26 Algorithms and Combinatorics

Topics in Discrete Mathematics

Dedicated to Jarik Nešetřil on the Occasion of his 60th Birthday

M. Klazar
J. Kratochvíl
M. Loebl
J. Matoušek
R. Thomas
P. Valtr
Editors

Springer

Algorithms and Combinatorics 26

Martin Klazar · Jan Kratochvíl
Martin Loebl · Jiří Matoušek
Robin Thomas · Pavel Valtr

Editors

Topics in Discrete Mathematics

Dedicated to Jarik Nešetřil on the Occasion of his 60th Birthday

With 62 Figures

Springer

Editors

Martin Klazar
Jan Kratochvíl
Martin Loebl
Jiří Matoušek
Pavel Valtr

Department of Applied Mathematics
and Institute for Theoretical
Computer Science ITI
Faculty of Mathematics and Physics
Charles University, Prague
Malostranské nám. 25
118 00 Praha 1, Czech Republic

Robin Thomas

School of Mathematics
Georgia Institute of Technology
Atlanta, GA 30332-0160, USA

Library of Congress Control Number: 2006926241

Mathematics Subject Classification (2000):
05C10, 05C15, 05C30, 05C55, 05C65, 05C70, 05C85

ISSN 0937-5511

ISBN-10 3-540-33698-2 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-33698-3 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media

springer.com

Typesetting: by the authors, copyedited by Helena Nyklová using a Springer LaTeX macro package
Production: LE-TeX Jelonek, Schmidt & Vöckler GbR, Leipzig
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper 46/3100YL - 5 4 3 2 1 0

To our teacher, colleague and friend

Preface

The purpose of this book is twofold. We would like to offer our readers a collection of high quality papers in selected topics of Discrete Mathematics, and, at the same time, celebrate the 60th birthday of Jarik Nešetřil. Since our discipline has experienced an explosive growth during the last half century, it is impossible to cover all of its recent developments in one modest volume. Instead, we concentrate on six topics, those closest to Jarik's interests. We have invited leading experts and close friends of Jarik's to contribute to this endeavor, and the response has been overwhelmingly positive. We were fortunate to receive many outstanding contributions. They are divided into six parts.

Contents

The topics of the first part are rather diverse, including Algebra, Geometry, and Numbers and Games. Michael E. Adams and Aleš Pultr consider rigidity (lack of nontrivial homomorphisms) of algebraic structures, and they construct $2^{\aleph_0}$ rigid countable Heyting algebras. Vitaly Bergelson, Hillel Furstenberg, and Benjamin Weiss introduce a new notion of "large" sets of integers, piecewise-Bohr sets, and they show, in particular, that the sum of two sets of positive upper density is piecewise Bohr. Christopher Cunningham and Igor Kriz investigate a generalization of the Conway number games to more than two players and construct games with any given value. Miroslav Fiedler solves extremal geometric questions, namely, the shape of n-dimensional unit-volume simplices that maximize the length of a Hamilton cycle or path in their graph. The paper of Václav Koubek and Jiří Sichler in universal algebra concerns the relation of Q-universality and finite-to-finite universality of algebras. Christian Krattenthaler studies a simplicial complex associated to a colored root system, the generalized cluster complex, and proves a generalization of remarkable relations, discovered by Chapoton, concerning certain face counts.

Part II contains contributions in Ramsey theory. Ron Graham and József Solymosi give an elementary proof that an $n \times n$ integer grid colored by fewer than $\log \log n$ colors contains a monochromatic vertex set of an equilateral right triangle. András Gyárfás, Miklós Ruszinkó, Gábor N. Sárközy, and Endre Szemerédi use the regularity lemma for constructing coverings of edge-r-colored complete bipartite graphs by vertex-disjoint monochromatic cycles. Neil Hindman and Imre Leader consider a variant of partition-regularity of systems of linear equations, where they look for nonconstant solutions. Alexandr Kostochka and Naeem Sheikh construct infinitely many graphs for which the ratio of the induced Ramsey number to the weak induced Ramsey number is bounded away from 1, answering a question of Łuczak and Gorgol. Pavel Pudlák applies the recent Bourgain–Katz–Tao theorem on sums and products in finite fields to an explicit construction of 3-colorings of complete bipartite graphs with no large monochromatic complete bipartite subgraphs.

Topics in graph and hypergraph theory begin with Part III. József Balogh, Béla Bollobás, and Robert Morris consider the enumeration of ordered graphs not containing any ordered subgraph from a fixed (possibly infinite) set. The contribution of Zoltán Füredi, Kyung-Won Hwang, and Paul Weichsel is best described by its title: A proof and generalizations of the Erdős–Ko–Rado theorem using the method of linearly independent polynomials. Tomáš Kaiser, Daniel Král', and Serguei Norine prove that in any cubic bridgeless graph at least 60% of edges can be covered by two matchings, a result related to a conjecture of Berge and Fulkerson. Brendan Nagle, Vojtěch Rödl, and Mathias Schacht apply the hypergraph regularity method, a recent hypergraph generalization of the Szemerédi regularity lemma, to extremal problems for hypergraphs. Colin McDiarmid, Angelika Steger, and Dominic Welsh define addable graph classes, which include planar graphs and many other natural classes, and show that the probability of a random graph from such a class being connected is bounded away from both 0 and 1.

The papers in Part IV deal with graph homomorphisms. Noga Alon and Asaf Shapira survey the role of homomorphisms in recent results on constant-time probabilistic testing of graph properties. Christian Borgs, Jennifer Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi look at the number of homomorphisms $G \to H$ from various perspectives such as graph isomorphism, reconstruction, probabilistic property testing, and statistical physics. Josep Díaz, Maria Serna, and Dimitrios Thilikos investigate an algorithmic problem, the fixed-parameter complexity of testing the existence of a homomorphism $G \to H$, where H is fixed, G is the input, and the number of preimages of certain vertices of H is restricted. Pavol Hell considers the Dichotomy Conjecture, stating that every class of constraint satisfaction problems specified by a fixed relational structure H is either polynomial-time solvable or NP-complete, establishes special cases, and connects the problem to graph colorings.

Part V is concerned mostly with generalized graph colorings. Those in the paper by Glenn Chappell, John Gimbel, and Chris Hartman are path-colorings of planar graphs. Dwight Duffus, Vojtěch Rödl, Bill Sands, and Norbert Sauer consider the minimum chromatic number of graphs and hypergraphs of large girth that cannot be homomorphically mapped to a specified graph or hypergraph, obtaining a new probabilistic hypergraph construction in the process. Mickaël Montassier, André Raspaud, and Weifan Wang prove acyclic 4-choosability of planar graphs with excluded cycles of certain lengths. Xuding Zhu presents an authoritative survey of the circular chromatic number, a parameter introduced by Vince in 1988 that carries more information than the chromatic number itself. The contribution of Claude Tardif sticks to the usual chromatic number and provides an algorithmic version of a special case of the celebrated Hedetniemi conjecture.

Part VI on graph embeddings opens with the paper by Hubert de Fraysseix and Patrice Ossona de Mendez, who consider embeddings of multigraphs in the k-dimensional Euclidean space such that automorphisms correspond to isometries and present an elegant characterization of such embeddings. Bojan Mohar extends an intriguing result of Youngs on quadrangulations of the projective plane, and constructs the first explicit family of infinitely many 5-critical graphs on a fixed surface. János Pach and Géza Tóth relate the torus crossing number of a graph to the planar crossing number. The survey of Jozef Širáň deals with the classification of regular maps (maps possessing the highest level of symmetry – their automorphism groups act transitively on the set of flags) and explains its intriguing connections to other branches of mathematics.

Presented in a part of its own comes the last article written by Jørgen Bang-Jensen, Bruce Reed, Mathias Schacht, Robert Šámal, Bjarne Toft and Ulrich Wagner about six problems posed by Jarik Nešetřil and their current status. This last paper is just a small example of the enormous influence Jarik has had on other researchers.

Dedication

Jarik Nešetřil is a scientist and an artist of extraordinary breadth and vision. His publication record and other achievements, including over a half-dozen textbooks and monographs, an honorary doctorate and an academy membership, speak for themselves. Equally important is Jarik's tireless work with students and younger colleagues. He founded the Prague Combinatorics Seminar, which helped shape the careers of several generations of Czech mathematicians and computer scientists. Among them, the present editors greatly benefited from Jarik's guidance, ideas, and endless enthusiasm. We would like to express our appreciation and wish him many more productive years filled with success and satisfaction.

Acknowledgement. Many people helped us with this volume. We are indebted to the referees, who generously gave their time and effort in order to improve the presentation of the contributions. In preparation of the final version we were greatly assisted by our technical editor Helena Nyklová, whose meticulous copyediting is warmly appreciated. We also thank Ms. J. Borkovcová for her kind permission to reprint the photograph of Jarik. Finally, we would like to thank the Institute for Theoretical Computer Science and Department of Applied Mathematics of Charles University for their support.[1]

Prague and Atlanta,
13th March 2006

Martin Klazar
Jan Kratochvíl
Martin Loebl
Jiří Matoušek
Robin Thomas
Pavel Valtr

[1] In the framework of Czech research grants 1M0021620808 and MSM0021620838.

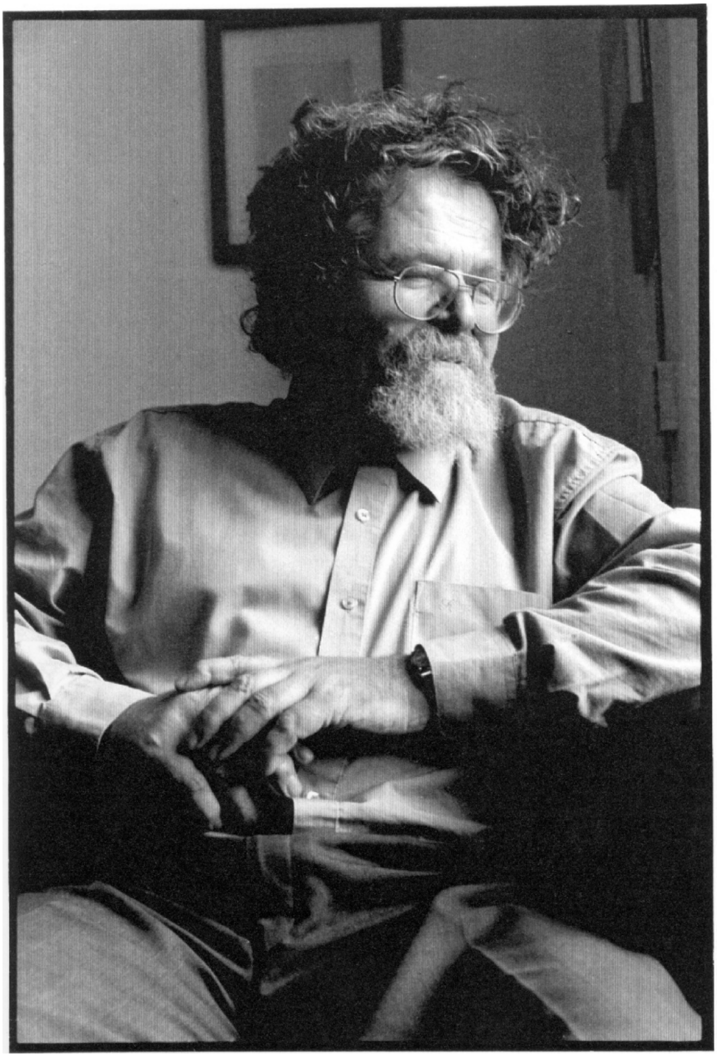

Photo of Jarik Nešetřil by Stanislav Tůma, © J. Borkovcová

Contents

Part V Graph Colorings

Part VI Graph Embeddings

Part VII

Part I

Algebra, Geometry, Numbers

Countable Almost Rigid Heyting Algebras

Michael E. Adams[1] and Aleš Pultr[2]*

[1] Department of Mathematics, State University of New York, New Paltz, NY 12561, USA
adamsm@newpaltz.edu

[2] Department of Applied Mathematics and ITI, MFF, Charles University, Malostranské nám. 25, CZ 11800 Praha 1, Czech Republic
pultr@kam.ms.mff.uni.cz

Summary. For non-trivial Heyting algebras H_1, H_2 one always has at least one homomorphism $H_1 \to H_2$; if $H_1 = H_2$ there is at least one non-identical one. A Heyting algebra H is *almost rigid* if $|\,\mathsf{End}(H)| = 2$ and a system of almost rigid algebras $\mathcal{H}$ is said to be *discrete* if $|\,\mathsf{Hom}(H_1, H_2)| = 1$ for any two distinct $H_1, H_2 \in \mathcal{H}$. We show that there exists a discrete system of 2^ω countable almost rigid Heyting algebras.

AMS Subject Classification. 06D20, 18A20, 18B15.
Keywords. Heyting algebras, almost rigid, discrete system, Priestley duality.

Introduction

A *Heyting algebra*

$$(H; \vee, \wedge, \to, 0, 1)$$

is an algebra of type $(2, 2, 2, 0, 0)$ where $(H; \vee, \wedge, 0, 1)$ is a distributive $(0,1)$-lattice and the extra operation $x \to y$ satisfies the formula

$$z \leq x \to y \quad \text{iff} \quad x \wedge z \leq y.$$

That is to say, a Heyting algebra is a bounded relatively pseudocomplemented distributive lattice for which relative pseudocomplementation is taken to be a binary algebraic operation.

Since any two elements of a finite distributive lattice have a uniquely determined relative pseudocomplement, any finite distributive lattice can be viewed as a Heyting algebra. So too, any two elements of a Boolean algebra

* The second author would like to express his thanks for the support by the project LN 00A056 of the Ministry of Education of the Czech Republic.

have a uniquely determined relative pseudocomplement $a \to b = \neg a \vee b$ which also provides an example of a Heyting algebra.

For an algebra A, let $\mathsf{Aut}(A)$ (resp $\mathsf{End}(A)$) denote the group of automorphisms (resp. the monoid of endomorphisms) of A under the operation of composition. An algebra is *automorphism rigid* provided $|\mathsf{Aut}(A)| = 1$, that is, the only automorphism of A is the identity.

Independently, Jónsson [Jón51], Katětov [Kat51], Kuratowski [Kur26], and Rieger [Rie51] have shown that there exists a proper class of non-isomorphic automorphism rigid Boolean algebras. Since, as observed above, every Boolean algebra is relatively pseudocomplemented, there exists a proper class of non-isomorphic automorphism rigid Heyting algebras as well.

In sharp contrast with respect to their automorphism groups, as independently shown by Magill [Mag72], Maxson [Max72], and Schein [Sch70], Boolean algebras are uniquely recoverable from their endomorphism monoids. That is, for Boolean algebras B and B', if $\mathsf{End}(B) \cong \mathsf{End}(B')$, then $B \cong B'$; or, by the result of Tsinakis ([Tsi79]), for bounded relative Stone lattices which are principal, $\mathsf{End}(S) \cong \mathsf{End}(S')$ implies $S \cong S'$ as well. However, this is far from the case for general Heyting algebras, where endomorphisms can be very few.

There are necessary non-identical homomorphisms, though. Every non-trivial Heyting algebra has at least one minimal prime ideal. Furthermore, for each minimal prime ideal I of a Heyting algebra H, and any other Heyting algebra H', $\varphi(x) = 0$ for $x \in I$ and 1 otherwise determines an homomorphism $\varphi : H \to H'$. Such homomorphisms will be referred to as

trivial homomorphisms.

In particular, if $|H| \geq 3$, then $|\mathsf{End}(H)| \geq 2$. Dismissing the necessary trivial endomorphisms one defines an *almost rigid* Heyting algebra H as such that $|\mathsf{End}(H)| = 2$. Thus, by the preceding remarks, H is almost rigid if and only if $|H| \geq 3$, if H has exactly one minimal prime ideal and the only endomorphism other than the identity is associated with the minimal prime as indicated.

In [AKS85] it was shown that there exists a proper class of non-isomorphic almost rigid Heyting algebras. All the almost rigid Heyting algebras from [AKS85] have cardinality $\geq 2^\omega$. Taking into account that for $|H| \geq 4$ there is no almost rigid finite Heyting algebra (for any such H either there are at least two minimal prime ideals I and I' or else a minimal prime ideal I and another prime ideal I' which is minimal with respect to properly containing I; in the former case obviously $|\mathsf{End}(H)| \geq 3$, in the latter case, for $a \in I' \setminus I$, $\psi(x) = 0$ for $x \in I$, a for $x \in I' \setminus I$, and 1 otherwise determines an endomorphism $\psi \in \mathsf{End}(H)$ distinct from φ associated with I, and $|\mathsf{End}(H)| \geq 3$ again) this begs the question whether there are countable almost rigid Heyting algebras. This is answered in the positive in this article. Moreover, we show that

there exists a system $\mathcal{H}$ of 2^ω countable almost rigid Heyting algebras such that

- *for each A in $\mathcal{H}$, $\mathsf{End}(H) = \{\mathrm{id}_A, \tau_{AA}\}$, and*

– *for any two distinct* A, B *in* $\mathcal{H}$ *there is exactly one homomorphism* τ_{AB} : $A \to B$,

where the τ_{AB} *are unique trivial homomorphisms.*

For related background on Heyting or Boolean algebras see Balbes and Dwinger [BD74] or Koppelberg [Kop89].

1 Preliminaries

Let $(P, \leq)$ be a partially ordered set. For $Q \subseteq P$, set $\downarrow Q = \{x \in P \mid x \leq y \text{ for some } y \in Q\}$ and $\uparrow Q = \{x \in P \mid x \geq y \text{ for some } y \in Q\}$; for $Q = \{x\}$ we write just $\downarrow x$ and $\uparrow x$, respectively. A set Q is said to be *decreasing* or *increasing* if $Q = \downarrow Q$ or $Q = \uparrow Q$, respectively. For partially ordered sets P and P', a mapping $\varphi : P \longrightarrow P'$ is *order-preserving* providing $\varphi(x) \leq \varphi(y)$ whenever $x \leq y$.

A *Priestley space* $(P; \leq, \tau)$ is a partially ordered set $(P, \leq)$ endowed with a compact topology τ which is totally order-disconnected (namely, for any $x, y \in P$ such that $x \not\leq y$ there exists a clopen decreasing set $Q \subseteq P$ such that $x \notin Q$ and $y \in Q$).

As shown by Priestley [Pri70], [Pri72], the category of non-trivial distributive $(0,1)$-lattices together with all $(0,1)$-lattice homomorphisms is dually isomorphic to the category of Priestley spaces together with all continuous order-preserving maps. The equivalence functors are usually given as

$$\mathcal{P}(L) = \{x \mid L \neq x \text{ a prime ideal of } L\}, \quad \mathcal{P}(h)(x) = h^{-1}[x],$$
$$\mathcal{D}(X) = \{U \mid U = \downarrow U \subseteq X \text{ clopen }\}, \quad \mathcal{D}(f)(U) = f^{-1}[U];$$

$\mathcal{P}(L)$ is endowed with a suitable topology and ordered by inclusion.

Since every Heyting algebra is a distributive $(0,1)$-lattice, it is to be expected that the category of all non-trivial Heyting algebras is dually isomorphic to a well-defined subcategory of the category of all Priestley spaces. And indeed, the Priestley spaces X dual to Heyting algebras are precisely those with the additional property that $\uparrow U$ is clopen whenever U is clopen. Such Priestley spaces will be called

h-spaces,

and if L, M are Heyting algebras then the Heyting homomorphisms $h : L \to M$ correspond to the Priestley maps f such that, moreover,

$$f(\downarrow x) = \downarrow f(x).$$

Such maps will be referred to as

h-maps.

