the random vari- ables q_s are arranged in descending order $q_1 \leq q_2 \leq \ldots$, that is, reordered into the descending order statistics is distributed according to the Poisson-Dirichlet distribution invented by Kingman, where the random variables qs are Beta$(1, \theta)$, other random variables are used to generate a two-parameter generalization of the Ewens distribution by Pitman.

Economic Applications

A traditional approach to model two-sector economy

In 1989 Dixit has analyzed several economic problems, such as that of how to optimally allocate capital stocks among two sectors, and of assessing the effects of exchange rate changes to induce entries or exits of firms in some export industry. In a setting of a two-sector economy what Dixit derives is the price schedule, that is, the price as a function of the number of firms existing in one sector. When the relative price of the two goods crosses the price schedule from above or below, a move by one more firm into or from one sector to the other is triggered. This, however, is analyzed as a problem for a central planner of the economy, not as problems for individual firm managers. He does not say how a firm manager knows that it is his turn to enter or switch sectors. In spite of random prices his approach is basically deterministic. What are some of the objections to this analysis? First, as we already mentioned, there is no explanation about which of the firms decide to move. Also, there is apparently no uncertainty as to which firm switches. Problems of imperfect or incomplete information and externalities among firms (agents) are cleverly hidden or abstracted away in his analysis.

An Alternative Approach: Basic Setup

Aoki (2002b) gives one potential applications of the Ewens sampling formula in finance.

Aoki (1996, 2002a) presents several examples of some alterntive approaches to that sketched above. Basically, our approach focuses on the random partitions of the set of firms into clusers induced by subsets formed by firms of the same types, and utilizes the conditional probability specifications for new entries and exits to derive equilibrium distributions, when they exist, for cluster sizes. We use the master equation (backward Chapman-Kolomorov equation) as the dynamic equation for the probabilities of state vectors.[8] Given the total number of agents, N, and the number of possible types, K, both of which are assumed in this paper to be known and finite for ease of explanation, we examine how the N-set, that is, the set $\{1, 2, \ldots N\}$ is partitioned into K clusters, or subsets. This partition is treated as a random exchangeable partition in the sense of Zabell (1992).

Holding times

Jump Markov processes stay at each state it visits for a while, called sojourn or holding time, before it jumps to another state. Holding times are exponentially distributed. Given a number of agents wishing to jump, one with the minimum holding time actually can jump. This notion is applied in Aoki (2002a, Chapt. 8) to a model of economy with several sectors. Each sector faces a fraction of the aggregate outputs of the economy as its demand. In his model some agents are in excess supplies and others are in excess demands. Those in the excess supply conditions wish to reduce their production, and those in excess demands wish to expand the production. The aggregate outputs a ect demand conditions each agent faces. Therefore, as soon as one agent adjusts its output first, that changes the aggregate output, and possibly the demand conditions each agent faces. Consequently, sets of agents with positive or negative aggregate demand conditions generally change as one agent actually jump to a new state. The notion of holding time is therefore useful as a conceptual device in choosing which of the agents actually can carry out their intended decisions in condiitons with externalities.

Power Laws and volatility switching

Aoki (2000, 2002) has examples of an asset market in which two dominant clusters of agents trade. This market exhibits power laws for returns. When two types of agents switch between two strategies, returns exhibit switchings of volatility as well.

Sluggish responses of dynamics as power laws, that is, decay of the form $t^{-\alpha}$ for some positive α rather than exponential decay $e^{-\lambda t}$ are found in models in which states are arranged as trees and transition rates between states are the functions of ultrametric distance as in Aoki (1996, Sec. 7.1). As the number of layers of tree nodes increase, reduce transition rates appropriately. In the limit we obtain power laws decays, not exponential decays. See Ogielski and Stein (1985), or Aoki (1996, p. 157).

[8] In a closed two-sector model the scalar variable of the number of firms in sector one, say, serves as the state variable. In an open model with K sectors, a K-dimensional vector is used.

1.5 Concluding Remarks

This paper proposes a finitary approach to economic modeling, that is to start with a finite number of agents with discrete choice sets, and with explicit transition rates. It discusses several entry and exit transition rates in economic models. In particular, it presented Ewens and related distributions as candidates for distributions of cluster sizes formed by a large number of economic agents who interact in a market. This distribution seems to be very useful in economic modelings, although we have only a few examples so far. However, see Arratia and Tavaré (1992), and Kingman (1980). These and other investigations strongly suggest that the Ewens' and related distributions are robust and ubiquitous.

Carlton (1999) discusses some estimation issues of two-parameter Poisson-Dirichlet distribution.

Although no application is described in this paper, Aoki (2002a, 2002b) has one simple application in which stocks of a holding company is traded by a large number of agents. With $\theta =: 3$, two largest groups are shown to capture nearly 80 per cent of the market shares and hence dominate the market excess demands for the shares, which in turn determine the stationary distributions of price. In this way it is also possible to relate the tail distribution of the market clearing prices with entry and exit assumptions.

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