567 LECTURE NOTES IN ECONOMICS AND MATHEMATICAL SYSTEMS

Akira Namatame · Taisei Kaizouji
Yuuji Aruga (Editors)

The Complex Networks of Economic Interactions

Essays in Agent-Based Economics and Econophysics

Springer

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Preface

Understanding the mechanism of a socio-economic system requires more than an understanding of the individuals that comprise the system. It also requires understanding how individuals interact with each other, and how the aggregated outcome can be more than the sum of individual behaviors. This book contains the papers fostering the formation of an active multi-disciplinary community on socio-economic systems with the exciting new fields of agent-based modeling and econophysics.

We especially intend to increase the awareness of researchers in many fields with sharing the common view many economic and social activities as collectives of a large-scale heterogeneous and interacting agents.

Economists seek to understand not only how individuals behave but also how the interaction of many individuals leads to complex outcomes. Agent-based modeling is a method for studying socio-economic systems exhibiting the following two properties: (1) the system is composed of interacting agents, and (2) the system exhibits emergent properties, that is, properties arising from the interactions of the agents that cannot be deduced simply by aggregating the properties of the system's components. When the interaction of the agents is contingent on past experience, and especially when the agents continually adapt to that experience, mathematical analysis is typically very limited in its ability to derive the outcome.

Many physicists have contributed to a better understanding of large-scale properties of socio-economic systems, and they open the new research field, "econophysics". An international scientific development has started to gain new insight into the dynamics of socio-economic systems by using methods originally developed in statistical physics and complex theory. This book also covers the current achievements in this rapidly changing field.

This book contains selected papers presented at the 9-th International Workshop on Heterogeneous Interacting Agents (WEHIA), which was held at Kyoto University, Japan, from May 27 to 29, 2004. From the broad spectrum of activities, leading experts presented important papers and numerous practical problems appear throughout this book. We also encouraged papers dealing with applications of agent-based modeling.

WEHIA was initiated as a result of the growing recognition of the importance of agent-based modeling to study large-scale socio-economic systems at University of Ancona, Italy in 1996. The annual series of WEHIA serve for sharing the most recent theoretical applications and methodological advances on agent-based approaches throughout economists, physicists, computer scientists, and other scientists in professionals. The main goals of WEHIA have been to promote interactions and cross-fertilization among different approaches to understanding complex and emergent behaviors and to mange large-scale socio-economic systems.

WEHIA confers especially to encourage papers at the cutting-edge of other approaches that are relevant socio-economic systems. By bringing together three different emerging fields, economics, echonophysics and computer science under the same umbrella, WEHIA stresses the expanding importance of importance close communication and cooperation of the three areas for the future scientific and technological development. The genuinely interdisciplinary approach will enable researchers and students to expand their socio-economic knowledge and to draw up concepts for future interdisciplinary academic achievement.

Based on the success of WEHIA for many years, the new association, "The society for Economic Science with Heterogeneous Interacting Agents" (ESHIA) (www.es-hia.org) will be launched in 2006. The official society journal, "Journal of Economic Interaction and Coordination" (JEIC) will be published from Springer in 2006. The new society, ESHIA especially features in-depth coverage of important areas and aims to contribute scientific ally in three directions: (1) To examine theoretical and methodological issues of agent-based modeling. (2) To discuss multi-agents based simulations and demonstrate applicability in order to study complex economic behaviors. (3) To contribute to develop methodological tools of agent-based modeling and apply them to complex economic and social problems.

We could solicit many high quality papers that reflect the result of the growing recognition of the importance of the areas. All papers have received a careful and supportive review, and we selected 22 papers out of 94. The contributions were submitted as a full paper and reviewed by senior researchers from the program committee. All authors revised their earlier versions presented at the workshop with reflecting criticisms and comments received at the workshop. The editors would like to thank the program committee for the careful review of the papers and the sponsors and volunteers for their valuable contribution. We hope that as a result of reading the book you will share with us the intellectual excitement and interest in this emerging discipline.