It is this dual equivalence that we will use in order to establish our result.

2 The Construction

The Posets

Set $X = \{n \in \mathbb{N} \mid n \geq 5\}$ and decompose this set as follows. Start with

$$X_1 = \{5\}, \quad \phi(2) = 5$$
$$X_2 = X_{2,1} = \{6, 7, 8, 9, 10\}, \quad \phi(3) = 10,$$

and further proceed inductively: if $X_k = \{\phi(k) + 1, \phi(k) + 2, \ldots, \phi(k + 1)\}$ is already defined (and, hence, $\phi(k)$ and $\phi(k + 1)$ too), take, for each element $\phi(k) + j \in X_k$, a set $X_{k+1,j}$ determined as follows

$X_{k+1,1} = \{\phi(k+1)+1, \ldots, \phi(k+1)+\phi(k)+1\}$, the first $\phi(k)+1$ natural numbers after $\phi(k + 1)$,
$X_{k+1,2} = \{\phi(k+1)+\phi(k)+2, \phi(k+1)+\phi(k)+3, \ldots, \phi(k+1)+2\phi(k)+3\}$, the next $\phi(k) + 2$ natural numbers after $\phi(k + 1) + \phi(k) + 1$,
where, in general, for $1 \leq j \leq \phi(k + 1) - \phi(k)$,
$X_{k+1,j} = \{\phi(k + 1) + (j - 1)\phi(k) + \binom{j}{2} + 1, \ldots, \phi(k + 1) + j\phi(k) + \binom{j+1}{2}\}$, the next $\phi(k) + j$ natural numbers after $\phi(k + 1) + (j - 1)\phi(k) + \binom{j}{2}$.

Then set

$$X_{k+1} = \{\phi(k + 1) + 1, \phi(k + 1) + 2, \ldots, \phi(k + 2)\} = \bigcup_{j=1}^{\phi(k+1)-\phi(k)} X_{k+1,j}.$$

For triples x, y, z of distinct elements belonging to the same X_k choose distinct

$$\tau(x, y, z) \notin X$$

and set

$$T = \{\tau(x, y, z) \mid x, y, z\}$$

and

$$Y = X \cup T \cup \{\omega\}$$

where ω is an element $\notin X \cup T$.

Now choose a countably infinite system $\mathbb{Q}$ of quadruples $\{x_1, x_2, x_3, x_4\}$ such that

- for every $q = \{x_1, x_2, x_3, x_4\}$, $q \subseteq X_k$ for some k, and
- if $p, q \in \mathbb{Q}$, $p \neq q$, then $p \cap q = \emptyset$.

For $A \subseteq \mathbb{Q}$ set

$$Z(A) = Y \cup A$$

and define an order $\sqsubseteq$ on $Z(A)$ by

$$\omega \sqsubseteq x \text{ for all } x \in Z(A),$$

and by transitivity from the successor relation $\prec$ where

$$x, y, z \prec \tau(x, y, z),$$
$$x_i \prec \{x_1, x_2, x_3, x_4\} \text{ for } \{x_1, x_2, x_3, x_4\} \in A,$$
$$\text{and for } x \in X_{k+1,j}, \quad x \prec \phi(k) + j.$$

Note that

$\uparrow x$ is finite for all $x \in Z(A) \setminus \{\omega\}$.

To simplify the notation define the *degree*

$$d\,(n) = n \quad \text{for} \quad n \in X,$$
$$d\,(\tau(x, y, z)) = 3,$$
$$d\,(q) = 4 \text{ for } q \in A;$$

for ω the degree is undefined.

The Topology

$Z(A)$ is endowed with the topology in which
U is open iff either $\omega \notin U$ or $Z(A) \setminus U$ is finite.
Thus we have

Observation 2.1. *The clopen sets are precisely the finite M not containing ω and the complements of such sets,and hence the $\uparrow M$ with clopen M are clopen.*
If x is not $\sqsubseteq y$ then $y \notin \uparrow x$ and $\uparrow x$ is clopen. Thus,
each $Z(A)$ with the order $\sqsubseteq$ and the topology just defined is an h-space.

From this we immediately obtain

Fact 2.2. *All the Heyting algebras $\mathcal{D}(Z(A))$ are countable.*

In the sequel, f will always be an h-map $Z(A) \to Z(B)$.

Lemma 2.3. *1. $f(\omega) = \omega$.*
2. If $f(M) = \{x\}$ for an infinite $M \subseteq Z(A)$ then $x = \omega$.

Proof. 1. $\{f(\omega)\} = f(\downarrow\omega) = \downarrow f(\omega)$; hence $\downarrow f(\omega)$ has just one element.
2. For an infinite M we have $\omega \in \overline{M}$. Thus, $\omega = f(\omega) \in f(\overline{M}) \subseteq \overline{f(M)} = \{x\}$. ◻

A *branch* of an element $x \in Z(A)$ is any $\downarrow y$ with $y \prec x$. Note that the degree $d(x)$ defined above is the number of distinct branches of x.

Lemma 2.4. *1. If $t = \tau(x_1, x_2, x_3)$ (resp. $q = \{x_1, x_2, x_3, x_4\} \in A$) and $f(x_i) \sqsubset f(t)$ (resp. $f(q)$) for all i then we cannot have $d(f(t)) \geq 4$ (resp. $d(f(q)) \geq 5$).*

2. *If $t = \tau(x_1, x_2, x_3)$ and $f(x_i) \sqsubset f(t)$ for all i then we cannot have two $f(x_i), f(x_j)$, $i \neq j$ in the same branch $\downarrow y$ of $f(t)$.*
3. *If $a_1, a_2, a_3 \prec a \in Z(A)$ are distinct, $f(a_1) \sqsubseteq f(a_2) \sqsubset f(a)$ and $f(a_3) \sqsubset f(a)$ then all the $f(a_i)$ are in the same branch of $f(a)$.*

Proof. 1. $\downarrow f(t) = f(\downarrow t) = \{f(t)\} \cup \downarrow f(x_1) \cup \downarrow f(x_2) \cup \downarrow f(x_3)$ (resp. $\{f(q)\} \cup \downarrow f(x_1) \cup \downarrow f(x_2) \cup \downarrow f(x_3) \cup \downarrow f(x_4)$) and hence $\downarrow f(t)$ cannot have more than three (resp. four) branches.
2. $\downarrow f(t)$ consists of at least three branches and hence it cannot be covered by $\downarrow y$ and just one more branch.
3. By 2, $a \in X \cup A$. Consider $t = \tau(a_1, a_2, a_3)$. By 2, $f(t) = f(a_2)$ or $f(t) = f(a_3)$. Then either $f(a_3) \sqsubseteq f(a_2)$ or $f(a_2) \sqsubseteq f(a_3)$. ◻

Observation 2.5. *The map*

$$const_\omega = const_\omega^{AB} : Z(A) \to Z(B)$$

defined by $const_\omega(x) = \omega$ for all $x \in Z(A)$ is an h-map.

(This is the Priestley image of the unique trivial homomorphism between the corresponding Heyting algebras, each of which has precisely one minimal prime ideal.)

3 The Result

Lemma 3.1. *For $a \neq \omega$ such that $f(a) \neq \omega$ one cannot have $d(f(a)) < d(a)$.*

Proof. If $f(b) \sqsubset f(a)$ for all $b \prec a$ we are led to a contradiction by 2.4.3: let C consist of all the $c \prec f(a)$. Then there have to be two distinct b_1, b_2 with $f(b_i) \sqsubseteq c$ for some $c \in C$ and we have an $x \sqsubset a$ such that $c = f(x)$. Now $x \sqsubseteq b$ for some $b \prec a$ and we have $f(b_1), f(b_2) \sqsubseteq f(b)$ (of course, b can be one of the b_i). Now by 2.4.3 all the $f(b)$ with $b \prec a$ are in the same branch of $f(a)$, a contradiction.

Thus, there is an $a_1 \prec a_0 = a$ such that $f(a_1) = f(a)$ and as $d(a_1) > d(a_0)$ we can repeat the procedure to obtain

$$a_0 \succ a_1 \succ a_2 \succ \cdots \quad \text{with } f(a_i) = f(a),$$

and by 2.3.2 we have $f(a) = \omega$ contradicting the assumption. ◻

Thus in particular

$$f[X \cup \{\omega\}] \subseteq X \cup \{\omega\}.$$

In the following four lemmas, f is, as before, an h-map $Z(A) \to Z(B)$, but since all the facts are relevant for the restriction $X \cup \{\omega\} \to X \cup \{\omega\}$ only (and since $\downarrow (X \cup \{\omega\}) = X \cup \{\omega\}$), we can use expressions such as $f(a) = a$, $f(f(a))$, or $f(\downarrow a) = \downarrow f(a)$.

Lemma 3.2. *If $a \in X$ and $f(a) \neq \omega$ then $f(a) \sqsubseteq a$ and $f(f(a)) = f(a)$. If $f(a) \sqsubset a$, we have an $a' \sqsubseteq a$ such that $a' \succ f(a) = f(a')$.*

Proof. We already know that $d(f(a)) \geq d(a)$, hence if $f(a) \neq \omega$ we have $f(a) \in X$. Suppose that $d(f(a)) > d(a)$. Then we cannot have $f(b) \sqsubset f(a)$ for all $b \prec a$ since in such a case

$$f(\downarrow a) = \{f(a)\} \cup \bigcup \{f(\downarrow b) \mid b \prec a\}$$

cannot cover $\downarrow f(a)$.

Hence for some $a_1 \prec a$ we have $f(a_1) = f(a)$. Now $d(a_1) > d(a)$. If we still have $d(f(a)) > d(a_1)$, we can repeat the argument and ultimately we obtain $a = a_0 \succ a_1 \succ \cdots \succ a_k$ with $f(a_i) = f(a)$, $d(a_i) < d(f(a))$ for $i < k$, and $d(a_k) = d(f(a))$ (by 3.1 we cannot have $d(a_k) > d(f(a))$). Since $d(a_k) \geq 5$, a_k and $f(a) = f(a_k)$ are in X and hence $a_k = f(a_k)$ by the equality of the degrees. ◻

Lemma 3.3. *If for an $a \in X$ one has $f(a) = a$ then f is identical on the whole of $\downarrow a$.*

Proof. Let $x \sqsubset a$ be an element with the shortest path $x \prec x_1 \prec \cdots \prec a$ such that $f(x) \sqsubset x$. As $\downarrow a = f(\downarrow a)$, $x = f(y) \sqsubset y$ for some $y \sqsubset a$. But then y is one of the x_i which is a contradiction, by Lemma 3.2. ◻

Lemma 3.4. *If there is an $x \in X$ such that $f(x) = b \sqsubset x$ then there is a $y \in X$ such that $f(y) = u \sqsubset y$ and b, u are incomparable.*

Proof. $f(b) = b$ and hence f is identical on $\downarrow b$. Choose $b_1 \neq b_2$, $b_i \prec b$; thus $f(b_i) = b_i$. Let $b_1, b_2 \in X_k$. Choose a $y \in X_k$ incomparable with b (since $b \sqsubset x \sqsubseteq 5$ there exist such incomparable elements). Set

$$t = \tau(b_1, b_2, y).$$

Now $b_i = f(b_i) \sqsubset f(t)$ (we cannot have an equality as b_i are incomparable) and hence $f(y) \neq \omega$ (else $f(\downarrow t) = \{f(t)\} \cup \downarrow b_1 \cup \downarrow b_2 \neq \downarrow f(t)$). Thus, $f(y) \sqsubseteq y$. We cannot have $f(y) = y$ for all such y: in such a case f would be identical on the whole of X_k which would fix all the elements above as well, including 5 and x, contradicting the assumption (if $f(a) = a$ for all $a \prec n \in X$ then $f(n) \sqsupseteq a$ for all $a \prec n$ and hence $f(n) \sqsupseteq n$; we cannot have $f(n) \sqsupset n$ though, since that would imply $d(f(n)) < d(n)$).

Thus there has to be some such y with $u = f(y) \sqsubset y$, and now u is incomparable with b. ◻

Lemma 3.5. *For $\omega \neq x \in X$ one cannot have $f(x) \neq \omega$ and $d(f(x)) > d(x)$.*

Proof. Suppose there is such an x. By 3.2 and 3.4 we can choose an instance of $b \prec a$ such that $f(a) = f(b) = b$ and that there exists a $u \in X$ incomparable with b such that $f(u) = u$ is in an X_l with $l \leq k$ where b is in X_k (this can be achieved but exchanging the b and u in 3.4 if necessary). Consider a $c \prec a$, $c \neq b$ and a general $z \neq b, c$ in X_k. Set $t = \tau(b, c, z)$. Since $f(c) \sqsubseteq f(a) = f(b)$ we cannot have (see 2.4.2)

$$f(b), f(c), f(z) \sqsubset f(t).$$

Now $f(t)$ cannot be equal to $f(z)$ and distinct from the others since then $b = f(b) \sqsubset f(t) = f(z)$ and hence, z being in the same X_k as b, $d(f(z)) < d(z)$ contradicting 3.1. Thus we have $f(t)$ equal to either $f(c)$ or $f(b)$ and hence $f(z) \sqsubseteq f(b) = b$.

Thus, $f(X_k) \subseteq\downarrow b$. Take a $v \sqsubseteq u$ in X_k. Then $f(v) = v$ by 3.3 and we have a contradiction: v cannot be in $\downarrow b$ since u and b are incomparable and the subposet $(X, \sqsubseteq)$ of $Z(A)$ is a tree. □

Lemma 3.6. *Let $f : Z(A) \to Z(B)$ be an h-map. Then either $f(X) = \{\omega\}$ or $f(n) = n$ for all $n \in X$.*

Proof. By 3.5 and 3.1, $f(5) = \omega$ or $f(5) = 5$. Hence $f(5) = 5$ and, by 3.3, f is identical on $X =\downarrow 5$. □

Theorem 3.7. *Let $f : Z(A) \to Z(B)$ be an h-map. Then either f is* const_ω *or it is the inclusion map $Z(A) \subseteq Z(B)$.*

On the other hand, any inclusion $A \subseteq B$, with $A, B \subseteq \mathbb{Q}$ can be extended to an inclusion h-map $Z(A) \subseteq Z(B)$.

Proof. If $f(5) = \omega$ then $f(X) = \{\omega\}$ by monotonicity. Now if for $y \in T$ or $y \in A$ one should have $f(y) = x \neq \omega$ we had $\downarrow x \neq f(\downarrow y) = \{f(y)\} \cup \{\omega\}$. Thus, also $f(y) = \omega$.

If $f(5) \neq \omega$ then f is identical on X. This also fixes T, since for $t = \tau(x, y, z)$, $x, y, z \sqsubseteq f(t)$ and equality is impossible since x, y, z are incomparable, and since any other element greater then all the x, y, z has too many branches to be covered by $f(\downarrow t)$. Finally for $q = \{x_1, x_2, x_3, x_4\} \in A$ one cannot have $f(q) \in T$ by 3.1. Since the x_i are incomparable, $f(q)$ coincides with none of the $f(x_i) = x_i$ and hence, by 2.4.1, $f(q) \notin X$. By monotonicity, $f(q) \neq \omega$ and hence $f(q) \in B$. But there is only one $p \in \mathbb{Q}$ such that $x_i \sqsubset p$ for $i = 1, 2, 3, 4$, namely q itself. The second statement is obvious. □

As above, denote by $\mathbb{N}$ the set of all natural numbers.

Theorem 3.8. *There exist countable almost rigid Heyting algebras $H(A)$ associated with the subsets $A \subseteq \mathbb{N}$ such that*

- *if $A \nsubseteq B$ there is no non-trivial homomorphism $H(A) \to H(B)$, and*
- *if $A \subseteq B$ there exists exactly one non-trivial homomorphism $H(A) \to H(B)$.*

Consequently there exist 2^ω countable almost rigid Heyting algebras such that the only homomorphism between any two distinct of them is the trivial one.

Proof. The first part immediately follows from 3.7 and 2.2.

For the second statement it suffices to observe that there are 2^ω many subsets of $\mathbb{N}$ such that no two of them are in inclusion.

For any $N \subseteq \mathbb{N}$ consider the set

$$\widetilde{N} = \{2n \mid n \in N\} \cup \{2n+1 \mid n \notin N\}.$$

Then $\widetilde{N}_1 \subseteq \widetilde{N}_2$ only if $N_1 = N_2$. ◻

References

[AKS85] M. E. Adams, V. Koubek, and J. Sichler, *Endomorphisms and homomorphisms of Heyting algebras*, Algebra Universalis 20 (1985), 167–178.

[BD74] R. Balbes, P. Dwinger, *"Distributive Lattices"*, University of Missouri Press, Columbia, Missouri, 1974.

[Jón51] B. Jónsson, *A Boolean algebra without proper automorphisms*, Proc. Amer. Math. Soc. 2 (1951), 766–770.

[Kat51] M. Katětov, *Remarks on Boolean algebras*, Colloq. Math. 2 (1951), 229–235.

[Kop89] S. Koppelberg, *"Handbook of Boolean Algebras I"*, North-Holland, Amsterdam, 1989.

[Kur26] C. Kuratowski, *Sur la puissance de l'ensemble des "nombres de dimensions" de M. Fréchet*, Fund. Math. 8 (1926), 201–208.

[Mag72] K. D. Magill, *The semigroup of endomorphisms of a Boolean ring*, Semigroup Forum 4 (1972), 411–416.

[Max72] C. J. Maxson, *On semigroups of Boolean ring endomorphisms*, Semigroup Forum 4 (1972), 78–82.

[Pri70] H. A. Priestley, *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. 2 (1970), 186–190.

[Pri72] H. A. Priestley, *Ordered topological spaces and the representation of distributive lattices*, Proc. London Math. Soc. **324** (1972), 507–530.

[Rie51] L. Rieger, *Some remarks on automorphisms of Boolean algebras*, Fund. Math. 38 (1951), 209–216.

[Sch70] B. M. Schein, *Ordered sets, semilattices, distributive lattices, and Boolean algebras with homomorphic endomorphism semigroups*, Fund. Math. 68 (1970), 31–50.

[Tsi79] C. Tsinakis, *Brouwerian Semilattices Determined by their Endomorphism Semigroups*, Houston J. Math. 5 (1979), 427–436.

Piecewise-Bohr Sets of Integers and Combinatorial Number Theory

Vitaly Bergelson[1⋆], Hillel Furstenberg[2], and Benjamin Weiss[3]

[1] Department of Mathematics, Ohio State University, Columbus, OH 43210, USA
vitaly@math.ohio-state.edu
[2] Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel
harry@math.huji.ac.il
[3] Institute of Mathematics, Hebrew University of Jerusalem, Jerusalem, Israel
weiss@math.huji.ac.il

Summary. We use ergodic-theoretical tools to study various notions of "large" sets of integers which naturally arise in theory of almost periodic functions, combinatorial number theory, and dynamics. Call a subset of **N** a Bohr set if it corresponds to an open subset in the Bohr compactification, and a piecewise Bohr set (PWB) if it contains arbitrarily large intervals of a fixed Bohr set. For example, we link the notion of PWB-sets to sets of the form A+B, where A and B are sets of integers having positive upper Banach density and obtain the following sharpening of a recent result of Renling Jin.

Theorem. If A and B are sets of integers having positive upper Banach density, the sum set A+B is PWB-set.

AMS Subject Classification. 37A45.
Keywords. Bohr sets, Banach density, measure preserving system, Kronecker factor.

1 Introduction to Some Large Sets of Integers

In combinatorial number theory, as well as in dynamics, various notions of "large" sets arise. Some familiar notions are those of sets of positive (upper) density, syndetic sets, thick sets (also called "replete"), return-time sets (in dynamics), sets of recurrence (also known as Poincaré sets), (finite or infinite) difference sets, and Bohr sets. We will here introduce the notion of "piecewise-Bohr" sets (or PWB-sets), as well as "piecewise-Bohr$_0$" sets (or PWB$_0$-sets), and we'll show how they arise in some combinatorial number-theoretic questions.

We begin with some basic definitions and elementary considerations. We'll say that a subset $A \subset \mathbb{Z}$ has *positive upper* (Banach) *density*, $d^*(A) > 0$,

⋆ V. Bergelson was partially supported by NSF, grant DMS-0345350.

if for some $\delta > 0$, there exist arbitrarily large intervals of integers $J = \{a, a+1, \dots, a+l-1\}$ with $\frac{|J \cap A|}{|J|} \geq \delta$. (Here $|S|$ is the cardinality of the set S; $d^*(A) = \text{l.u.b.}\{\delta \text{ as above}\}$.) Syndetic sets are special cases of sets with positive upper density. Namely, A is *syndetic* if for some l, every interval J of integers with $|J| \geq l$ intersects A. Clearly $d^*(A) \geq 1/l$ in this case. We'll say a set A is *thick* if it contains arbitrarily long intervals; thus A is syndetic $\Leftrightarrow$ $\mathbb{Z}\backslash A$ is not thick $\Leftrightarrow$ $A \cap B \neq \emptyset$ for any thick set B. For any distinct r integers $\{a_1, a_2, \dots, a_r\}$ the set $\{a_j - a_i | 1 \leq i < j \leq r\}$ is called an r-*difference* set or a Δ_r-set. Every thick set contains some r-difference set for every r. This is obvious for $r = 2$, and inductively, if A is thick and if A contains the $(r-1)$-difference set formed from $\{a_1, \dots, a_{r-1}\}$, by choosing a_r in the middle of a large enough interval in A, we can complete this to an r-difference set. It follows that for any r, a set that meets every r-difference set is syndetic. An example of this is the set of (non-zero) differences $A - A = \{x - y : x, y \in A, x \neq y\}$ when A has positive upper density. For if $d^*(A) > 1/r$ and if the numbers $a_1, a_2, \dots, a_r$ are distinct, the sets $A + a_1, A + a_2, \dots, A + a_r$ cannot be disjoint; so, for some $1 \leq i < j \leq r$, $a_j - a_i \in A - A$. One conclusion which is behind much of our subsequent discussion is that if A has positive upper density, then $A - A$ is syndetic. We shall see in §3 that $d^*(A) > 0$ implies that $A - A$ is a *piecewise-Bohr* set.

Definition 1.1. *$S \subset \mathbb{Z}$ is a* Bohr *set if there exists a trigonometric polynomial $\psi(t) = \sum_{k=1}^{m} c_k e^{i\lambda_k t}$, with the λ_k real numbers, such that the set*

$$S' = \{n \in \mathbb{Z} : \operatorname{Re}\psi(n) > 0\}$$

is non-empty and $S \supset S'$. When $\psi(0) > 0$ we say S is a Bohr$_0$ set. (Compare with [Bilu97]).

The fact that a Bohr set is syndetic is a consequence of the almost periodicity of trigonometric polynomials. It is also a consequence of the "uniform recurrence" of the Kronecker dynamical system on the m-torus

$$(\theta_1, \theta_2, \dots \theta_m) \longrightarrow (\theta_1 + \lambda_1, \theta_2 + \lambda_2, \dots, \theta_m + \lambda_m).$$

Indeed, it is not hard to see that a set $S \subset \mathbb{Z}$ is Bohr if and only if there exist $m \in \mathbb{N}$, $\alpha \in \mathbb{T}^m$ and an open set $U \subset \mathbb{T}^m$ such that $S \supset \Big\{n \in \mathbb{Z} : n\alpha \in U\Big\}$.

Alternatively we can define Bohr sets and Bohr$_0$ sets in terms of the topology induced on the integers $\mathbb{Z}$ by imbedding $\mathbb{Z}$ in its Bohr compactification. Namely, a set in $\mathbb{Z}$ is Bohr if it contains an open set in the induced topology, and it is Bohr$_0$ if it contains a neighborhood of 0 in this topology.

We can apply the foregoing observations regarding $A - A$ to dynamical systems. We shall be concerned with *measure preserving systems* $(X, \mathcal{B}, \mu, T)$, where $(X, \mathcal{B}, \mu)$ is a probability space, $T: X \to X$ a measurable measure preserving transformation. We assume (for simplicity) that the system is *ergodic*

($T^{-1}A = A$ for $A \in \mathcal{B} \Rightarrow \mu(A)\mu(X\backslash A) = 0$). The ergodic theorem then ensures that for $A \in \mathcal{B}$ with $\mu(A) > 0$, the orbit $\{T^n x\}_{n\in\mathbb{Z}}$ of almost every x visits A along a set of times $V(x,A) = \{n : T^n x \in A\}$ of positive density. If we set $R_1(A) = \{n : A \cap T^{-n}A \neq \emptyset\}$ (the return time set of A), then for any x, $R_1(A) \supset V(x,A) - V(x,A)$. Hence $R_1(A)$ is syndetic. We can define a smaller set $R(A) = \{n : \mu(A \cap T^{-n}A) > 0\} = R(A')$ where $A' = A \setminus \bigcup\{(A \cap T^{-n}A) : \mu(A \cap T^{-n}A) = 0\}$, and it follows that $R(A)$ is also syndetic. This can be seen directly as well (and for arbitrary measure preserving systems), but the present argument illustrates the connection of dynamics to combinatorial properties of sets. We shall call sets containing sets of the form $R(A)$, where $\mu(A) > 0, RT$-sets (for return time). A set meeting every RT-set is called a Poincaré set since Poincaré's recurrence theorem gives content to the property by implying that $R(A)$ is never empty for $\mu(A) > 0$ even if T is not ergodic. These are also known in the literature as *intersective* sets. (See [Ruz82]). Much is known about these (see [Fur81], [B-M86], [BH96], [BFM96]). In particular $\{n^r; n = 1, 2, \dots\}$ is a Poincaré set for each $r = 1, 2, 3, \dots$

For a family $\mathcal{F}$ of subsets of $\mathbb{Z}$ it is customary to denote by $\mathcal{F}^*$ the dual family: $\mathcal{F}^* = \{S \subset \mathbb{Z} : \forall S' \in \mathcal{F}, S \cap S' \neq \emptyset\}$. Note that $\{\text{syndetic}\} = \{\text{thick}\}^*$, $\{\text{thick}\} = \{\text{syndetic}\}^*$ and $\{RT\} = \{\text{Poincaré}\}^*, \{\text{Poincaré}\} = \{\text{RT}\}^*$.

We have seen above that a Δ_r^*-set is necessarily syndetic. One of our objectives is to sharpen this statement.

We will need the notion of a "PW-$\mathcal{F}$" set for a family $\mathcal{F}$ of subsets of $\mathbb{Z}$. "PW" stands for "piecewise" and if $S \in \mathcal{F}$ and Q is a thick set then we shall say $S \cap Q$ is PW-$\mathcal{F}$ (or $S \cap Q \in$ PW-$\mathcal{F}$). Clearly this notion is useful only for families of syndetic sets. "PW-syndetic" is itself a useful notion. Van der Waerden's theorem [GRS80] implies that syndetic sets contain arbitrarily long arithmetic progressions. In fact this is true for PW-syndetic sets. Unlike the family of syndetic sets, the latter have the "divisibility" property: if S is PW-syndetic and $S = S_1 \cup S_2 \cup \dots \cup S_k$ is a finite partition, then some S_i is PW-syndetic, see [Bro71]. A recent result of Renling Jin [Jin02] is the following:

Theorem 1.2. *If $A, B \subset \mathbb{Z}$ and $d^*(A) > 0$, $d^*(B) > 0$, then $A + B$ is PW-syndetic.*

We will sharpen this to

Theorem I. *If $A, B \subset \mathbb{Z}$ and $d^*(A) > 0$, $d^*(B) > 0$, then $A + B$ is a PW-Bohr set (PWB-set).*

In particular $d^*(A) > 0$ will imply that $A - A$ is a PW-Bohr set. More precisely it is a PW-Bohr$_0$ (PWB$_0$)-set. This will also follow from our earlier observation that it is a Δ_r^*-set for sufficiently large r, and from

Theorem II. *For each $r \geq 2$, a Δ_r^*-set is PW-Bohr$_0$.*

It is not hard to see that the prefix "PW" is indispensable in these theorems. For example $A = \bigcup[10^n, 10^n + n]$ has $d^*(A) = 1$ but $A + A$ is not syndetic. Also since $x^3 + y^3 = z^3$ has no solution in non-zero integers, it follows that the set of non-cubes $S = \mathbb{Z}\backslash\{n^3; n = \pm1, \pm2, \pm3, \ldots\}$ is a Δ_3^* set. But by Weyl's equidistribution theorem S is not a Bohr_0-set. (See Theorem 4.1 below for a stronger form of this observation.)

From Theorem I we shall deduce the following result which should be compared with a theorem due to Ruzsa ([Ruz82], Theorem 3) which states that if $d^*(A) > 0$, then $A + A - A$ is a Bohr set. (Both Ruzsa's theorem and our result can be viewed as improvements on a theorem of Bogoliouboff ([Bog39], [Føl54]) which implies that if $d^*(A) > 0$, then $A - A + A - A$ is a Bohr set.)

Corollary 1.3. *If A, B, C are three subsets of $\mathbb{Z}$ with positive upper density and one of them is syndetic, then $A + B + C$ is a Bohr set.*

2 Measure Preserving Systems, Time Series, and Generic Schemes

In this section we introduce a basic tool which will be needed repeatedly: the correspondence between data given on large intervals of time ("time series") and measure preserving dynamical systems. This tool has been used previously under the name "correspondence principle" (see e.g., [Ber96]) and here we present it in a more general form. We repeat the definition of a measure preserving system which was given informally in §1.

Definition 2.1. *A* measure preserving system *is a quadruple $(X, \mathcal{B}, \mu, T)$ where $(X, \mathcal{B}, \mu)$ is a probability space where we assume $\mathcal{B}$ is countably generated, and T is a measurable, invertible, and measure preserving map, $T: X \to X$. The system is* ergodic *if every measurable T-invariant set has measure 0 or 1.*

For a measurable function $f: X \to \mathbb{C}$ we denote by Tf the function $Tf(x) = f(Tx)$. We take note of the ergodic theorem (see, for example, [Kre85]):

Theorem 2.2. *If $(X, \mathcal{B}, \mu, T)$ is a measure preserving system and $f \in L^1(X, \mathcal{B}, \mu)$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} T^n f = \overline{f}$$

exists almost everywhere. If $f \in L^p(X, \mathcal{B}, \mu)$, $1 \le p < \infty$, the convergence is in L^p as well. If the system is ergodic then $\overline{f} = \int f d\mu$ a.e., so that the average of the sequence $\{f(T^n x)\}$ equals a.e. the average of f over X.

Sequences of the form $\{f(T^n x)\}_{a\leq n\leq b}$ are referred to as "time series". In a certain sense the ergodic theorem enables one to reconstruct a dynamical system from "time series data". We shall make this precise in the notion of "generic schemes" which we proceed to define. In the next definitions the indices l and r range over the natural numbers.

Definition 2.3. *An* array *is a sequence* $\{J_l\}$ *of intervals of integers,* $J_l = \{a_l, a_l+1, \dots, b_l\}$ *for which* $|J_l| = b_l - a_l + 1 \to \infty$ *as* $l \to \infty$.

Definition 2.4. *A* scheme $(\{J_l\}, \{\xi_r^l\})$ *is an array* $\{J_l\}$ *together with a doubly indexed set of complex-valued functions* $\{\xi_r^l\}$ *where, for each* r, $\xi_r^l(n)$ *is defined for* $n \in J_l$ *and, for each* r, *the functions* $\{\xi_r^l; l = 1, 2, \dots\}$ *are uniformly bounded. For* $n \notin J_l$ *we take* $\xi_r^l(n) = 0$. *The* $\{\xi_r^l\}$ *will be referred to as* time series. *They are defined on all of* $\mathbb{Z}$ *but only the values on* J_l *have significance. The following notion relates closely to that of a "stationary stochastic process".*

Definition 2.5. *A* process $(X, \mathcal{B}, \mu, T, \Phi)$ *consists of a measure preserving system* $(X, \mathcal{B}, \mu, T)$ *together with an at most countable ordered set* $\Phi = \{\varphi_1, \varphi_2, \dots\}$ *of* L^∞*-functions on* X *such that* $\mathcal{B}$ *is the* σ*-algebra generated by the functions of* Φ *and their translates under* T. *(When the* φ_i *are complex valued we assume* Φ *closed under conjugation). A process is ergodic if the underlying measure preserving system is ergodic.*

Finally we have

Definition 2.6. *A scheme* $(\{J_l\}, \{\xi_r^l\})$ *is* generic *for a process* $(X, \mathcal{B}, \mu, T, \Phi)$ *if for every* m *and for every choice of* $i_1, i_2, \dots, i_m$ *and* $j_1, j_2, \dots, j_m$ *(the indices here need not be distinct):*

$$\lim_{l\to\infty} \frac{1}{|J_l|} \sum_{n\in J_l} \xi_{i_1}^l(n+j_1)\xi_{i_2}^l(n+j_2)\cdots\xi_{i_m}^l(n+j_m) \tag{1}$$
$$= \int_X T^{j_1}\varphi_{i_1} T^{j_2}\varphi_{i_2} \cdots T^{j_m}\varphi_{i_m}\, d\mu$$

It will be convenient to introduce the countable family Φ^* consisting of the products appearing in (1):

$$\Phi^* = \{\psi = T^{j_1}\varphi_{i_1} T^{j_2}\varphi_{i_2} \cdots T^{j_m}\varphi_{i_m}\}$$

The corresponding time series have the form

$$\zeta^l(n) = \xi_{i_1}^l(n+j_1)\xi_{i_2}^l(n+j_2)\cdots\xi_{i_m}^l(n+j_m),$$

and when (1) holds, we say that $\{\zeta^l\}$ *represents* ψ.

It will be convenient in the sequel to regard Φ^* as the increasing union of finite sets, $\Phi^* = \bigcup_{h=1}^{\infty} \Phi_h^*$. The subscript h has no significance other than as an index with $\Phi_1^* \subset \Phi_2^* \subset \cdots \subset \Phi_h^* \subset \cdots$.

We note that the ergodic theorem implies that if $(X, \mathcal{B}, \mu, T)$ is ergodic, then for almost every $x_0 \in X$, the scheme $(\{J_l\}, \{\xi_r^l\}$ is generic for the process $(X, \mathcal{B}, \mu, T, \Phi)$ with $J_l = [1, l]$ and $\xi_r^l(n) = \varphi_r(T^n x_0)$ independently of l.

The main result of this section goes in the opposite direction, and will attach to an arbitrary scheme an ergodic process. First we need the notions of *subarrays* and *subschemes.*

Definition 2.7. *An array $\{H_l\}$ is a* subarray *of $\{J_l\}$ if $l \to L_l$ is a monotone increasing function from $\mathbb{N}$ to $\mathbb{N}$ and H_l is a subinterval of J_{L_l}.*

Definition 2.8. *A scheme $(\{H_l\}, \{\eta_r^l\})$ is a* subscheme *of $(\{J_l\}, \{\xi_r^l\})$ if $\{H_l\}$ is a subarray of $\{J_l\} : H_l \subset J_{L_l}$, and η_r^l is the restriction of $\xi_r^{L_l}$ to H_l.*

Our main result in this section is

Theorem 2.9. *For any scheme $(\{J_l\}, \{\xi_r^l\})$ there exists a subscheme and an ergodic process for which the subscheme is generic.*

Proof. First we will pass to a subscheme which is generic for a process $(X, \mathcal{B}, \mu, T, \Phi)$ which is not necessarily ergodic. For each r, let $\Lambda_r \subset \mathbb{C}$ be a compact set with $\xi_r^l(n) \in \Lambda_r$ for all l and n. Let $\tilde{\Lambda} = \prod \Lambda_r$ and let $X = \tilde{\Lambda}^{\mathbb{Z}}$. We denote by ξ_r^l the point in $\Lambda_r^{\mathbb{Z}}$ with $\xi_r^l = (\ldots, \xi_r^l(-1), \xi_r^l(0), \xi_r^l(1), \ldots)$ and form $\tilde{\xi}^l = (\xi_1^l, \xi_2^l, \ldots) \in \tilde{\Lambda}^{\mathbb{Z}} = X$. X is a compact metrizable space and we form the measures

$$\nu_l = \frac{1}{|J_l|} \sum_{n \in J_l} \delta_{T^n \tilde{\xi}^l} \tag{2}$$

where $T : X \to X$ denotes the shift map $T\omega(n) = \omega(n+1)$. Since $|J_l| \to \infty$, any weak limit of a subsequence of ν_l is T-invariant, and we let ν be some such limit: $\nu = \lim \nu_{L_l}$. It is not hard to see that $(\{J_{L_l}\}, \{\xi_r^{L_l}\})$ is generic for the process $(X, \mathcal{B}, \nu, T, \Phi)$ where $\mathcal{B}$ is the Borel σ-algebra of sets in X and $\Phi = \{\varphi_1, \varphi_2, \ldots\}$ with φ_r the functions on $\tilde{\Lambda}^{\mathbb{Z}}$ given by $\varphi_r(\omega) = \omega(0)(r)$. By ergodic decomposition there will be an ergodic measure μ whose support is a subset of the support of ν. Any point in the support of μ is a limit of points of the form $T^n \tilde{\xi}^l$ with $n \in J_l$ and $l \to \infty$, by (2). Since μ is ergodic, almost every point ω in its support is *generic* for μ, in the sense that averages of a given bounded measurable function along the orbit of ω tend to the integral of the function. In particular for functions in Φ^* we have:

$$\frac{1}{N} \sum_{n=k}^{k+N-1} T^{j_i} \varphi_{i_1} T^{j_2} \varphi_{i_2} \cdots T^{j_m} \varphi_{i_m}(T^n \omega) \longrightarrow \int T^{j_1} \varphi_{i_1} T^{j_2} \varphi_{i_2} \cdots T^{j_m} \varphi_{i_m} d\mu \tag{3}$$

uniformly for $|k| \le N$.

We can find N sufficiently large that the difference of the two sides in (3) is $< \varepsilon$ for all $T^{j_1} \varphi_{i_1} \cdots T^{j_m} \varphi_{i_m} \in \Phi_h^*$. We then choose $T^n \tilde{\xi}^l$ close enough to ω, $n \in J_l$, so that the difference of the two sides of (3) remains $< \varepsilon$ with ω replaced by $T^n \tilde{\xi}^l$. Since $n \in J_l$, assuming l sufficiently large, we will have

$H_l = [n+k, n+k+N-1] \subset J_l$ for some k with $|k| \leq N$. We now let $\varepsilon \to 0$, $h \nearrow \infty$, and choose an appropriate subsequence of l; rescrambling the information in (3) we find a subscheme $(\{H_l\}, \{\xi_r^l\})$ which is generic for $(X, \mathcal{B}, \mu, T, \Phi)$. ◻

Scholium to Theorem 2.9. If for some r,

$$\limsup_{l\to\infty} \frac{1}{|J_l|} \Big| \sum_{n\in J_l} \xi_r^l(n) \Big| > 0,$$

we can add the condition that the corresponding φ_r does not vanish a.e. This follows from the fact that the measure ν satisfies $\int \varphi_r d\nu \neq 0$ and so ν must have an ergodic component with $\int \varphi_r d\mu \neq 0$.