We are grateful to the many people who have made this symposium possible. First and foremost, we thank the authors for providing manuscripts on time and in a standard format. We also thank the many referees who gen-

erously contributed time and Dr. Hiroshi Sato to ensure the quality of the finished product.

Finally, we would like to acknowledge the support and encouragement of many peoples in helping us getting this book to be published. Especially the publication of this book and the 9th WEHIA are financially supported by the grant from the Commerative Organization for the Japan World Exposition ('70), Hayasibara Foundation, Kozo Keikaku Engineering Inc. We would like also thank for the grant-in-aid for Scientific Research (C) No.15201038, Japan Society for the promotion of Science (JSPS).

October 2005

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Contents

Part I

Econophysics

Five Years of Continuous-time Random Walks in Econophysics

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Summary. This paper is a short review on the application of continuos-time random walks to Econophysics in the last five years.

1 Introduction

Recently, there has been an increasing interest on the statistical properties of high-frequency financial data related to market microstructural properties [1, 2, 3, 4, 5, 6]. High-frequency econometrics is now well established after research on autoregressive conditional duration models [7, 8, 9, 10].

In high-frequency financial data not only returns but also waiting times between consecutive trades are random variables [11]. This remark is present in a paper by Lo and McKinlay [12], but it can be traced at least to papers on the application of compound Poisson processes [13] and subordinated stochastic processes [14] to finance. Models of tick-by-tick financial data based on compound Poisson processes can also be found in the following references: [15, 16, 17].

Compound Poisson processes are an instance of continuous-time random walks (CTRWs) [18]. The application of CTRW to economical problems dates back, at least, to the 1980s. In 1984, Rudolf Hilfer discussed the application of stochastic processes to operational planning, and used CTRWs as tools for sale forecasts [19]. The revisited and augmented CTRW formalism has been applied to high-frequency price dynamics in financial markets by our research group since 2000, in a series of three papers [20, 21, 22]. Other scholars have recently used this formalism [23, 24, 25]. However, already in 1903, the PhD thesis of Filip Lundberg presented a model for ruin theory of insurance companies, which was later developed by Cramér [26, 27]. The underlying stochastic process of the Lundberg-Cramér model is another example of compound Poisson process and thus also of CTRW.

Among other issues, we have studied the independence between log-returns and waiting times for stocks traded at the New York Stock Exchange in October 1999. For instance, according to a contingency-table analysis performed on General Electric (GE) prices, the null hypothesis of independence can be rejected with a significance level of 1 % [28]. We have also discussed the anomalous non-exponential behaviour of the unconditional waiting-time distribution between tick-by-tick trades both for future markets [21] and for stock markets [28, 29]. Different waiting-time scales have been investigated in different markets by various authors. All these empirical analyses corroborate the waiting-time anomalous behaviour. A study on the waiting times in a contemporary FOREX exchange and in the XIXth century Irish stock market was presented by Sabatelli *et al.* [30]. They were able to fit the Irish data by means of a Mittag-Leffler function as we did before in a paper on the waiting-time marginal distribution in the German-bund future market [21]. Kyungsik Kim and Seong-Min Yoon studied the tick dynamical behavior of the bond futures in Korean Futures Exchange (KOFEX) market and found that the survival probability displays a stretched-exponential form [31]. Finally, Ivanov *et al.* [33] confirmed that a stretched exponential fits well the survival distribution for NYSE stocks as we suggested in [28]. Moreover, just to stress the relevance of non-exponential waiting times, a power-law distribution has been recently detected by T. Kaizoji and M. Kaizoji in analyzing the calm time interval of price changes in the Japanese market [32]. We have offered a possible explanation of the anomalous waiting-time behaviour in terms of daily variable activity [29].

The aforementioned empirical results are important as market microstructural models should be able to reproduce such a non-exponential behaviour of waiting-time distributions in order to be realistic. However, the rest of this paper focuses on the theory and is divided as follows: in Section 2, CTRW theory is presented as applied to finance. Finally, in Sec. 3, a summary of main results is presented together with a discussion on the direction of future research.