We remark that in the case of ergodic processes, given a generic scheme, "many" subschemes will again be generic. This is made precise in the following: For any process $(X, \mathcal{B}, \mu, T, \Phi)$, Φ^* is countable and we fix an increasing family of finite sets $\Phi_h^* \subset \Phi^*$ increasing to Φ^*. Given a scheme $(\{J_l\}, \{\xi_r^l\})$ and fixing l, and letting $\varepsilon > 0$, we shall say that an interval $H \subset J_l$ is *ε-h-generic* for the process $(X, \mathcal{B}, \mu, T, \Phi)$ if (1) holds approximately; i.e, if for every $\psi \in \Phi_h^*$ and corresponding time series $\zeta^l(n)$.

$$\left| \frac{1}{|H|} \sum_{n\in H} \zeta^l(n) - \int \psi d\mu \right| < \varepsilon. \tag{4}$$

Assume now a process $(X, \mathcal{B}, \mu, T, \Phi)$ given with $\Phi^* = \bigcup \Phi_h^*$ as above, and let $(\{T_l\}, \{\xi_r^l\})$ be a generic scheme for the process.

Proposition 2.10. *If $(X, \mathcal{B}, \mu, T, \Phi)$ is an ergodic process, then for any $\varepsilon > 0$ and $h \in \mathbb{N}$ there exists $p_0 \in \mathbb{N}$ so that for any $p \geq p_0$ there exists a positive number $l_0(\varepsilon, h, p)$ so that for $l > l_0(\varepsilon, h, p)$, at least $(1-\varepsilon)(|J_l| - p + 1)$ of the $(|J_l| - p + 1)$ intervals of length p in J_l are ε-h-generic for the process.*

Letting p and l grow we see, according to the proposition, that the intervals J_l can be replaced by many choices of subintervals, and the scheme will remain generic. It is easy to see that this is not true for non-ergodic processes (where time series have different statistical behavior along different intervals of time).

Proof of Proposition 2.10. It suffices to treat a single function and the corresponding time series. For if for each of the $|\Phi_h^*|$ functions in Φ_h^* we have $(1-\varepsilon_1)(|J_l| - p + 1)$ "ε_1-generic" intervals with $\varepsilon_1 |\Phi_h^*| < \varepsilon$, the number of intervals common to all of these will not be less than $(1-\varepsilon)(|J_l| - p + 1)$, and these intervals are ε_1-h-generic, and so also ε-h-generic. So let $\psi \in \Phi^*$. Ergodicity assures that for p large, $\frac{1}{p} \sum_{q=0}^{p-1} T^q \psi$ is L^2-close to $\int \psi d\mu$, and so

$$\int \left(\frac{1}{p} \sum_{q=0}^{p-1} T^q \psi \right)^2 d\mu - \left(\int \psi d\mu \right)^2$$

is small. Fix p and set $\eta(n) = \frac{1}{p}\sum_{q=0}^{p-1}\zeta(n+q)$. η and ζ have the same long-term averages,

$$\frac{1}{|J_l|}\sum_{n\in J_l}\left(\eta(n) - \int\psi d\mu\right)^2 = \frac{1}{J_l}\sum_{n\in J_l}\eta(n)^2 - 2\left(\frac{1}{|J_l|}\sum_{n\in J_l}\eta(n)\right)\left(\int\psi d\mu\right) + \left(\int\psi d\mu\right)^2$$

$$\longrightarrow \int\left(\frac{1}{p}\sum_{q=0}^{p-1}T^q\psi\right)^2 d\mu - \left(\int\psi d\mu\right)^2$$

which is small for large p. But this implies that most $\eta(n)$ are close to $\int\psi d\mu$ as asserted in the proposition. □

3 Some Examples of PW-Bohr Sets

3.1 Fourier Transforms

Our first example of PW-Bohr sets will lead to three more in the following subsections.

Theorem 3.1. *Let ω be a non-negative measure on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with a non-trivial discrete (atomic) component, and let $\hat{\omega}$ denote its Fourier transform: $\hat{\omega}(n) = \int_{\mathbb{T}} e^{2\pi i t n}d\omega(t)$. If*

$$S = \{n : \operatorname{Re}\hat{\omega}(n) > 0\},$$

then S is a PW-Bohr$_0$ set.

Proof. Let ω_d denote the discrete component of ω: $\omega_d = \sum_{\lambda\in\Lambda}\omega(\{\lambda\})\delta_\lambda$ where Λ consists of all the atoms of ω . Let Λ_0 be a finite subset of Λ so that $\omega_d(\Lambda_0) > \frac{3}{4}\omega_d(\Lambda)$. Set

$$\psi(\tau) = \sum_{\lambda\in\Lambda_0}\omega_d(\lambda)e^{2\pi i\lambda\tau}$$

and let B_0 be the Bohr$_0$ set: $B_0 = \{n : \operatorname{Re}\psi(n) > \frac{2}{3}\omega_d(\Lambda_0)\}$. The measure $\omega - \omega_d$ is continuous and so by Wiener's theorem (see [Kre85], p.96)

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\left|\hat{\omega}(n) - \hat{\omega}_d(n)\right|^2 = 0$$

It follows that $Q' = \left\{n : \left|\hat{\omega}(n) - \hat{\omega}_d(n)\right| \geq \frac{1}{3}\omega_d(\Lambda_0)\right\}$ has density 0 so that $Q = \mathbb{Z}\backslash Q'$ is a thick set.

In $B_0 \cap Q$,

$$\begin{aligned}\operatorname{Re}\hat{\omega}(n) &> \operatorname{Re}\hat{\omega}_d(n) - \tfrac{1}{3}\omega_d(\Lambda_0)\\ &\geq \operatorname{Re}\psi(n) - \omega_d(\Lambda\backslash\Lambda_0) - \tfrac{1}{3}\omega_d(\Lambda_0)\\ &> \operatorname{Re}\psi(n) - \tfrac{1}{4}\omega_d(\Lambda) - \tfrac{1}{3}\omega_d(\Lambda_0)\\ &> \tfrac{2}{3}\omega_d(\Lambda_0) - \tfrac{1}{3}\omega_d(\Lambda_0) - \tfrac{1}{3}\omega_d(\Lambda_0) = 0\end{aligned}$$

so that $B_0 \cap Q \subset S$. It follows that S is PWB$_0$. □

3.2 Positive Definite Sequences

Theorem 3.2. *Let $\{a(n)\}_{n\in\mathbb{Z}}$ be a positive definite sequence of non-negative reals for which* $\lim \frac{1}{2N+1}\sum_{n=-N}^{N} a(n) > 0$. *Then $S = \{n : a(n) > 0\}$ is a PWB_0 set.*

Proof. By Herglotz's theorem $a(n) = \hat{\omega}(n)$ for some non-negative measure ω on $\mathbb{T}$ [Hel83], and the hypothesis of the theorem implies that $\omega\{0\} > 0$. The previous theorem applies and so S is PWB$_0$. □

3.3 Return Time Sets

A consequence of the foregoing is that RT-sets are PW-Bohr$_0$ sets. Recall a return time set has the form $S \supset R(A) = \{n\colon \mu(A \cap T^{-n}A) > 0\}$ where $(X, \mathcal{B}, \mu, T)$ is a measure preserving system, $A \in \mathcal{B}$ and $\mu(A) > 0$. If $a(n) = \mu(A \cap T^{-n}A)$ we can write $a(n) = \int fT^n f d\mu$ with $f = 1_A$ and T is a unitary operator. It is easily checked that $\sum_{m,n=1}^{N} a(n-m)x_n\overline{x}_m \geq 0$ for any $x_1, x_2, \ldots, x_N$, and so $\{a(n)\}$ is a positive definite sequence. We also have

$$\frac{1}{2N+1}\sum_{n=-N}^{N}\int fT^n f d\mu \longrightarrow \int fP_T f d\mu,$$

where P_T is the self-adjoint projection of $L^2(X, \mathcal{B}, \mu)$ to the subspace of T-invariant functions. Since $\int P_T f d\mu = \mu(A)$, it follows that $P_T f \neq 0$, and since $\int fP_T f d\mu = \int fP_T^2 f d\mu = \int (P_T f)^2 d\mu > 0$ the hypotheses of Theorem 3.2 are fulfilled. This proves

Theorem 3.3. *RT sets are PW-Bohr$_0$.*

3.4 Difference Sets of Sets of Positive Upper Density

Proposition 3.4. *Let $\{J_l\}$ be an array and, for each l, let $S_l \subset J_l$ with $|S_l| > \delta|J_l|$ for fixed $\delta > 0$. Then $\bigcup(S_l - S_l)$ is PW-Bohr$_0$.*

This leads immediately to

Theorem 3.5. *If $d^*(S) > 0$, then $S - S$ is PWB_0 for $S \subset \mathbb{Z}$*

Proof of Proposition 3.4. We form the scheme $(\{J_l\}, \{\xi^l\})$, where the usual index r is suppressed since it takes only one value, and we define $\xi^l(n) = 1_{S_l}(n)$. We pass to a subscheme which is generic for a process $(X, \mathcal{B}, \mu, T, \{\varphi\})$ where, according to the scholium following Theorem 2.2, φ is not almost everywhere 0. By the construction ($\Lambda = \{0, 1\}$), φ takes on the values 0, 1 and so $\varphi = 1_A$ for $A \in \mathcal{B}$, $\mu(A) > 0$. By definition of a generic scheme

$$\mu(A \cap T^{-k}A) = \int \varphi T^k \varphi d\mu = \lim \frac{1}{|H_l|} \sum_{n \in H_l} \xi^l(n)\xi^l(n+k)$$

which will be > 0 only if $k \in \bigcup(S_l - S_l)$. This proves the proposition. □

In the sequel we will use a stronger version of Proposition 3.4. Let us say that a set Q is *uniformly thick* if for every $l \in \mathbb{N}$, $\exists l' \in \mathbb{N}$ so that every interval J of length l' meets Q in a set containing an interval of length l. This will happen if $\frac{1}{N} \sum_{j=n+1}^{n+N} 1_Q(j) \to 1$ uniformly in n. If ω is a continuous measure on $\mathbb{T}$ then Wiener's Theorem can be sharpened to

$$\frac{1}{N} \sum_{j=n+1}^{n+N} |\hat{\omega}(j)|^2 \longrightarrow 0$$

uniformly in n. Using this in the proof of Theorem 3.1 we find that the set S of that theorem is the intersection of a Bohr$_0$-set and a uniformly thick set. If we call a set a UPW-Bohr$_0$ set if it contains intersection of a Bohr set and a uniformly thick set, we can replace PW-Bohr$_0$ throughout this section by UPW-Bohr$_0$. For later reference we re-write Proposition 3.4 in its strengthened form as

Proposition 3.6. *Let $\{J_l\}$ be an array and for each l, let $S_l \subset J_l$ with $|S_l| > \delta|J_l|$ for fixed $\delta > 0$. Then $\bigcup(S_l - S_l)$ is a UPW-Bohr$_0$ set.*

4 The Hierarchy of Families of Large Sets

We consider the following families of "large sets":

(a) B_0 = Bohr$_0$ sets
(b) RT = return time sets
(c) $\bigcup \Delta_r^*$ = sets which for some r meet every $(S - S)\backslash\{0\}$ provided $|S| \geq r$
(d) PWB$_0$ = piecewise Bohr$_0$ sets
(e) PWB = piecewise Bohr sets
(f) PW Syn = piecewise syndetic sets
(g) PD = sets of positive upper Banach density = $\{S : d^*(S) > 0\}$

It is easily seen that the first of these families is contained in the second, the second in the third, the fourth in the fifth and the fifth in the sixth. That $\bigcup \Delta_r^* \subset \mathrm{PWB}_0$ is the content of our Theorem II to be proved in §9. In fact all these inclusions are proper, and in this section we shall show that (b) $\neq$ (c), (c) $\neq$ (d) and (e) $\neq$ (f). The fact that (a) $\neq$ (b) follows from work of I. Kříž [Kříž87] and that (f) $\neq$ (g) is an exercise.

Theorem 4.1. *There are Δ_3^*-sets which do not contain RT-sets. So (b) $\neq$ (c).*

Proof. We use the fact ([Fur81], [Sár78]) that for every $r = 1, 2, \dots$ the set $P_r = \{n^r\}_{n\in\mathbb{Z}}$ is a Poincaré set; i.e., it meets every return time set. Hence $\mathbb{Z}\backslash P_r$ does not contain any RT-set. On the other hand, when $r \geq 3, \mathbb{Z}\backslash P_r$ is a Δ_3^*-set. For, by Fermat's theorem, for any distinct a, b, c, we cannot have $b - a$, and $c - b$ as well as $c - a = (b - a) + (c - b)$ all in P_r. ◻

To prove that (c) $\neq$ (d) we produce a set of density 0 in $\mathbb{Z}$ that contains a Δ_r-set for every r. The complement of this set cannot belong to any Δ_r^*. On the other hand, the complement of a set of density 0 contains arbitrarily long intervals, and so is thick, and in particular it is PWB_0. So we take as a Δ_r-set a set of the form

$$D_r = \{-rq_r, -(r-1)q_r, \dots, -q_r, 0, q_r, \dots, (r-1)q_r, rq_r\}$$

Choosing $q_r = r^3$ we can check that the density of $\bigcup D_r$ is 0. This proves

Theorem 4.2.

$$\bigcup \Delta_r^* \neq PWB_0$$

Finally we have (e) $\neq$ (f) by the following:

Theorem 4.3. *There are syndetic sets that are not PWB.*

Proof. We use considerations from topological dynamics. Let $\Omega = \{0,1\}^{\mathbb{Z}}$ and define the shift T on Ω by $T\omega(n) = \omega(n+1)$. If $M \subset \Omega$ is a minimal closed T-invariant subset, $M \neq \{0\}$, then for any $\omega \in M$, $\{n : \omega(n) = 1\}$ is syndetic. We can choose M so that the system (M,T) is weakly mixing ([Fur81]). Let $\xi \in M$ and set $S = \{n : \xi(n) = 1\}$. Assume S is PWB; then $S = Q \cap P$ where Q is thick and P is a Bohr set. If $\eta = 1_P$ then ξ and η agree on arbitrarily long intervals and for some $\{n_k\}, \lim T^{n_k}\xi = \lim T^{n_k}\eta$. Let $L = \overline{\{T^n\eta\}_{n\in\mathbb{Z}}}$ be the closed invariant set generated by η (so that $M \cap L \neq \emptyset$). Since M is minimal, $M \subset L$. By definition of a Bohr set there is a torus $\mathbb{T}^m$, a rotation $R : \mathbb{T}^m \to \mathbb{T}^m$, $R(\theta) = \theta + \alpha$, and an open set $U \subset \mathbb{T}^m$ so that $R^n(0) \in U \Rightarrow \eta(n) = 1$. Let $A = \{\omega : \omega(0) = 1\}$; then $R^n(0) \in U \Rightarrow T^n\eta \in A$. Let $Z \subset \mathbb{T}^m$ be the closed subgroup of $\mathbb{T}^m$ generated by α. By [Fur67] (Z, R) and (M,T) are disjoint, and since both are minimal, $Z \times M$ is minimal for $R \times T$. This implies that $\{(R^n(0), T^n\eta)\}$ is dense in $Z \times M$. But from the foregoing, when the first coordinate is in U the other is in A. It follows that $U \times A$ is dense in $U \times M$; hence $M = A$ and $\xi \equiv 1$. Choosing M non-degenerate gives us the example we seek. ◻

5 The Sum Set of Positive Density Sets

In this section we will prove Theorem I which asserts that the sum set of two sets A, B with positive upper density is a PW-Bohr set.

We begin with an elementary lemma.

Lemma 5.1. *Let $J, J' \subset \mathbb{Z}$ be intervals of length l, l' respectively. Let $S \subset J$, $S' \subset J'$ be subsets satisfying $|S| \geq \delta l$, $|S'| \geq \delta' l'$. We can find an interval L and a subset $R \subset L$ so that for some t, $S + S' \supset R - R + t$ and such that $|R| \geq \frac{\delta\delta'}{2}|L|$.*

Proof. Without loss of generality we suppose $l \leq l'$. For each $t \in \mathbb{Z}$, form $R_t = S \cap (t - S')$. $|R_t|$ equals the number of points of $S \times S'$ lying on the line $x + y = t$. The number of such lines meeting $S \times S'$ doesn't exceed $l + l'$, and so for some t,

$$|R_t| \geq \frac{|S \times S'|}{l + l'} \geq \frac{\delta\delta' l l'}{l + l'} \geq \frac{\delta\delta'}{2} l.$$

Take $R = R_t$ so that $R - R \subset S + (S' - t)$, and take $L = J$. □

Theorem I will now follow from

Theorem 5.2. *Let $\{J_k\}$ be an array (Def. 2.3), and let $S_k \subset J_k$, with $|S_k| > \delta |J_k|$ where $\delta > 0$. Let $\{t_k\}$ be an arbitrary set of integers. The set $A = \bigcup_{k=1}^{\infty} (S_k - S_k + t_k)$ is PW-Bohr.*

Our next step is to reduce Theorem 5.2 to a special case in which the sets S_k are related. For two sets of integers S', S'', let us write $S' \prec S''$ if for some $c \in Z$, $S' + c \subset S''$. Clearly $S' \prec S''$ implies that $S' - S' \subset S'' - S''$.

Lemma 5.3. *Theorem 5.2 is true in general if it is true for the case that $S_k \prec S_{k+1}$ for each $k = 1, 2, 3, \ldots$.*

Proof. We consider the general case of an arbitrary array $\{J_k\}$ with subsets $S_k \subset J_k$. We follow the procedure in the proof of Proposition 3.4 based on Theorem 2.2 to obtain a subscheme of $(\{J_k\}, \{1_{S_k}\})$ generic for an ergodic process $(X, \mathcal{B}, \mu, T, 1_A)$ with $\mu(A) > 0$. Reindexing and renaming sets we suppose that $(\{J_k\}, \{1_{S_k}\})$ is generic for the above process. Note that the hypothesis of genericity implies that we will still have $|S_k| > \delta' |J_k|$ for some positive δ'. We now pass to a further subscheme for which $S'_k \prec S'_{k+1}$. This is done as follows. Removing a set of measure 0 from A we can assume that any non-empty intersection $A \cap T^{-\tau_1} A \cap T^{-\tau_2} A \cap \cdots \cap T^{-\tau_r} A$ has positive measure. It follows from the ergodic theorem that there exist points x with $T^{\tau_n} x \in A$ for a sequence $\tau_1 < \tau_2 < \cdots < \tau_n < \cdots$ (depending on x) with $\lim \frac{\tau_n}{n} < \infty$. Thus $A \cap T^{-\tau_1} A \cap T^{-\tau_2} A \cap \cdots \cap T^{-\tau_r} A$ is non-empty for each r and by our assumption $\mu(A \cap T^{-\tau_1} A \cap T^{-\tau_2} A \cap \cdots \cap T^{-\tau_r} A) > 0$ for each r. By genericity of $(\{J_k\}, \{1_{S_k}\})$ this implies that translating $\{0, \tau_1, \tau_2, \cdots, \tau_r\}$ by some c_r we will obtain a subset of some $S_k : \{c_r, c_r+\tau_1, c_r+\tau_2, \cdots, c_r+\tau_r\} \subset S_{k(r)}$. We now set $J'_r = [c_r, c_r+\tau_r] \subset J_{k(r)}$ and $S'_r = \{c_r, c_1+\tau_1, c_r+\tau_2, \cdots, c_r+\tau_r\}$. Then $S'_r \prec S'_{r+1}$ and since $S'_r \subset S_{k(r)}$, $\bigcup(S_k - S_k + t_k) \supset \bigcup(S'_r - S'_r + t_{k(r)})$. At the same time $\lim \frac{\tau_n}{n} < \infty$ so that $\exists \alpha > 0$ with $r = |S'_r| \geq \alpha |J'_r|$. □

We show now how Theorem 5.2 follows from Proposition 3.6.