2 Theory

Random walks have been used in finance since the seminal thesis of Bachelier [34], a work completed at the end of the XIXth century, more than a hundred years ago. After a rather long period in which the ideas of Bachelier were neglected, they were further developed until recent times [35, 36].

Our approach to random walks in finance is related to that of Clark [14] and to the introductory part of Parkinson's paper [37]. It is a purely phenomenological approach. There is no assumption on the rationality or the behaviour of trading agents and it is not necessary to assume the validity of the efficient market hypothesis [38, 39]. However, as briefly discussed above, even in the absence of a *microfoundation*, a phenomenological model can still be useful to corroborate or falsify the consequences of behavioural or other assumptions on markets. The model itself can be corroborated or falsified by empirical data.

In order to model tick-by-tick data, we use the so-called continuous-time random walk (CTRW), where time intervals between successive steps are random variables, as discussed by Montroll and Weiss [18]. In physics, CTRWs have been introduced as models of diffusion with instantaneous jumps from one position to the next. For this reason they can be used as models of price dynamics as well.

Let $S(t)$ denote the price of an asset at time t. In a real market with a double-auction mechanism, prices are fixed when buy orders are matched with sell orders and a transaction (trade) occurs. It is more convenient to refer to returns rather than prices. For this reason, we shall take into account the variable $x(t) = \log S(t)$: the logarithm of the price. For a small price variation $\Delta S = S(t_{i+1}) - S(t_i)$, the return $r = \Delta S/S(t_i)$ and the logarithmic return $r_{log} = log[S(t_{i+1})/S(t_i)]$ virtually coincide.

CTRWs are essentially point processes with reward [40]. The point process is characterized by a sequence of independent identically distributed (i.i.d.) positive random variables τ_i, which can be interpreted as waiting times between two consecutive events:

$$t_n = t_0 + \sum_{i=1}^{n} \tau_i; \;\; t_n - t_{n-1} = \tau_n; \;\; n = 1, 2, 3, \ldots; \;\; t_0 = 0. \tag{1}$$

The rewards are (i.i.d.) not necessarily positive random variables: ξ_i. In the usual physical intepretation, the ξ_is represent the jumps of a diffusing particle, and they can be n-dimensional vectors. Here, only the 1-dimensional case is studied, but the extension of many results to the n-dimensional case is straightforward. The position x of the particle at time t is given by the following random sum (with $N(t) = \max\{n : \; t_n \leq t\}$ and $x(0) = 0$):

$$x(t) = \sum_{i=1}^{N(t)} \xi_i. \tag{2}$$

In the financial interpretation outlined above, the ξ_i's have the meaning of log-returns, whereas the *positions* or rewards $x(t)$ represent log-prices at time t. Indeed, the time series $\{x(t_i)\}$ is characterised by $\varphi(\xi, \tau)$, the *joint probability density* of log-returns $\xi_i = x(t_{i+1}) - x(t_i)$ and of waiting times $\tau_i = t_{i+1} - t_i$. The joint density satisfies the normalization condition $\int \int d\xi d\tau \varphi(\xi, \tau) = 1$. It must be again remarked that both ξ_i and τ_i are assumed to be independent and identically distributed (i.i.d.) random variables. This strong assumption is useful to derive limit theorems for the stochastic processes described by CTRWs. However, in financial time series, the presence of volatility clustering, as well as correlations between waiting times do falsify the i.i.d hypothesis. The reader interested in a review on correlated random variables in finance is referred to chapter 8 in McCauley's recent book [41].

In general, log-returns and waiting times are not independent from each other [28]. By probabilistic arguments (see [18, 21, 42]), one can derive the following integral equation that gives the probability density, $p(x,t)$, for the particle of being in position x at time t, conditioned by the fact that it was in position $x=0$ at time $t=0$:

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \int_{-\infty}^{+\infty} \varphi(x-x', t-t')\,p(x',t')\,dt'\,dx', \tag{3}$$

where $\delta(x)$ is Dirac's delta function and $\Psi(\tau)$ is the so-called survival function. $\Psi(\tau)$ is related to the marginal waiting-time probability density $\psi(\tau)$. The two marginal densities $\psi(\tau)$ and $\lambda(\xi)$ are:

$$\begin{aligned} \psi(\tau) &= \int_{-\infty}^{+\infty} \varphi(\xi,\tau)\,d\xi \\ \lambda(\xi) &= \int_0^{\infty} \varphi(\xi,\tau)\,d\tau, \end{aligned} \tag{4}$$

and the survival function $\Psi(\tau)$ is:

$$\Psi(\tau) = 1 - \int_0^{\tau} \psi(\tau')\,d\tau' = \int_{\tau}^{\infty} \psi(\tau')\,d\tau'. \tag{5}$$

Both the two marginal densities and the survival function can be empirically derived from tick-by-tick financial data in a direct way.

The integral equation, eq. (3), can be solved in the Laplace-Fourier domain. The Laplace transform, $\widetilde{g}(s)$ of a (generalized) function $g(t)$ is defined as:

$$\widetilde{g}(s) = \int_0^{+\infty} dt\;\mathrm{e}^{-st}\,g(t)\,, \tag{6}$$

whereas the Fourier transform of a (generalized) function $f(x)$ is defined as:

$$\widehat{f}(\kappa) = \int_{-\infty}^{+\infty} dx\,\mathrm{e}^{i\kappa x}\,f(x)\,. \tag{7}$$

A generalized function is a distribution (like Dirac's δ) in the sense of S. L. Sobolev and L. Schwartz [43].

One gets:

$$\widehat{\widetilde{p}}(\kappa,s) = \widetilde{\Psi}(s)\,\frac{1}{1-\widehat{\widetilde{\varphi}}(\kappa,s)}, \tag{8}$$

or, in terms of the density $\psi(\tau)$:

$$\widehat{\widetilde{p}}(\kappa,s) = \frac{1-\widetilde{\psi}(s)}{s}\,\frac{1}{1-\widehat{\widetilde{\varphi}}(\kappa,s)}, \tag{9}$$

as, from eq. (5), one has:

$$\Psi(s) = \frac{1 - \widetilde{\psi}(s)}{s}. \tag{10}$$

In order to obtain $p(x,t)$, it is then necessary to invert its Laplace-Fourier transform $\widehat{\widetilde{p}}(\kappa, s)$. As we shall see in the next subsection, for log-returns independent from waiting times, it is possible to derive a series solution to the integral equation (3).

2.1 Limit Theorems: The Uncoupled Case

In a recent paper, Gorenflo, Mainardi and the present author have discussed the case in which log-returns and waiting times are independent [42]. It is the so-called uncoupled case, when it is possible to write the joint probability density of log-returns and waiting times as the product of the two marginal densities:

$$\varphi(\xi, \tau) = \lambda(\xi)\psi(\tau) \tag{11}$$

with the normalization conditions $\int d\xi \lambda(\xi) = 1$ and $\int d\tau \psi(\tau) = 1$.

In this case the integral master equation for $p(x,t)$ becomes:

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \psi(t-t')\,\left[\int_{-\infty}^{+\infty} \lambda(x-x')\,p(x',t')\,dx'\right]\,dt' \tag{12}$$

This equation has a well known general explicit solution in terms of $P(n,t)$, the probability of n jumps occurring up to time t, and of the n-fold convolution of the jump density, $\lambda_n(x)$:

$$\lambda_n(x) = \int_{-\infty}^{+\infty} \ldots \int_{-\infty}^{+\infty} d\xi_{n-1} \ldots d\xi_1 \lambda(x - \xi_{n-1}) \ldots \lambda(\xi_1). \tag{13}$$

Indeed, $P(n,t)$ is given by:

$$P(n,t) = \int_0^t \psi_n(t-\tau)\Psi(\tau)\,d\tau \tag{14}$$

where $\psi_n(\tau)$ is the n-fold convolution of the waiting-time density:

$$\psi_n(\tau) = \int_0^{\tau} \ldots \int_0^{\tau_1} d\tau_{n-1} \ldots d\tau_1 \psi(\tau - \tau_{n-1}) \ldots \psi(\tau_1). \tag{15}$$

The n-fold convolutions defined above are probability density functions for the sum of n independent variables.