Proof of Theorem 5.2. According to the foregoing lemma, we may assume that for each k, $S_k - S_k \subset S_{k+1} - S_{k+1}$. For each $m = 1, 2, 3, \ldots$, let $k(m)$ be chosen so that $(S_k - S_k) \cap [-m, m]$ is a fixed set for $k \geq k(m)$. Write $S'_m = S_{k(m)}$ and $t'_m = t_{k(m)}$; we will show that $\bigcup(S'_m - S'_m + t'_m)$ is PW-Bohr. By Proposition 3.6 $\bigcup(S'_m - S'_m)$ is UPW-Bohr$_0$; i.e., it contains the intersection of a Bohr$_0$ set H and a *uniformly* thick set Q. Thus there is a trigonometric polynomial $\psi(t) = \sum_{j=1}^{N} a_j e^{i\lambda_j t}$ with $\mathrm{Re}\,\psi(0) > 0$ such that for any $n \in Q$, if $\mathrm{Re}\,\psi(n) > 0$ then $n \in \bigcup(S'_m - S'_m)$. Form $\psi_m(t) = \psi(t - t'_m)$ and pass to a subsequence $\{m_p\}$ so that these converge uniformly to a polynomial $\psi'(t)$. Let $0 < \alpha < \mathrm{Re}\,\psi(0)$. By almost periodicity of $\psi(t)$ it follows that $\mathrm{Re}\,\psi'(n) > \alpha$ on a non-empty (and therefore syndetic) set of n. We can suppose that the subsequence $\{m_p\}$ is such that $\mathrm{Re}\,\psi'(n) > \alpha$ implies $\mathrm{Re}\,\psi(n - t'_{m_p}) > 0$ for each p. Form the set $Q' = \bigcup([-m_p, m_p] \cap Q + t'_{m_p})$. Suppose $\mathrm{Re}\,\psi'(n) > \alpha$ with $n \in Q'$. Then for some p, $n - t'_{m_p} \in [-m_p, m_p] \cap Q$ and $\mathrm{Re}\,\psi(n - t'_{m_p}) > 0$. It follows that $n - t'_{m_p} \in (\bigcup(S'_m - S'_m)) \cap [-m_p, m_p]$. By the choice of $\{S'_m\}$ this implies $n \in S'_{m_p} - S'_{m_p} + t'_{m_p}$. Since Q is uniformly thick, for large p, $[-m_p, m_p] \cap Q$ contains large intervals and this implies that Q' is a thick set. This proves that $\bigcup(S'_m - S'_m + t'_m)$ is a PW-Bohr set. □

This completes the proof of Theorem I.

Corollary 5.4 (Corollary 1.3 of §1). *If $A, B, C, \subset \mathbb{Z}$ are three sets with positive upper density, one of which is syndetic, then $A+B+C$ is a Bohr set.*

This will follow from the Theorem I together with the following lemma:

Lemma 5.5. *If R is a PW-Bohr set and S is syndetic in $\mathbb{Z}$ then $R+S$ is Bohr.*

Proof. A translate of R will be PW-Bohr$_0$ and the opposite translate of S is syndetic, so we can assume that R is a PW-Bohr$_0$ set. This means that there is a torus $\mathbb{T}^m$, an $\alpha \in \mathbb{T}^m$, a neighborhood U of 0 in $\mathbb{T}^m$ and a thick set Q with $R \supset \{n : n\alpha \in U\} \cap Q$. Let V be a neighborhood of 0 in $\mathbb{T}^m$ with $V - V \subset U$ and let $\beta_1, \beta_2, \ldots \beta_k \in \mathbb{T}^m$ so that $\mathbb{T}^m = \bigcup_{l=1}^{k}(\beta_l + V)$.

We claim that for some $l, 1 \leq l \leq k$, $S + R \supset \{n : n\alpha \in \beta_l + V\}$ which implies that $S+R$ is a Bohr set. Assume this isn't so; then for each l, $\exists x_l$ with $x_l\alpha \in \beta_l + V$ and $x_l \notin S+R$. Let $S_l = S \cap \{n : n\alpha \in \beta_l + V\}$ so that $S = \bigcup S_l$. We have $x_l \notin S+R$ and so $x_l - S_l \cap R = \varnothing$. Since $x_l\alpha \in \beta_l + V$ and $S_l\alpha \subset \beta_l + V$ we have $(x_l - S_l)\alpha \subset U$. Now $R \supset \{n : n\alpha \in U\} \cap Q$ so $(x_l - S_l) \cap R = \varnothing$

implies that $(x_l - S_l) \subset Q^c$, the complement of Q. Equivalently $S_l \subset (x_l - Q)^c$, so $S = \bigcup S_l \subset \left(\bigcap (x_l - Q)\right)^c$. But the intersection of finitely many translates of a thick set is thick whereas S is syndetic. This contradiction proves our assertion. □

6 Kronecker-complete Processes

The remaining sections are directed to giving a proof of Theorem II of §1. The crucial step in this proof is a proposition to be proved in §8 which generalizes the fact (Theorem 3.5) that $d^*(A) > 0$ implies that $A - A$ is PWB_0. To achieve this generalization we will use once more the correspondence described in §2 between schemes and processes. Another ingredient that will enter is the point spectrum of an ergodic system, i.e., the eigenvalues of the operator T on the L^2-space of the system. It will be of importance that in a scheme generic for a process for which non-trivial eigenvalues exist, the eigenfunctions are also represented. This leads to the notion dealt with in this section of a "Kronecker-complete process."

We begin by recalling the notion of the "Kronecker factor" of an ergodic system: Let $(X, \mathcal{B}, \mu, T)$ be an ergodic measure preserving system. There is a compact abelian group Z and an element $\alpha \in Z$ whose multiples $\{n\alpha\}$ are dense in Z, and a map $\pi: X \to Z$ which is measurable and measure preserving with respect to Haar measure dz on Z, and such that for a.e. $x \in X$, $\pi(Tx) = \pi(x) + \alpha$. If $\chi \in \hat{Z}$ is a character on Z then $f = \chi \circ \pi$ is an eigenfunction of T: $f(Tx) = \chi(\pi(x) + \alpha) = \chi(\alpha) f(x)$, and *every* eigenfunction of T in $L^2(X, \mathcal{B}, \mu)$ is a multiple of one derived from a character. (Z, α) is unique up to isomorphism and is called the *Kronecker factor* of $(X, \mathcal{B}, \mu, T)$. The eigenvalues of T are $\{\chi(\alpha)\}_{\chi \in \hat{Z}}$, so that $Z \cong$ the dual group to the (discrete) group of eigenvalues of T. The system $(X, \mathcal{B}, \mu, T)$ is *weakly mixing* if and only if there are no eigenvalues other than 1 if and only if Z is the trivial one-element group. The discussion in this section will be vacuous in the case of weakly mixing systems.

We turn to processes. When we speak of an eigenfunction f we will assume $f \not\equiv 0$.

Definition 6.1. *A process* $(X, \mathcal{B}, \mu, T, \Phi)$ *is* Kronecker-complete *if it is ergodic and if every eigenfunction of* T *is proportional to some function in* Φ*.*

Note that for an ergodic system, if $Tf = \lambda f$ for a measurable f, it is easily seen that $|\lambda| = 1$ and that $|f(x)|$ is constant a.e., so that $f \in L^\infty(X, \mathcal{B}, \mu)$. Also note that $Tf_1 = \lambda f_1, Tf_2 = \lambda f_2$ implies that f_1/f_2 is invariant so that by ergodicity, f_1, f_2 are proportional. Thus a process is Kronecker-complete if Φ contains *some* eigenfunction for each eigenvalue. Under our standing hypothesis that $\mathcal{B}$ is a countably generated σ-algebra, the set of eigenvalues is at most countable. As a result we can always "complete" a non-Kronecker-complete process. The principal result in this section states that if a scheme is generic for a non-Kronecker-complete process, by augmenting the process and the scheme and passing to a subscheme, we will obtain a scheme generic for a Kronecker-complete process.

Theorem 6.2. *Let $(\{J_l\},\{\xi_r^l\})$ be generic for an ergodic process $(X,\mathcal{B},\mu,T,\Phi)$. Denote by Λ the subgroup of the unit circle S^1 consisting of eigenvalues of T on $L^2(X,\mathcal{B},\mu)$. We can find eigenfunctions ψ_λ for each $\lambda \in \Lambda$ and a subscheme $(\{H_k\},\{\eta_r^k\})$ so that setting $\eta_\lambda^k(n) = \lambda^n$ independent of k and letting $\Psi = \{\psi_\lambda\}_{\lambda\in\Lambda}$, the process $(X,\mathcal{B},\mu,T,\Phi\cup\Psi)$ will be Kronecker-complete, and the scheme $(\{H_k\},\{\xi_r^k\}\cup\{\eta_\lambda^k\})$ will be generic for $(X,\mathcal{B},\mu,T,\Phi\cup\Psi)$.*

In the weak mixing case we merely need to adjoin the function 1 to the process and to the scheme. In the general case we proceed by successively adjoining eigenfunctions, passing to a subarray at each stage. We will thus obtain a sequence of subarrays which is "decreasing" and a sequence $\Phi_n = \Phi\cup\{\Psi_{\lambda_1},\Psi_{\lambda_2},\dots,\Psi_{\lambda_n}\}$ of sets of functions with the corresponding $\{\eta_r^{(k)}\}\cup\{\eta_{\lambda_1},\eta_{\lambda_2},\dots,\eta_{\lambda_n}\}$ of representative time series. Our final scheme is obtained by choosing from successive schemes intervals that are "ε-h-generic" for the final process $(X,\mathcal{B},\mu,T,\Phi\cup\Psi)$ with $\varepsilon \searrow 0$, $h \nearrow \infty$. Such intervals will be found in the array for Φ_n with n sufficiently large.

Adjoining a single eigenfunction will also entail a procedure of successive approximation. We assume given a scheme $(\{J_l\},\{\xi_r^l\})$ generic for $(X,\mathcal{B},\mu,T,\Phi)$ and we wish to adjoin an eigenfunction for the eigenvalue λ. Fix an eigenfunction $f, Tf = \lambda f$, with $|f| = 1$. Since we have fixed the representative time series for the eigenfunction as η_λ where $\eta_\lambda(n) = \lambda^n$, the corresponding φ_λ to be adjoined will be some multiple *cf*, $|c| = 1$. Our task is to find subintervals of J_l that give better and better representation for the augmented $\Phi\cup\{cf\}$ in a sense analogous to ε-h-genericity (§2). In our procedure of successive approximation we can let c vary, since a subsequence will converge to a fixed value for which the intervals that have been found will still provide good representation. We form Φ^* from Φ as in §2, and express Φ^* as a union $\Phi^* = \bigcup \Phi_h^*$ of increasing finite subsets. Now $\{cf\}$ enters the picture and we say that the interval $J \subset J_l$ is "ε-h-m-generic" for $(X,\mathcal{B},\mu,T,\Phi\cup cf)$ if for every $\varphi\in\Phi_h^*$ and the corresponding time series ξ^l, and for a an integer with $0\le a\le m$,

$$\left|\frac{1}{|J|}\sum_{n\in J}\xi^l(n)\lambda^{an} - \int_X \varphi\cdot c^a f^a d_\mu\right| < \varepsilon. \tag{5}$$

Note that for $a = 0$ this is ε-h-genericity. What will be shown for the proof of the theorem is the existence of ε-h-m-generic intervals inside J_l for large l for arbitrary ε, h, m, and putting these together we obtain the subscheme that is sought.

In establishing (5) we will use the following lemma.

Lemma 6.3. *Let $\alpha_1, \alpha_2, \ldots, \alpha_N$ be N complex numbers and form, for $a, b = 0, 1, 2, \ldots$, the averages*

$$u(a,b) = \frac{1}{N}\sum_{i=1}^{N} \alpha_i^a \overline{\alpha}_i^b.$$

There is a function $\delta(\varepsilon, p) > 0$ for $\varepsilon > 0$ and $p \in \mathbb{N}$ so that if $|u(a,b) - 1| < \delta(\varepsilon, p)$ for $0 \le a, b \le p$, then $\exists \beta$ so that

$$\frac{1}{N}\sum_{i=1}^{N} |\alpha_i - \beta|^{2p} < \varepsilon$$

Proof. We form the average

$$\begin{aligned}
\frac{1}{N^2}\sum_{i,j=1}^{N} |\alpha_i - \alpha_j|^{2p} &= \frac{1}{N^2}\sum_{i,j=1}^{N} (\alpha_i - \alpha_j)^p (\overline{\alpha}_i - \overline{\alpha}_j)^p \\
&= \frac{1}{N^2}\sum_{q=0}^{p}\sum_{q'=0}^{p}\sum_{i,j=1}^{N} (-1)^{2p-q-q'} \tbinom{p}{q}\tbinom{p}{q'} \alpha_i^q\, \overline{\alpha}_i^{q'}\, \alpha_j^{p-q}\, \overline{\alpha}_j^{p-q'} \\
&= \sum_{q,q'=0}^{p} (-1)^{2p-q-q'} \tbinom{p}{q}\tbinom{p}{q'} u(q,q') u(p-q,\, p-q')
\end{aligned}$$

The latter expression is continuous in the $(p+1)^2$ expressions $\{u(q,q'), 0 \le q,\, q' \le p\}$ and we can evaluate it for $u(q,q') = 1$ by setting all $\alpha_i = 1$. Since the expression in question vanishes when $\alpha_i = 1$, it follows that we can find $\delta(\varepsilon, p) > 0$ so that the hypothesis of the lemma implies

$$\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{N}\sum_{i=1}^{N} |\alpha_i - \alpha_j|^{2p}\right) < \varepsilon.$$

But this implies that for some index j the inside average is $< \varepsilon$, so with $\beta = \alpha_j$ we get the desired result. □

Proof of Theorem 6.2. We have seen that to prove the theorem we have to show the existence of long intervals J inside J_l for sufficiently large l, for which (5) is valid, where φ ranges over Φ_h^*, f is an eigenfunction $Tf = \lambda f$, and the $\xi^l(n)$ are the time series representing φ in the respective J_l, and the exponent "a" ranges from 1 to m.

Our assumption in Definition 2.5 that the functions of Φ generate the σ-algebra $\mathcal{B}$ for the process $(X, \mathcal{B}, \mu, T, \Phi)$ implies that linear combinations of functions in Φ^* will approximate any function in $L^p(X, \mathcal{B}, \mu)$ in the L^p-norm, for any p, $1 \le p < \infty$. We wish to approximate f and for any $\varepsilon_1 > 0$ we can find σ in the linear space spanned by Φ^* with $||\sigma - f||_{L^q} < \varepsilon_1$ where $q = q(m) \ge 8$ will be made explicit further on. Taking appropriate combinations of the time series $\zeta^l(n)$ representing σ in the given scheme, we find that

$$\frac{1}{|J_l|}\sum_{n\in J_l}\left(\zeta^l(n)\right)^r\left(\overline{\zeta^l(n)}\right)^s\left(\zeta^l(n+k)\right)^t\left(\overline{\zeta^l(n+k)}\right)^u \longrightarrow \int_X \sigma^r\overline{\sigma}^s T^k(\sigma^t\overline{\sigma}^u)d\mu. \tag{6}$$

We're going to apply Lemma 6.3 with $p=2$ to the $N=K|J_l|$ numbers:

$$\alpha_{k,n}=\lambda^{-k}\zeta(n+k)\overline{\zeta(n)} \qquad 0\le k\le K-1,\quad n\in J_l$$

where $\zeta=\zeta^l$. K will be arbitrary and l will be large. We have

$$u(a,b)=\frac{1}{|J_l|}\frac{1}{K}\sum_{k=0}^{K-1}\sum_{n\in J_l}\lambda^{(b-a)k}\zeta(n+k)^a\overline{\zeta(n+k)}^b\zeta(n)^b\overline{\zeta(n)}^a$$

When l is large this is close to $\int_X \frac{1}{K}\sum_{k=0}^{K-1}\lambda^{(b-a)k}\sigma^b\overline{\sigma}^a T^k(\sigma^a\overline{\sigma}^b)d\mu$. The latter expression will be within ε_2 of

$$\int_X \frac{1}{K}\sum_{k=0}^{K-1}\lambda^{(b-a)k}f^b\overline{f}^aT^k(f^a\overline{f}^b)d\mu=\int_X\frac{1}{K}\sum_{k=0}^{K-1}\lambda^{(b-a)k}f^{b-a}T^k(f^{a-b})d\mu=1$$

where $\varepsilon_2=\varepsilon_2(\varepsilon_1)\to 0$ as $\varepsilon_1\to 0$, using the fact that σ is close to f in L^8 and the total exponent in the integrals above is $2a+2b\le 8$, and the fact that $T^kf=\lambda^kf$. Having chosen ε_1 sufficiently small, we find by Lemma 6.3 that for l large we can find β_l so that

$$\frac{1}{K}\frac{1}{J_l}\sum_{k,n}\left|\lambda^{-k}\zeta(n+k)\overline{\zeta(n)}-\beta_l\right|^4<\varepsilon_0 \tag{7}$$

where ε_0 is given.

We wish to use (7) to estimate

$$\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}\left|\zeta(n+k)\overline{\zeta(n)}\zeta(n)-\lambda^k\beta_l\zeta(n)\right|^2=$$
$$\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}\left|(\lambda^{-k}\zeta(n+k)\overline{\zeta(n)}-\beta_l\right|^2|\zeta(n)|^2\le\sqrt{\varepsilon_0}\theta_l$$

where $\theta_l^2=\frac{1}{|J_l|}\sum_{n\in J_l}|\zeta(n)|^4=\frac{1}{|J_l|}\sum_{n\in J_l}\zeta(n)^2\,\overline{\zeta(n)}^2$, and by (6), $\theta_l^2\to\int|\sigma|^4d\mu$ as $l\to\infty$. Since $\|\sigma-f\|_4<\varepsilon_1$ the latter expression is $<(1+\varepsilon_1)^4$ and we can assume this ≤ 4. We get for large l

$$\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}\left|\zeta(n+k)|\zeta(n)|^2-\lambda^k\beta_l\zeta(n)\right|^2<2\sqrt{\varepsilon_0}. \tag{8}$$

Finally we wish to use this to estimate

$$\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}\left|\zeta(n+k)-\lambda^k\beta_l\zeta(n)\right|^2$$

and for this we need an estimate of

$$\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}\Big|\zeta(n+k)|\zeta(n)|^2-\zeta(n+k)\Big|^2. \tag{9}$$

As $l\to\infty$, (9) approaches

$$\int\Big(|\sigma|^4\,T^k|\sigma|^2-2|\sigma|^2\,T^k\,|\sigma|^2\;+T^k|\sigma|^2\Big)\,d\mu. \tag{10}$$

The corresponding expression for f instead of σ vanishes so that for some C, the expression in (10) is bounded by $C\varepsilon_1$, and the same will be true for (9) when l is large. Combining this with (8) gives

$$\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}\Big|\zeta(n+k)-\lambda^k\beta_l\zeta(n)\Big|^2<\varepsilon_3=\varepsilon_3(\varepsilon_1)$$

for large l, where $\varepsilon_3(\varepsilon_1)\to 0$ for $\varepsilon_1\to 0$.