Using the Laplace-Fourier method and recalling the properties of Laplace-Fourier transforms of convolutions, one gets the following solution of the integral equation [44, 42, 45, 46]:

$$p(x,t) = \sum_{n=0}^{\infty} P(n,t)\lambda_n(x) \tag{16}$$

Eq. (16) can also be used as the starting point to derive eq. (12) via the transforms of Fourier and Laplace, as it describes a jump process subordinated to a renewal process [14, 47].

Let us now consider the following pseudodifferential equation, giving rise to anomalous relaxation and power-law tails in the waiting-time probability:

$$\frac{d^\beta}{d\tau^\beta}\Psi(\tau) = -\Psi(\tau),\ \ \tau > 0,\ \ 0 < \beta \leq 1;\ \ \Psi(0^+) = 1, \tag{17}$$

where the operator d^β/dt^β is the Caputo fractional derivative, related to the Riemann–Liouville fractional derivative. For a sufficiently well-behaved function $f(t)$, the Caputo derivative is defined by the following equation, for $0 < \beta < 1$:

$$\frac{d^\beta}{dt^\beta}f(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\beta}\,d\tau - \frac{t^{-\beta}}{\Gamma(1-\beta)}f(0^+), \tag{18}$$

and reduces to the ordinary first derivative for $\beta = 1$. The Laplace transform of the Caputo derivative of a function $f(t)$ is:

$$\mathcal{L}\left(\frac{d^\beta}{dt^\beta}f(t);\, s\right) = s^\beta \widetilde{f}(s) - s^{\beta-1}f(0^+). \tag{19}$$

If eq. (19) is applied to the Cauchy problem of eq. (17), one gets:

$$\widetilde{\Psi}(s) = \frac{s^{\beta-1}}{1+s^\beta}. \tag{20}$$

Eq. (20) can be inverted, giving the solution of eq. (17) in terms of the Mittag-Leffler function of parameter β [48, 49]:

$$\Psi(\tau) = E_\beta(-\tau^\beta), \tag{21}$$

defined by the following power series in the complex plane:

$$E_\beta(z) := \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\beta n+1)}. \tag{22}$$

The Mittag-Leffler function is a possible model for a fat-tailed survival function. For $\beta = 1$, the Mittag-Leffler function coincides with the ordinary exponential function. For small τ, the Mittag-Leffler survival function coincides with the stretched exponential:

$$\Psi(\tau) = E_\beta(-\tau^\beta) \simeq 1 - \frac{\tau^\beta}{\Gamma(\beta+1)} \simeq \exp\{-\tau^\beta/\Gamma(\beta+1)\},\ 0 \leq \tau \ll 1, \tag{23}$$

whereas for large τ, it has the asymptotic representation:

$$\Psi(\tau) \sim \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta)}{\tau^{\beta}}, \; 0 < \beta < 1, \; \tau \to \infty. \tag{24}$$

Accordingly, for small τ, the probability density function of waiting times $\psi(\tau) = -d\Psi(\tau)/d\tau$ behaves as:

$$\psi(\tau) = -\frac{d}{d\tau} E_{\beta}(-\tau^{\beta}) \simeq \frac{\tau^{-(1-\beta)}}{\Gamma(\beta)}, \; 0 \leq \tau \ll 1, \tag{25}$$

and the asymptotic representation is:

$$\psi(\tau) \sim \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta+1)}{\tau^{\beta+1}}, \; 0 < \beta < 1, \; \tau \to \infty. \tag{26}$$

The Mittag-Leffler function is important as, without passage to the diffusion limit, it leads to a time-fractional master equation, just by insertion into the CTRW integral equation. This fact was discovered and made explicit for the first time in 1995 by Hilfer and Anton [50]. Therefore, this special type of waiting-time law (with its particular properties of being singular at zero, completely monotonic and long-tailed) may be best suited for approximate CTRW Monte Carlo simulations of fractional diffusion.