Using the Hilbert space inequality

$$||u||^2-||v||^2\le\Big(||u||+||v||\Big)\,||u-v||$$

we find for large l

$$\left|\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}|\zeta(n+k)|^2-\frac{1}{K}\frac{1}{|J_l|}\,|\beta_l|^2\,\sum_{k,n}|\zeta(n)|^2\right|\le C'\sqrt{\varepsilon_3}$$

from which it follows that $|\beta_l|\to 1$. To summarize the foregoing, we have shown that for any $\varepsilon>0$ we can find a function σ with time series $\zeta^l(n)$ and γ_l with $|\gamma_l|=1$ so that for l sufficiently large, and any K,

$$\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}\Big|\zeta^l(n+k)-\lambda^k\gamma_l\zeta^l(n)\Big|^2<\varepsilon.$$

To apply this to (5) we let $1\le a\le m$ and we estimate for a time series $\xi^l(n)$

$$\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}\left|\Big(\zeta^l(n+k)\Big)^a-\lambda^{ak}\gamma_l\Big(\zeta^l(n)\Big)^a\right|\,\Big|\xi^l(n+k)\Big| \tag{11}$$

Writing $x^a-y^a=(x-y)(x^{a-1}+x^{a-2}y+\cdots+y^{a-1})$ we obtain for large l that the expression in (11) is bounded by $M\sqrt{\varepsilon}$ where

$$M^2=a\left(\sum_{j=0}^{a-1}\frac{1}{K}\frac{1}{|J_l|}\sum_{k,n}|\zeta^l(n+k)|^{2(a-j-1)}|\zeta^l(n)|^{2j}|\xi^l(n+k)|^2\right)$$

If ξ^l represents the function φ, the limit of the foregoing expression, as $l \to \infty$, is

$$\frac{a}{K}\sum_{k=0}^{K-1}\int T^k|\sigma|^{2(a-j-1)}|\sigma|^{2j}T^k|\varphi|^2 d\mu,$$

and provided $q(m) \geq 2m+2$ with $\|\sigma - f\|_{L^q} < 1$, the expression in (11) will be bounded by $C'\|\varphi\|_{L^q}\sqrt{\varepsilon},\ C' = C'(m)$.

In all the estimates for averages over $0 \leq k < K,\ n \in J_l$, if the overall average is $< \theta$, then for at least half of the $n \in J_l$, the average over k cannot exceed 2θ. For large l, we let $N_l \subset J_l$ consist of the n with $\{n, n+1, \ldots, n+K-1\} \subset J_l$ and

$$\left|\frac{1}{K}\sum_{k=0}^{K-1}\left(\left(\zeta^l(n+k)\right)^a - \lambda^{ak}\left(\gamma_l\zeta^l(n)\right)^a\right)\xi^l(n+k)\right| < 2C'\|\varphi\|_{L^q}\sqrt{\varepsilon}. \quad (12)$$

We now refer to Theorem 2.9 applied to the functions $\sigma^a_\varphi,\ 1 \leq a \leq m$, $\varphi \in \Phi^*_h$ which are in the linear span of Φ^*. These functions are represented in the given scheme by $\left(\zeta^l(n)\right)^a\xi^l(n)$, and with $\delta > 0$ given, there will be a K so that for sufficiently large l, the inequalities

$$\left|\frac{1}{K}\sum_{k=0}^{K-1}\left(\zeta^l(n+k)\right)^a\xi^l(n+k) - \int\sigma^a\varphi d\mu\right| < \delta \quad (13)$$

hold for most $n \in J_l$ provided $|J_l| \gg K$. This implies that (12) and (13) will hold simultaneously for most $n \in N_l$ for which we will then have

$$\left|\frac{1}{K}\sum_{k=0}^{K-1}\lambda^{ak}\left(\gamma_l\zeta^l(n)\right)^a\xi^l(n+k) - \int\sigma^a\varphi d\mu\right| < \delta + 2C'\|\varphi\|_{L^q}\sqrt{\varepsilon}.$$

Set $c_{l,n} = \lambda^n\gamma_l^{-1}\zeta^l(n)^{-1}$ and we can write

$$\left|\frac{1}{K}\sum_{k=0}^{K-1}\lambda^{a(n+k)}\xi^l(n+k) - c^a_{l,n}\int\sigma^a\varphi d\mu\right| < |c_{l,n}|^a\left(\delta + 2C'\|\varphi\|_{L^q}\sqrt{\varepsilon}\right) \quad (14)$$

We write $n \in N'_l$ if (14) is valid.

$$\left|\frac{1}{K}\sum_{k=0}^{K-1}\lambda^{a(n+k)}\ \xi^{(l)}(n+k) - c^a_{l,n}\int_X f^a\varphi d\mu\right| < \quad (15)$$
$$|c_{l,n}|^a\left(\delta + 2C'\|\varphi\|_{L^q}\sqrt{\varepsilon}\right) + |c_{l,n}|^a c''\|\varphi\|_{L^{m+1}}\|\sigma - f\|_{L^{m+1}}$$

If J is the interval $\{n, n+1, \ldots, n+k-1\}$ then (15) has the form (5) if the right hand side can be made small and if $|c_{l,n}|$ is close to 1. All this can be achieved by choosing σ with $\|\sigma - f\|_{L^q}$ small, and finding $n_l \in J_l$ for which (15) holds with $|c_{l,n}| = |\gamma_l|^{-1}|\zeta^l(n)|^{-a}$ close to 1. The domain of n is N_l' which depends on ζ^l, but $|N_l'|/|J_l|$ is bounded from below. It suffices to show that by choosing $\|\sigma - \delta\|_{L^q}$ small we will have (for ζ^l representing σ) $\Big||\zeta^{(l)}(n)| - 1\Big| < \theta$ for a preassigned $\theta > 0$ for most $n \in J_l$. But this follows from the fact that

$$\frac{1}{K}\sum_{n\in J_l}\Big(|\zeta^l(n)|^2 - 1\Big)^2 \longrightarrow \int \Big(|\sigma|^2 - 1\Big)^2 d\mu$$

as $l \to \infty$ and the latter expression is small if $\|\sigma - f\|$ is small. With this we have completed the proof of Theorem 6.2. □

Corollary 6.4. *If an ergodic process is Kronecker-complete, it has a generic scheme whereby eigenfunctions are represented by the time series $c_\lambda \lambda^n$ for all intervals of the array $\{J_l\}$.*

Suppose now that we have a generic scheme for a Kronecker-complete process, $(X, \mathcal{B}, \mu, T, \Phi)$ and let $\Lambda \subset S^1$ be the group of eigenvalues of the process. If we identify the Kronecker factor of $(X, \mathcal{B}, \mu, T)$ with $Z = \hat{\Lambda}$ we can define a *canonical map* $\pi\colon X \to Z$. Namely for $\lambda \in \Lambda$ there is a unique eigenfunction φ_λ on X with $T\varphi_\lambda = \lambda\varphi_\lambda$, and which is represented in the scheme by $\eta_\lambda(n) = \lambda^n$. We set $\alpha \in Z = \hat{\Lambda}$ to correspond to the inclusion map of $\Lambda \to S^1 : \alpha(\lambda) = \lambda$. Notice that since $\eta_{\lambda_1\lambda_2} = \eta_{\lambda_1}\eta_{\lambda_2}$ we will have $\varphi_{\lambda_1\lambda_2} = \varphi_{\lambda_1}\varphi_{\lambda_2}$. This means that for a.e. $x \in X$, $\varphi_{\lambda_1\lambda_2}(x) = \varphi_{\lambda_1}(x)\varphi_{\lambda_2}(x)$ so that if we define $\pi(x)(\lambda) = \varphi_\lambda(x)$, then for a.e. $x, \pi(x) \in \hat{\Lambda} = Z$. Moreover $\pi(Tx)(\lambda) = \varphi_\lambda(Tx) = \lambda\varphi_\lambda(x) = \alpha(\lambda)\pi(x)(\lambda) = (\alpha + \pi(x))(\lambda)$; so $\pi(Tx) = \pi(x) + \alpha$. The mapping π is measurable since all φ_λ are measurable, and so the foregoing gives an explicit map of X to its Kronecker factor. This map will play a role in §7.

Note that for $\lambda \in \Lambda$, the eigenfunction φ_λ on X can be identified with $\chi \circ \pi$, where χ is the character on Z given by $\chi(z) = z(\lambda)$ where Z is identified with $\hat{\Lambda}$, since $\chi(\pi(x)) = \pi(x)(\lambda) = \varphi_\lambda(x)$ by definition of π. Since the time series representing φ_λ is $\lambda^n = \chi(n\alpha)$, we conclude:

Proposition 6.5. *Given a scheme generic for a Kronecker-complete process $(X, \mathcal{B}, \mu, T, \Phi)$, if π is the canonical map of X to its Kronecker factor (Z, α) then for any continuous function ψ on Z, $\psi \circ \pi$ can be adjoined to Φ, and it will be represented by the time series $\{\psi(n\alpha)\}$.*

Proof. ψ can be approximated uniformly by linear combinations of $\{\varphi_\lambda\}$. □

7 Weighted Ergodic Averages for Kronecker-complete Processes

Let $(X, \mathcal{B}, \mu, T, \Phi)$ be a Kronecker-complete process and $(\{J_l\}, \{\xi_r^l\})$ a generic scheme. We shall show how to evaluate L^2-limits of weighted ergodic averages

$$\frac{1}{|J_l|}\sum_{n\in J_l}\xi^l(n)T^n f$$

for $f \varepsilon L^2(X,\mathcal{B},\mu)$ and ξ^l representing a function $\varphi\in\Phi$. By our assumption $(X,\mathcal{B},\mu,T)$ is ergodic so that $\frac{1}{N}\sum_{n=0}^{N-1}T^n f\to\int f d\mu$ in L^2. Since T is a contraction we can write

$$\frac{1}{|J_l|}\sum_{n\in J_l}T^n f\longrightarrow\int f d\mu$$

for any array $\{J_l\}$. This will be generalized for processes that are Kronecker-complete, except that the limits taken are weak L^2-limits.

Recall from §6 the notion of Kronecker factor and the canonical map $\pi: X\to Z$ where Z is a compact abelian group and $\pi(Tx)=\pi(x)+\alpha$. All eigenfunctions on X are, up to constant multiples, of the form $\chi\circ\pi$ where χ is a character on Z. The set of all functions in $L^2(X,\mathcal{B},\mu)$ of the form $\psi\circ\pi,\psi\in L^2(Z)$ form a subspace that is spanned by eigenfunctions. If $f\in L^2(X,\mathcal{B},\mu)$ we denote by $E(f|Z)$ the unique function in $L^2(Z)$ so that $E(f|Z)\circ\pi$ denotes the orthogonal projection of f to the subspace $L^2(Z)\circ\pi$. $E(f|Z)=0\Leftrightarrow f$ is orthogonal to all eigenfunctions in $L^2(X,\mathcal{B},\mu)$. We will make use of an operation on $L^1(Z)$ related to (but not the same as) convolution:

$$f_1\Box f_2(z)=\int_Z f_1(z+u)f_2(u)du$$

Proposition 7.1. *Let $(\{J_l\},\{\xi_r^l\})$ be generic for the Kronecker-complete process $(X,\mathcal{B},\mu,T,\Phi)$, let $f\in L^2(X,\mathcal{B},\mu)$, and let $\varphi\in\Phi$ be represented by the time series ξ^l. Then*

$$\frac{1}{J_l}\sum_{n\in J_l}\xi^l(n)T^n f\xrightarrow{w}\Big[E(f|Z)\Box E(\varphi|Z)\Big]\circ\pi \tag{16}$$

where $\xrightarrow{w}$ signifies weak convergence in $L^2(X,\mathcal{B},\mu)$.

Proof. It suffices to consider two cases: (a) $E(f|Z)=0$, (b) f is an eigenfunction.

In the first case, for any g in $L^2(X,\mathcal{B},\mu)$, the sequence $\{\int T^n f\cdot g d\mu\}$ satisfies

$$\frac{1}{N}\sum_{k=n+1}^{n+N}\left|\int T^k f\cdot g d\mu\right|^2\underset{N\to\infty}{\longrightarrow}0$$

uniformly in n, so that the left hand side of (16) goes to 0 weakly, and the proposition is verified. We turn to case (b) with $f=\varphi_\lambda$. To $\lambda\in\Lambda$ we associate the character χ on $\hat{\Lambda}$ with $\chi(z)=z(\lambda)$. Then $\chi\circ\pi(x)=\pi(x)(\lambda)=\varphi_\lambda(x)=f(x)$, and $E(f|Z)=\chi$. In this case the right hand side of (16) is $[\chi \Box E(\varphi|Z)] \circ \pi = \left(\int\limits_Z E(\varphi|Z)\chi dz \right) \chi \circ \pi$. We evaluate the left hand side of (16):

$$\frac{1}{|J_l|} \sum_{n \in J_l} \xi^l(n) T^n f = \frac{1}{|J_l|} \sum_{n \in J_l} \lambda^n \xi^{(l)}(n) f$$

which by genericity converges to $\left(\int \varphi_\lambda \varphi d\mu \right) f$. Since $\varphi_\lambda \in L^2(Z) \circ \pi$, we can replace φ by its projection to this subspace which is $E(\varphi|Z) \circ \pi$. Since $\varphi_\lambda = \chi \circ \pi$ we now have

$$\int\limits_X \varphi_\lambda \cdot \varphi d\mu = \int\limits_Z \chi E(\varphi|Z) dz$$

and since $f = \chi \circ \pi$, this proves the proposition. □

8 A Condition for PW-Bohr$_0$

We know from Theorem 3.5 that if $d^*(S) > 0$ for a subset $S \subset \mathbb{Z}$, then $S - S$ is PW-Bohr$_0$. We can rephrase this as saying that if for each $s \in S$, $S - s \cap B = \emptyset$ for a subset $B \subset \mathbb{Z}$, then the complement of B is PW-Bohr$_0$. In this section we show that it will suffice for this conclusion that $d^* \Big((S - s) \cap B \Big) = 0$ for each $s \in S$. In §9 we'll see how this leads to a proof of Theorem II.

Proposition 8.1. *Let $A \subset \mathbb{Z}$ and $B = \mathbb{Z} \backslash A$ and let $S \subset \mathbb{Z}$ with $d^*(S) > 0$. If for every $s \in S$, $d^* \Big((S - s) \cap B \Big) = 0$, then A is a PW-Bohr$_0$ set.*

Proof. Let $\{J_l\}$ be an array with $\frac{|J_l \cap S|}{|J_l|} \to \beta > 0$. Set $\xi_1^l(n) = 1_A(n)$, $\xi_2^l = 1_B(n), \xi_3^l(n) = 1_S(n)$ and consider the scheme $(\{J_l\}\ \{\xi_1^l, \xi_2^l, \xi_3^l\})$. By Theorem 2.9 we can find a subscheme generic for an ergodic process $(X, \mathcal{B}, \mu, T, \Phi)$ where Φ includes $\varphi_1, \varphi_2, \varphi_3$ which are respectively represented by $\xi_1^l, \xi_2^l, \xi_3^l$. By the scholium to Theorem 2.9 we can assume φ_3 is not a.e. 0. Since $(\xi_i^{(l)})^2 = \xi_i^{(l)}$ we find $\varphi_i^2 = \varphi_i$ a.e. and so φ_i take values 0, 1. We write $\varphi_1 = 1_{\tilde{A}}$, $\varphi_2 = 1_{\tilde{B}}$, $\varphi_3 = 1_{\tilde{S}}$ with $\tilde{A}, \tilde{B}, \tilde{S} \subset X$, $\mu(\tilde{S}) > 0$, and $\tilde{A} \cup \tilde{B} = X$. Using Theorem 6.2 we can also assume that the process $(X, \mathcal{B}, \mu, T, \Phi)$ is Kronecker-complete and that the eigenfunctions $\{\varphi_\lambda\}$ of the process are represented by time series $\eta_\lambda(n) = \lambda^n$. We will also make use of the canonical map $\pi: X \to Z$, where (Z, α) is the Kronecker factor of $(X, \mathcal{B}, \mu, T)$.

We now apply Proposition 7.1 to this subscheme generic for the Kronecker-complete process with $\varphi_1, \varphi_2, \varphi_3 \in \Phi$, and where we again denote the array of intervals by $\{J_l\}$. We will take $f = \varphi = 1_{\tilde{S}} = \varphi_3$ which is represented by $\xi_3^l(n) = 1_S(n)$. We conclude that in the weak L^2-topology,

$$\frac{1}{|J_l|}\sum_{n\in J_l} 1_S(n)T^n 1_{\tilde{S}} \longrightarrow (f\Box f)\circ\pi \tag{17}$$

where $f = E(1_{\tilde{S}}|Z)$. The function f is bounded and non-negative with $\int f dz = \mu(\tilde{S}) > 0$ so it is non-trivial. We note that since $f \in L^\infty(Z)$, the function $F = f\Box f$ is continuous on Z.

We turn now to the hypothesis that $d^*\Big((S-s)\cap B\Big) = 0$ for $s \in S$. This implies that

$$\frac{1}{|J_l|}\sum_{n\in J_l} 1_B(n)1_S(n+s) \longrightarrow 0$$

or

$$\int 1_{\tilde{B}}T^s 1_{\tilde{S}} d\mu = 0.$$

In particular, averaging over $s \in S$:

$$\frac{1}{|J_l|}\sum_{n\in J_l} 1_S(n)\int T^n 1_{\tilde{S}}1_{\tilde{B}}\, d\mu \longrightarrow 0. \tag{18}$$

But by (17), the limit in (18) is

$$\int F\circ\pi\cdot 1_{\tilde{B}}\, d\mu \tag{19}$$

and so the latter integral vanishes. We again apply the generic scheme where according to Corollary 6.4, $F\circ\pi$ is represented by $\{F(n\alpha)\}$, a non-negative almost periodic sequence with

$$\frac{1}{|J_l|}\sum_{n\in J_l} F(n\alpha) \longrightarrow \int F dz > 0$$

Since the integral in (19) vanishes we can write

$$\frac{1}{|J_l|}\sum_{n\in J_l} F(n\alpha)1_B(n) \longrightarrow 0$$

Let H be the Bohr_0 set for which $F(n\alpha) > \delta$ where $\delta > 0$ is chosen so that H is non-empty. Then

$$\frac{\sum_{n\in J_l} 1_H(n)1_B(n)}{\sum_{n\in J_l} 1_H(n)} \longrightarrow 0$$

whence

$$\frac{\sum_{n\in J_l} 1_{H\cap A}(n)}{\sum_{n\in J_l} 1_H(n)} \longrightarrow 1.$$

This implies that there are arbitrarily long intervals $L_l \subset J_l$ for which $H\cap L_l = H\cap A\cap L_l \subset A$. Hence $H\cap\bigcup L_l \subset A$ from which it follows that A is PW-Bohr_0. This proves Proposition 8.1. ◻

9 Application to Δ_r^*-sets

We shall apply the foregoing results to prove Theorem II of §1. We recall that a subset $A \subset \mathbb{Z}$ is a Δ_r^*-set, $r = 2, 3, \ldots$ if for distinct numbers $x_1, x_2, \ldots, x_r$, some difference $x_j - x_i$, $i < j$ belongs to A. More generally we will need

Definition 9.1. *If $S \subset \mathbb{Z}$ we shall write $A \in \Delta_r^*(S)$ if for $x_1, x_2, \ldots, x_r \in S$, $x_i \neq x_j$ for $i \neq j$, there exists $i < j$ with $x_j - x_i \in A$.*

In the sequel, A and B denote complementary sets in $\mathbb{Z}$, $B = \mathbb{Z} \backslash A$. If $0 \in B$ we denote by B' the set $B \backslash \{0\}$.

Lemma 9.2. *The following are equivalent for a set $S \subset \mathbb{Z}$:*

(a) $A \in \Delta_{r+1}^*(S)$

(b) $A \in \Delta_r^*\Big(B' \cap (S - s)\Big)$ *for every $s \in S$.*

Proof. (a) $\Rightarrow$ (b): Suppose $x_1, x_2, \ldots, x_r \in B' \cap (S - s)$. Form the $(r+1)$-tuple $s, s + x_1, s + x_2, \ldots, s + x_r$ and apply (a). (b) $\Rightarrow$ (a): Let $x_0, x_1, x_2, \ldots, x_r$ be distinct elements in S. If $\{x_1 - x_0,\, x_2 - x_0, \ldots, x_r - x_0\}$ doesn't meet A, then this is an r-tuple in $B' \cap (S - x_0)$ and we can apply (b). □

We recall Theorem II:

Theorem II. *For any $r = 2, 3, \ldots$, if A is a Δ_r^*-set then A is a PW-Bohr$_0$.*

Proof. We assume A is not PW-Bohr$_0$. By Proposition 8.1 this will imply that whenever $d^*(S) > 0$ there must be some $s \in S$ with $d^*\Big(B \cap (S - s)\Big) > 0$. This will give us an inductive procedure to obtain sets S_i with $d^*(S_i) > 0$. Start with $S_0 = \mathbb{Z}$ and we find $d^*(B) > 0$. Set $S_1 = B'$, there exists $s_1 \in S_1$ with $d^*(B \cap (S_1 - s_1)) > 0$. Set $S_2 = B' \cap (S_1 - s_1)$ and continue with $S_{k+1} = B' \cap (S_k - s_k)$, $s_k \in S_k$. Now apply the foregoing lemma. $A \in \Delta_r^* \Leftrightarrow A \in \Delta_r^*(\mathbb{Z}) \Rightarrow A \in \Delta_{r-1}^*\Big(B' \cap (\mathbb{Z} - s_0)\Big) = \Delta_{r-1}^*(S_1) \Rightarrow A \in \Delta_{r-2}^*\Big(B' \cap (S_1 - s_1)\Big) = \Delta_{r-2}^*(S_2) \Rightarrow \cdots$ We continue with $A \in \Delta_{r-k}^*(S_k)$ for $k = 0, 1, \ldots, r-2$. Finally $A \in \Delta_2^*(S_{r-2})$. At each stage we have $d^*(S_k) > 0$. But $d^*(S_{r-2}) > 0 \Rightarrow S_{r-2} - S_{r-2}$ is PW-Bohr$_0$; and $A \in \Delta_2^*(S_{r-2}) \Rightarrow A \supset S_{r-2} - S_{r-2}$. This proves the theorem. □

References

[Ber96] V. Bergelson, Ergodic Ramsey theory – an update in *Ergodic Theory of $\mathbb{Z}^d$-actions*, London Math. Soc. Lecture Notes Series 228 (1996), 1-61.

[BH96] V. Bergelson, I. J. Håland, Sets of recurrence and generalized polynomials, Convergence in Ergodic Theory and Probability (Columbus, OH, 1993) 91–110, *Ohio State University Math. Res. Inst. Publ.* 5, de Gruyter, Berlin, 1996.

[BFM96] V. Bergelson, H. Furstenberg, R. McCutcheon, IP-sets and polynomial recurrence, *Ergodic Theory and Dynamical Systems* **16** (1996) no. 5, 963–974.

[Bilu97] Y. Bilu, Addition of sets of integers of positive density, *Journal of Number Theory* **64** (1997) 233-275.

[B-M86] A. Bertrand-Mathis, Ensembles intersectifs et recurrence de Poincaré, *Israel J. Math.* **55** (1986), 184–198.

[Bog39] N. Bogoliouboff, Sur quelques propriétés arithmetiques des presque periodes, *Ann. Chaire Phys. Math. Kiev* **4** (1939), 195–205.

[Bro71] T. C. Brown, An interesting combinatorial method in the theory of locally finite semigroups, *Pacific Journal of Math.* **36** (1971), 285–289.

[Føl54] E. Følner, Generalization of a theorem of Bogoliouboff to topological abelian groups, *Mathematica Scandinavica* **2** (1954), 5–19.

[Fur67] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation, *Math. Systems Theory* **1** (1967), 1–49.

[Fur81] H. Furstenberg, *Recurrence in Ergodic Theory and Combinatorial Number Theory*, Princeton, Princeton Univ. Press, 1981.

[GRS80] R. L. Graham, B. L. Rothschild, J. H. Spencer, *Ramsey Theory*, John Wiley, New York (1980).

[Hel83] H. Helson, *Harmonic Analysis*, Addison-Wesley Reading, MA, (1983).

[Jin02] R. Jin, The sumset phenomenon, *Proc. Amer. Math. Soc.* **130** (2002), no. 3, 855–861.

[Kre85] U. Krengel, *Ergodic Theorems*, De Gruyter Studies in Mathematics, 6, Walter de Gruyter & Co., Berlin, (1985) viii+357.

[Kříž87] I. Kříž, Large independent sets in shift-invariant graphs: solution of Bergelson's problem, *Graphs Combin.* **3** (1987) no. 2, 145–158.

[Ruz82] I. Ruzsa, Uniform distribution, positive trigonometric polynomials and difference sets, *Seminar on Number Theory*, 1981/1982, Exp. No. 18, 18 pp. Univ. Bordeaux I, Talence, 1982.

[Sár78] A. Sárközy, On difference sets of sequences of integers, III, *Acta Math. Acad. Sci. Hungar.* **31** (1978) no. 3-4, 355–386.

A Generalization of Conway Number Games to Multiple Players

Christopher Cunningham* and Igor Kriz*

Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, USA
ccccc@umich.edu, ikriz@math.lsa.umich.edu

Summary. We define an analogue of the the concept of J. H. Conway's number games for games of multiple players. We define the value of such number game as an element of a vector space over the Conway field. We interpret the value in terms of the strategy of the game, and prove that all possible values in the vector space can occur.

AMS Subject Classification. 91A06, 91A46, 91A70.
Keywords. Multiple player games, deterministic games, number games, strategy, Conway field.

1 Introduction

There are many different mathematical meanings of the word 'game'. Regardless of the kind of games we consider, people agree that games of n players are much more difficult to understand for $n > 2$ than for $n \leq 2$. In this paper, we consider deterministic 'combinatorial' games, i.e. games where each player in each position has a well defined set of moves, which, in a fixed way, change the position to another position (in fact, it is clear that there is no point in distinguishing between positions and games, so we can substitute the word 'game' for the word 'position' everywhere). For some recent work on combinatorial games, see [Now02]. The main result of this paper is to analyze a certain, very special, class of combinatorial games for multiple players.

The definition given above, of course, describes only the 'static' aspect of the rules of a game. The 'dynamic' aspects refer to how the game is actually played. A play by play sequence of moves in a game will be called a 'match'. The dynamical rules of matches which we will consider specify a certain order of the set of players; the players shall move repeatedly in the same order of play until a certain player cannot move, at which point the match shall end.

* The authors were supported by NSF grant DMS 0305853.

The player who cannot move shall then be declared the loser of the match. We shall consider only games where there is no possibility of infinite matches.

Even for such deterministic games, however, it is difficult to make any conclusions about the course of matches for $n > 2$. The reason is that unlike the case of $n = 2$, there is no natural order of preference of the outcomes of the game from the point of view of the i'th player. While the i'th player obviously prefers not to lose, there is no natural reason why he should a priori prefer one particular other player to lose. Yet, such preferences will determine strategies, and ultimately the outcome of a match. Preferences can even change throughout the course of a match. Thus, it is usually said that few strategic conclusions about deterministic games of $n > 2$ players can be made without introducing non-deterministic concepts, perhaps even non-mathematical concepts (e.g. psychology).

The purpose of this paper is to look at a certain very special class of deterministic games of n players, for which certain strategical conclusions can be made in a rigorous mathematical setting, without introducing outside concepts. The motivation for introducing our particular class of games is that they generalize the 'number games' for 2 players from J. H. Conway's famous book [Con01]. For this reason, we shall call this class of games *number games for n players.*

Conway number games for 2 players are games to which one can assign a value which is a 'number'. Here the word number means element of a certain ordered field, known as the Conway field $\mathcal{C}$ (also known as the surreal numbers [Con01, Con72, Knu74]). The Conway field contains, among other things, all ordinal numbers, as well as any other ordered field: it is a foundational-level object of set theory (in fact, Conway introduces his own approach to formal set theory based on number games in [Con01]).

However, this is not the aspect of number games which we will be most interested in. Rather, the main point of number games is that 'no player can possibly improve his own position by making a move'. This, of course, needs precise definition, (in particular in reference to the word 'improve'). Definitions will be provided later. We shall, however, remark here that one could adopt the point of view that for this reason, number games are generally strategically uninteresting, since the very meaning of strategy is being able to take advantage of one's own move in the best possible way. In number games, one always makes one's own position worse by moving. One can also take another point of view; in some sense, number games are *pure consumer models*: we can think of moving in a number game as using up one's assets (resources). The player who uses up his resources first dies.

With this in mind, it becomes interesting to try to define analogues of Conway number games for n players, and analyze what, if any, strategic conclusions one can make for such games. In this paper, we do give one possible definition of such number games for n players. The set of (equivalence classes of) such games is an $n-1$-dimensional vector space over the Conway field. It is somewhat surprising the set of number games of n players has many of the formal properties of number games of 2 players. Also, we shall be able to make certain strategic conclusions for number games: in particular, each match will have a well defined 'loser' who can always be defeated if all the other players act 'in concert'.

This paper is organized as follows: In the next section, we shall present basic definitions and facts about games and matches which do not involve numbers. In Section 3, we shall introduce number games, and prove what we can say about their strategic analysis. Section 4 contains, in some sense, our hardest result, namely constructing number games of n players with any given value. The paper has two appendices. In Appendix 1 (section 4), we draw some diagrams visualizing our concepts for games of three players. This may be helpful to the reader in understanding what we mean. In Appendix 2 (section 4), we show why our definition of number game cannot be simplified in one obvious way.

2 Games and Matches

In this paper, a *game with set T of players* is defined recursively as follows:

1. The empty set $\emptyset$ is a game (also called 0).
2. If G_i are sets of games for all $i \in T$, then the tuple $G = (G_i)_{i \in T}$ is a game. (G_i is the set of possible moves of the player i in the game G; a game is identified with its initial position.)
3. Every game can be obtained by 1, 2 in a possibly transfinite number of steps. (This means that for every ordinal α, we have an "α'th generation of games", the 0'th generation being the empty game. For a game G of the α'th generation, every element of every G_i must be a game of generation $< \alpha$.)

Obviously, only the cardinality of the set T matters. We shall mostly consider the case $T = \{1, \ldots, n\}$ (in which case we shall simply speak of *games of n players*), but it is useful to allow other T's, notably $T \subset \{1, \ldots, n\}$.

We shall now introduce our main strategic concept for games of n players. It is important to notice that this concept does not involve dynamic aspects of games, i.e. matches.

Specifically, we shall inductively define

$$G <_S 0$$

for a non-empty set $S \subseteq T$ if the following conditions hold:

1. If $i \in S$ then for all $H \in G_i$, $H <_{\{i\}} 0$.
2. If $i \in S$ and $j \notin S$ then there exists an $H \in G_j$ and a set U with $i \in U \subseteq S \cup \{j\}$, such that $H <_U 0$.

We shall write

$$G \sim 0$$

if $G <_T 0$. The set S will be called the *set of possible losers of the game G*. We shall justify this terminology at the end of this section.

When working with games of n players, we shall use a notational convention analogous to that established in Conway's book [Con01], and denote a "general member" of the set G_i by G^i. Thus, for example, instead of referring to something that is true for *all* $H \in G_i$, we instead refer to something that is true for *all* G^i.

The members of G_i, or using the new convention, *the* G^i, are referred to as *i's options* in the game G.

In this notation, the above definition reads as follows:

Inductively define $G <_S 0$ for a non-empty set $S \subseteq T$ if the following conditions hold:

1. If $i \in S$ then all $G^i <_{\{i\}} 0$.
2. If $i \in S$ and $j \notin S$ then there exists a G^j and a set U with $i \in U \subseteq S \cup \{j\}$, such that $G^j <_U 0$.

Remark. When we are considering games of two players, we are in the context of Conway [Con01]. There, players are denoted by L and R, so $T = \{L, R\}$. Conway notes that it is easy to prove that all games of 2 players are of one of the following types: $0, L, R, F$. For 0, the first player loses, in F, the first player wins, in L (resp. R) the player L (resp. R) wins no matter whose move it is. In the above notation, $S = \{L, R\}$ for type 0, $S = R$ for type L, $S = L$ for type R and S does not exist for type F. The proof is left to the reader as an exercise.

Lemma 2.1. *For each game G, there exists at most one S such that $G <_S 0$.*

Proof. Induction: Note that

$$0 <_S 0 \text{ if and only if } S = T.$$

Assume the statement true for all G^i for all $i \in T$. Then if

$$G <_S 0,$$

note that $i \notin S$ if and only if there exists a G^i such that it is not true that $G^i <_{\{i\}} 0$, which uniquely characterizes S. □

We now explain the dynamical significance of $G <_S 0$. To this end, we must define matches. Assume now that T is finite, and that we have a bijection

$$\sigma : \{1, \ldots, n\} \to T.$$

Such bijection will be called an *order of play.* A match according to the order of play σ is a sequence of games

$$(G(j))_{j=1,\ldots,N}$$

where $G(1) = G$,

$$G(j+1) \in G(j)_{k(j)} \text{ where } k(j) = \sigma(j'),\ j' \equiv j \mod n,$$

$$G(N)_{k(N)} = \emptyset.$$

Then $k(N)$ is called *the loser of the match.* Note that

$$(G(j))_{j=2,\ldots,N}$$

is a match according to the order of play σ' where

$$\sigma'(j) = \sigma(j+1) \text{ for } j < n,$$

$$\sigma'(n) = \sigma(1).$$

Note also that by part 3 of our definition of game, it is impossible to have an infinite match, i.e. an infinite sequence satisfying the properties of a match without the N. (Proof: induction.)

We now define inductively our main dynamic strategic concept. A player i is called the *loser of a game G according to the order of play σ* if

1. If $\sigma(1) = i$ then i is the loser of all its options G^i according the order of play σ'.
2. If $\sigma(1) \neq i$, then there exists a $G^{\sigma(1)}$ such that i is the loser of $G^{\sigma(1)}$ according to the order of play σ'.

Intuitively speaking, this means that i will lose any match according to the order of play σ, provided that all the other players act "in concert".

Proposition 2.2. *Suppose $G <_S 0$ and suppose that σ is any order of play. Let j be minimal such that $\sigma(j) \in S$. Then $\sigma(j)$ is the loser of the game G according to the order of play σ.*

Proof. Induction. If $\sigma(1) = i$, then always $G^i <_{\{i\}} 0$, so the induction hypothesis applies. If $\sigma(1) \neq i$, then there exists a $G^{\sigma(1)}$ such that $G^{\sigma(1)} <_U 0$ for some $i \in U \subseteq S \cup \{\sigma(1)\}$. Note that i satisfies the induction hypothesis with G replaced by $G^{\sigma(1)}$, and σ replaced by σ'. □

With this new dynamic significance applied to our previous definitions, the definitions can be formulated in a more intuitive way. If we find a set $S \subseteq T$ with $G <_S 0$, then the set S is the set of players who, for some order of play σ, would definitely lose the game if the others acted in concert. This is the reason the set S can be thought of as the set of possible losers of the game G. This can yield intuitive versions of 1 and 2 of the previous definition. $G <_S 0$ means:

1. If $i \in S$, then player i is the *only* possible loser of each of i's options.
2. If $i \in S$ but $j \notin S$, then player j has an option of which i is a possible loser. In addition, this option must not add any new possible losers, except possibly player j himself.

For the purposes of the next section, we shall now define the *sum of games*: Define inductively

$$G + H$$

by

$$(G + H)_i = \{G + H^i\} \cup \{G^i + H\}.$$

The sum of games is understood as follows: playing $G + H$ is the same as playing the games G and H side by side, so that i's options in the game $G+H$ should be to either "move in G" or "move in H." If player i chooses to move in G, he chooses an option G^i of the game G, and the game progresses to the position $G^i + H$. Similarly, moving in H moves the game to some position $G+H^i$. Thus, the set of i's options is defined to be the set $\{G+H^i\}\cup\{G^i+H\}$.

3 Number Games

We continue to assume that T is finite, of cardinality n. We shall work with T-tuples of real numbers (or more generally T-tuples of elements of any ordered field F)

$$g = (g_i)_{i\in T} \tag{1}$$

which satisfy

$$\sum_{i\in T} g_i = 0.$$

Obviously, the set of all such T-tuples is an $n-1$-dimensional vector space over F, which we shall denote by F_T. For $S \subseteq T$, and for the T-tuple (1), we now write

$$g <_S 0 \tag{2}$$

for the unique set S of all $i \in T$ with

$$g_i = \min_{k\in T} g_k.$$

Note that S is always non-empty. We also write

$$g \leq_S 0$$

if $g <_U 0$ for some $U \supseteq S$. Note that $g <_T 0$ is equivalent to $g \leq_T 0$ which is equivalent to $g \sim 0$. We shall write

$$g <_S h$$

if $g - h <_S 0$, and similarly for $\leq_S$. By abuse of notation, we write $<_i$ instead of $<_{\{i\}}$.

Lemma 3.1. *If $g <_i h$, then $g_i - g_j < h_i - h_j$ for all $j \neq i$.*

Proof. $g <_i h$ means $g - h <_i 0$, i.e. $g_i - h_i < g_j - h_j$ for all $j \neq i$. □

Below, we shall need the following construction. For $i \in T$, consider the function

$$p_i : F_T \to F_{T-\{i\}}$$

given by

$$p_i(g) = \left(g_j + \frac{g_i}{n-1}\right)_{j \in T-\{i\}}.$$

(In some cases, we shall also use the symbol $p_i g$ instead of $p_i(g)$.)

The function p_i takes n-tuples in F_T and creates $n-1$-tuples in $F_{T-\{i\}}$ in the most natural way: it evenly divides up the strength of the i^{th} element among all the others.

We now proceed to number games. We begin by recalling briefly Conway number games of 2 players [Con01]. The main point is that to each pair of subsets

$$\langle A|B\rangle$$

of the Conway field $\mathcal{C}$, such that for all $a \in A$, $b \in B$ we have

$$a < b,$$

there is assigned an element

$$x = v\langle A|B\rangle \in \mathcal{C} \tag{3}$$

such that, for all $a \in A$, $b \in B$,

$$a < v\langle A|B\rangle < b.$$

More precisely, the Conway field can be defined inductively in this way. Once again, as in the case of games, for every ordinal α there is the "α'th generation" of elements of the Conway field; the 0'th generation consists of the number 0. The Conway field is linearly ordered, and for an element of the form (3) of generation α, A, B are considered its *defining sets* (such pair of sets is required to exist for x to be of the given generation) if all elements of the sets A, B are of generation $< \alpha$. One then refers to elements of A (resp. B) as x_L (resp. x_R), and writes

$$x = \langle x_L | x_R\rangle.$$

One defines addition in the Conway field inductively by

$$x + y = \langle x_L + y, x + y_L | x_R + y, x + y_R\rangle.$$

Now elements represented as (3) in different ways may however be equal. Inductively, an element x of a generation α is ≥ 0 (resp. ≤ 0) if there is no

x_R which is ≤ 0 (resp. x_L which is ≥ 0). One puts $x = 0$ if $0 \leq x \leq 0$. One defines inductively

$$-x = \langle -x_R | -x_L \rangle$$

and $x = y$ if $x + (-y) = 0$. Multiplication is then defined by

$$xy = \langle x_L y + x y_L - x_L y_L, x_R y + x y_R - x_R y_R | x_L y + x y_R - x_L y_R, x_R y + x y_L - x_R y_L \rangle$$

(based on $(x - x_L)(y - y_L) > 0$ etc.).

One must prove that this indeed works. We refer the reader to [Con01] for details, but the following two properties are crucial for our purposes (they follow quite directly from the inductive definition outlined above):

1. $v(\emptyset) = 0$ and $v(G + H) = v(G) + v(H)$ where one defines
$$\langle A|B\rangle + \langle C|D\rangle = \langle v\langle A|B\rangle + C | v\langle C|D\rangle + B\rangle.$$
2. If $C \supseteq A$ and $D \supseteq B$ and for each $x \in C$ (resp. $y \in D$) $x < v\langle A|B\rangle$ (resp. $v\langle A|B\rangle < y$) then
$$v\langle C|D\rangle = v\langle A|B\rangle.$$

Using this, we define inductively a *number game of* T *players* as a game G for which there exists an n-tuple

$$v(G) \in \mathcal{C}_T$$

such that

1. For all $i \in T$, all G^i are number games, and $v(G^i) <_i v(G)$.
2. For all $i \neq j \in T$,
$$v_i(G) - v_j(G) = v\langle \{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v(G)\} | \\ \{v_i(G^j) - v_j(G^j) : p_j v(G^j) \leq_i p_j v(G)\}\rangle$$

On first glance, the conditions given in the definition of a number game do not seem natural, but intuitive meaning can be given to them. First, the n-tuple $v(G) = (v_1, v_2, \ldots, v_n)$ gives the strengths of the positions of each player. Larger, positive values of v_i indicate better positions for player i; smaller, negative values indicate worse positions.

Thus, the first statement, that $v(G^i) <_i v(G)$, can be understood as follows: Player i's move from G to G^i not only hurts player i's position; it hurts player i's position more than anyone else's position. This seems natural, as moving in a number game should never "improve" one's position compared to any other player.

The second statement defines the quantity $v_i - v_j$ for each i and j, which is understood to be *i's advantage over j in the game G*. This advantage is defined as the number $v\langle A|B\rangle$, where A is a set of possible advantages i could have after moving, and B is the set of possible advantages j could have after moving. This means that i's advantage in G is more than any advantage he would have after choosing one of his own options G^i, but less than the advantage he would gain were his opponent to move to any G^j.

This would completely explain the definition, however, the sets A and B have an additional restriction on them. Take, for example, the definition of the set A, which contains a condition further restricting its members:

$$A = \{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v(G)\}.$$

The condition $v(G^i) \leq_j v(G)$ (disregarding the p_i's) would mean that the move from G to G^i must hurt player j's position the most. The $\leq$ allows for this to be nonstrict, namely, that the move may hurt other players just as much. However, in a number game, i's moves from G to G^i must hurt player i the most, so the condition would never be true without considering the p_i.

Recall that the function p_i takes n-tuples and creates $n-1$-tuples, with player i's strength equally distributed among all other players. So, the restriction $p_i v(G^i) \leq_j p_i v(G)$ means that the inequality is true once player i is no longer considered, namely, that the move from G to G^i must hurt player j at least as much as everyone else, with player i himself excluded. Such moves G^i are called *i's anti-j options*, since they do as much damage to player j as possible.

So, number games can be understood as games G for which each player has a well-defined strength of position, given by the n-tuple $v(G)$. G having a *well-defined* strength means that:

1. In the game G, each player's options must be number games, and a player's move must always damage his own position the most.
2. In the game G, *i's advantage over j* is the Conway field element $\langle A|B\rangle$, where A is the set of all advantages player i could have if he chose an *anti-j move*, and B is the set of all advantages player i could have if his opponent chose an *anti-i move*. Thus, player i's advantage over j only depends on the i- and j-options that are primarily directed against one another.

Lemma 3.2. *The T-tuple $v(G)$, if it exists, is uniquely determined.*

Proof. By (1) of the definition of number game and Lemma 2.1, for all G^i we have

$$v_i(G^i) - v_j(G^i) < v_i(G) - v_j(G)$$

while for all G^j we have

$$v_i(G^j) - v_j(G^j) > v_i(G) - v_j(G).$$

By property (2) of Conway games, (2) of the definition of number games then implies

$$v_i(G) - v_j(G) = v\langle v_i(G^i) - v_j(G^i)|v_i(G^j) - v_j(G^j)\rangle \tag{4}$$

which recursively determines $v(G)$. ◻

Lemma 3.3. *A sum of number games is a number game.*

Proof. By induction, both conditions (1), (2) are obviously additive. In particular, in (2), the right hand side for a sum of games contains the Conway sum of the right hand sides of (2) of the individual games, so we can use properties (1) and (2) of Conway games. □

Corollary 3.4. *(of (4))*

$$v(G+H) = v(G) + v(H).$$ □

Proposition 3.5. *If G is a number game and $v(G) <_S 0$, then $G <_S 0$.*

Recall that $v(G) <_S 0$ means that for each $i \in S$, $v_i = \min_{k \in T} v_k$. So, this will show that those players with the least strength of position are exactly those players who will lose a match of this game for some order of play, if all others act in concert.

Proof. Induction. By the induction hypothesis, condition (1) in the definition of number games implies condition (1) for $G <_S 0$.

Suppose condition (2) for number games is valid for G. Choose $i \notin S$, $j \in S$. Then, by definition of $v(G) <_S 0$,

$$v_i(G) > v_j(G).$$

By (2) for number games and properties of Conway games, there is an option G^i such that

$$v_i(G^i) \geq v_j(G^i),\ p_i v(G^i) \leq_j p_i v(G). \tag{5}$$

The second condition implies that

$$v_j(G^i) - v_k(G^i) \leq v_j(G) - v_k(G) \leq 0 \text{ for all } k \neq i, j, \tag{6}$$

so together with (5) this implies that

$$v_j(G^i) = min\{v_p(H) | p \in T\}.$$

On the other hand, if $k \notin S$, (6) implies that

$$v_j(G^i) < v_k(G^i).$$

Thus, $G^i <_T 0$ for some $j \in T \subseteq S \cup \{i\}$, as required in condition 2 for $G <_S 0$. □

Remark. Since for $g \in \mathcal{C}_T$, there is always a unique $S \subseteq T$, $S \neq \emptyset$ with $g <_S 0$, the converse of the Proposition is also true.

Corollary 3.6. *If G is a number game and $v(G) = 0$ then $G \sim 0$.* □

Corollary 3.7. *A number game G has an inverse, i.e. a game H such that $G + H \sim 0$.*

Proof. The symmetric group Σ_T obviously acts on number games by permuting players. Now we obviously have

$$v\left(\sum_{\sigma\in\Sigma_T} \sigma G\right) = \sum_{\sigma\in\Sigma_T} \sigma v(G) = 0,$$

so

$$\sum_{\sigma\in\Sigma_T} \sigma G \sim 0$$

by the previous Corollary. □

4 The Existence Theorem

In this section, we prove that number games of arbitrary values exist.

Theorem 4.1 (Existence theorem). *For every $g \in \mathcal{C}_T$ there exists a number game G with*

$$v(G) = g.$$

Proof. We begin by constructing games that are "worth one move" to each player, then from there games that are "worth x moves" to each player for any $x > 0 \in \mathcal{C}$. Sums and inverses of these games will then be enough to construct a game with $v(G) = g$ for any $g \in \mathcal{C}_T$.

First, we construct the game ${}^{(\mathbf{i})}\mathbf{1}$ for each $i \in T$, the *game worth one move to player i*. It is constructed by

$$\begin{aligned} {}^{(\mathbf{i})}\mathbf{1}_i &= \{0\}, \\ {}^{(\mathbf{i})}\mathbf{1}_j &= \emptyset \text{ for } j \neq i. \end{aligned}$$

In this game, player i has only one option: the move to the zero game. No other players have any options. This is a number game with

$$v({}^{(\mathbf{i})}\mathbf{1}) = {}^{(i)}v \in \mathbb{C}_T$$

where

$${}^{(i)}v_i = \frac{n-1}{n}, \qquad {}^{(i)}v_j = -\frac{1}{n}.$$

For example, for $T = \{1, 2, 3\}$, this construction yields three games:

$$
\begin{aligned}
{}^{(1)}\mathbf{1}: &\quad v({}^{(1)}\mathbf{1}) = (\ \ 2/3, -1/3, -1/3)\\
{}^{(2)}\mathbf{1}: &\quad v({}^{(2)}\mathbf{1}) = (-1/3, \ \ 2/3, -1/3)\\
{}^{(3)}\mathbf{1}: &\quad v({}^{(3)}\mathbf{1}) = (-1/3, -1/3, \ \ 2/3)
\end{aligned}
$$

To demonstrate how to check that a game is a number game, we will check that indeed the games ${}^{(i)}\mathbf{1}$ above are number games, with $v(G) = ((n-1)/n, -1/n, \ldots, -1/n)$. Obviously, it suffices to consider $i = 1$.

To prove ${}^{(1)}\mathbf{1}$ is a number game, we need to check the two conditions. First, we must check that all the options of G are number games, which they are, and additionally, that $v(G^i) <_i v(G)$ for all i and all G^i.

There is only one option to check, namely $G^i = \mathbf{0}$ and $i = 1$. Indeed, $\mathbf{0}$ is a number game, with $v(\mathbf{0}) = (0,0,\ldots,0)$. So, to check that $v(\mathbf{0}) <_1 v({}^{(1)}\mathbf{1})$, we need only that

$$
\begin{aligned}
v(\mathbf{0}) - v({}^{(1)}\mathbf{1}) &<_1 0\\
(0,0,\ldots,0) - ((n-1)/n, -1/n, \ldots, -1/n) &<_1 0\\
(-(n-1)/n, 1/n, \ldots, 1/n) &<_1 0
\end{aligned}
$$

which is true. So, condition (1) for number games is satisfied here.

Now, to check condition (2), we need to make sure that the definition's $v_i - v_j$ match up with what we claimed they were by setting $v(G) = ((n-1)/n, -1/n, \ldots, -1/n)$.

First, check $v_1 - v_2$. We should get that $v_1 - v_2 = (n-1)/n + 1/n = 1$. Indeed,

$$
\begin{aligned}
v_1 - v_2 &= v\langle \{v_1(G^1) - v_2(G^1) : p_1 v(G^1) \leq_2 p_1 v({}^{(1)}\mathbf{1})\}\\
&\qquad |\{v_1(G^2) - v_2(G^2) : p_2 v(G^2) \leq_1 p_2 v({}^{(1)}\mathbf{1})\}\rangle\\
&= v\langle \{v_1(G^1) - v_2(G^1) : p_1 v(G^1) \leq_2 p_1 v({}^{(1)}\mathbf{1})\} | \emptyset\rangle\\
&= v\langle \{v_1(\mathbf{0}) - v_2(\mathbf{0}) : p_1 v(\mathbf{0}) \leq_2 p_1 v({}^{(1)}\mathbf{1})\} | \emptyset\rangle\\
&= v\langle 0 - 0 | \emptyset\rangle \qquad (\text{we have } (0,\ldots,0) = p_1 v(\mathbf{0}) \leq_2 p_1 v({}^{(1)}\mathbf{1}) = (0,\ldots,0))\\
&= v\langle 0 | \emptyset\rangle\\
&= 1.
\end{aligned}
$$

Checking $v_1 - v_i$ is similar for other i. And, checking $v_i - v_j$ for $i, j \neq 1$ ($i \neq j$) is easy, since we want $v_i - v_j = 0$, and indeed, it is

$$
\begin{aligned}
&v\langle \{v_i(G^i) - v_j(G^i) : p_i v(G^i) \leq_j p_i v({}^{(1)}\mathbf{1})\} |\\
&\quad \{v_i(G^j) - v_j(G^j) : p_j v(G^j) \leq_i p_j v({}^{(1)}\mathbf{1})\}\rangle
\end{aligned}
$$

$$= v\langle\emptyset|\emptyset\rangle$$
$$= 0.$$

Now, we construct games ${}^{(\mathbf{i})}\mathbf{x}$ for all numbers $x \in \mathbb{C}$. The game ${}^{(\mathbf{i})}\mathbf{x}$ is *the game worth x moves to player i.*

Lemma 4.2. *Given $x \geq 0 \in \mathbb{C}$ and $i \in T$, there exists a number game ${}^{(\mathbf{i})}\mathbf{x}$ such that*

$$v({}^{(\mathbf{i})}\mathbf{x}) = x \cdot v({}^{(\mathbf{i})}\mathbf{1}).$$

For $x = 0$, the zero game satisfies this condition, so we need only consider $x > 0$. Once this construction is complete, the proof will be nearly finished.

Given $x \in \mathbb{C}$, we know that it is constructed by

$$x = v\langle L|R\rangle$$

for some sets L, R of simpler numbers in $\mathcal{C}$. In addition, since $x > 0$, we have that all $x^R > 0$. We can assume without loss of generality that all $x^L \geq 0$ as well, so by induction, we may assume that we have already constructed the "simpler" number games ${}^{(\mathbf{j})}\mathbf{x}^{\mathbf{L}}$ and ${}^{(\mathbf{j})}\mathbf{x}^{\mathbf{R}}$ for all $x^L \in L$, $x^R \in R$, and $j \in T$.

Now define an inverse ${}^{(\mathbf{j})}-\mathbf{x}^{\mathbf{R}}$ of ${}^{(\mathbf{j})}\mathbf{x}^{\mathbf{R}}$ as the sum of previously constructed number games:

$${}^{(\mathbf{j})}-\mathbf{x}^{\mathbf{R}} = \sum_{k \neq j} {}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}.$$

To prove that ${}^{(\mathbf{j})}-\mathbf{x}^{\mathbf{R}}$ is an inverse of ${}^{(\mathbf{j})}\mathbf{x}^{\mathbf{R}}$, note that by Corollary 3.7, we have that every game G has an inverse given by the sum of all the games that are the result of permuting the players of G. So, it suffices to show that ${}^{(\mathbf{j})}-\mathbf{x}^{\mathbf{R}}$ is the sum of the $(n! - 1)$ number games which are permutations of the game G. These permutations are still number games, and they are all given by ${}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}$ for some k:

$$\begin{aligned}
{}^{(\mathbf{j})}-\mathbf{x}^{\mathbf{R}} &= \left(((n-1)! - 1) \cdot {}^{(\mathbf{j})}\mathbf{x}^{\mathbf{R}}\right) + \left(((n-1)!) \cdot \sum_{k \neq j} {}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}\right) \\
&= \left(((n-1)! - 1) \cdot {}^{(\mathbf{j})}\mathbf{x}^{\mathbf{R}}\right) + \left(((n-1)! - 1) \cdot \sum_{k \neq j} {}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}\right) + \left(\sum_{k \neq j} {}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}\right) \\
&= \left(((n-1)! - 1) \cdot \sum_{\text{all k}} {}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}\right) + \left(\sum_{k \neq j} {}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}\right) \\
&= \left(((n-1)! - 1) \cdot \mathbf{0}\right) + \left(\sum_{k \neq j} {}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}\right) \\
&= \sum_{k \neq j} {}^{(\mathbf{k})}\mathbf{x}^{\mathbf{R}}.
\end{aligned}$$

Now consider the game G given by

$$G_i = \{{}^{(\mathrm{i})}\mathbf{x}^{\mathbf{L}}\},$$
$$G_j = \{{}^{(\mathrm{j})}-\mathbf{x}^{\mathbf{R}}\},\ \text{for } j \neq i.$$

Claim. G is the game ${}^{(\mathrm{i})}\mathbf{x}$ that satisfies the lemma, i.e. it is a number game with $v(G) = x \cdot v({}^{(\mathrm{i})}\mathbf{1})$.

To show that G is a number game with the given tuple, we need to check the conditions (1) and (2) for a number game.

For condition (1), we first need that each option of G is a number game, which is true by induction. Then we must show that each option G^k has $v(G^k) <_k v(G)$.

For player i, then, we must show that

$$v({}^{(\mathrm{i})}\mathbf{x}^{\mathbf{L}}) <_i x \cdot v({}^{(\mathrm{i})}\mathbf{1}).$$

But by induction, the left-hand side is given by

$$x^L \cdot v({}^{(\mathrm{i})}\mathbf{1}),$$

so it must be proven that

$$x^L \cdot v({}^{(\mathrm{i})}\mathbf{1}) <_i x \cdot v({}^{(\mathrm{i})}\mathbf{1})$$
$$(x^L - x) \cdot v({}^{(\mathrm{i})}\mathbf{1}) <_i 0.$$

This is true: since $x^L - x$ is negative, while the only positive entry of $v({}^{(\mathrm{i})}\mathbf{1})$ is in the i^{th} position, we have that the only negative value of the n-tuple is in the i^{th} position. So, it is $<_i 0$.

For other players j, we must show that

$$v({}^{(\mathrm{j})} - \mathbf{x}^{\mathbf{R}}) <_j x \cdot v({}^{(\mathrm{i})}\mathbf{1}).$$

But by induction, we already know $v({}^{(\mathrm{j})} - \mathbf{x}^{\mathbf{R}})$ is a number game, and since it is the inverse of ${}^{(\mathrm{j})}\mathbf{x}^{\mathbf{R}}$, we know $v({}^{(\mathrm{j})} - \mathbf{x}^{\mathbf{R}}) = -v({}^{(\mathrm{j})}\mathbf{x}^{\mathbf{R}})$. So, we need to show

$$v({}^{(\mathrm{j})} - \mathbf{x}^{\mathbf{R}}) <_j x \cdot v({}^{(\mathrm{i})}\mathbf{1})$$
$$-v({}^{(\mathrm{j})}\mathbf{x}^{\mathbf{R}}) <_j x \cdot v({}^{(\mathrm{i})}\mathbf{1})$$
$$-x^R \cdot v({}^{(\mathrm{j})}\mathbf{1}) <_j x \cdot v({}^{(\mathrm{i})}\mathbf{1})$$
$$-x^R \cdot v({}^{(\mathrm{j})}\mathbf{1}) - x \cdot v({}^{(\mathrm{i})}\mathbf{1}) <_j 0.$$