For processes with survival function given by the Mittag-Leffler function, the solution of the master equation can be explicitly written:

$$p(x,t) = \sum_{n=0}^{\infty} \frac{t^{\beta n}}{n!} E_{\beta}^{(n)}(-t^{\beta}) \lambda_n(x), \tag{27}$$

where:

$$E_{\beta}^{(n)}(z) := \frac{d^n}{dz^n} E_{\beta}(z).$$

The Fourier transform of eq. (27) is the characteristic function of $p(x,t)$ and is given by:

$$\hat{p}(\kappa,t) = E_{\beta}[t^{\beta}(\hat{\lambda}(\kappa) - 1)]. \tag{28}$$

If log-returns and waiting times are scaled according to:

$$x_n(h) = h\xi_1 + h\xi_2 + \ldots + h\xi_n, \tag{29}$$

and:

$$t_n(r) = r\tau_1 + r\tau_2 + \ldots + r\tau_n, \tag{30}$$

the scaled characteristic function becomes:

$$\hat{p}_{h,r}(\kappa,t) = E_{\beta}\left[\frac{t^{\beta}}{r^{\beta}}(\hat{\lambda}(h\kappa) - 1)\right]. \tag{31}$$

Now, if we assume the following asymptotic behaviours for vanishing h and r:

$$\hat{\lambda}(h\kappa) \sim 1 - h^{\alpha}|\kappa|^{\alpha}; \;\; 0 < \alpha \leq 2, \tag{32}$$

and

$$\lim_{h,r\to 0} \frac{h^\alpha}{r^\beta} = 1, \tag{33}$$

we get that:

$$\lim_{h,r\to 0} \hat{p}_{h,r}(\kappa, t) = \hat{u}(\kappa, t) = E_\beta[-t^\beta |\kappa|^\alpha]. \tag{34}$$

The Laplace transform of eq. (34) is:

$$\widetilde{\widehat{u}}(\kappa, s) = \frac{s^{\beta-1}}{|\kappa|^\alpha + s^\beta}. \tag{35}$$

Therefore, the well-scaled limit of the CTRW characteristic function coincides with the Green function of the following pseudodifferential *fractional* diffusion equation:

$$|\kappa|^\alpha \widetilde{\widehat{u}}(\kappa, s) + s^\beta \widetilde{\widehat{u}}(\kappa, s) = s^{\beta-1}, \tag{36}$$

with $u(x, t)$ given by:

$$u(x, t) = \frac{1}{t^{\beta/\alpha}} W_{\alpha,\beta}\left(\frac{x}{t^{\beta/\alpha}}\right), \tag{37}$$

where $W_{\alpha,\beta}(u)$ is given by:

$$W_{\alpha,\beta}(u) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\kappa\, \mathrm{e}^{-i\kappa u} E_\beta(-|\kappa|^\alpha), \tag{38}$$

the inverse Fourier transform of a Mittag-Leffler function [51, 52, 53, 54, 55].

For $\beta = 1$ and $\alpha = 2$, the fractional diffusion equation reduces to the ordinary diffusion equation and the function $W_{2,1}(u)$ becomes the Gaussian probability density function evolving in time with a variance $\sigma^2 = 2t$. In the general case ($0 < \beta < 1$ and $0 < \alpha < 2$), the function $W_{\alpha,\beta}(u)$ is still a probability density evolving in time, and it belongs to the class of Fox H-type functions that can be expressed in terms of a Mellin-Barnes integral as shown in details in ref. [52].

The scaling equation, eq. (33), can be written in the following form:

$$h \simeq r^{\beta/\alpha}. \tag{39}$$

If $\beta = 1$ and $\alpha = 2$, one recognizes the scaling relation typical of Brownian motion (or the Wiener process).

In the passage to the limit outlined above, $\widetilde{\widehat{p}}_{r,h}(\kappa, s)$ and $\widetilde{\widehat{u}}(\kappa, s)$ are asymptotically equivalent in the Laplace-Fourier domain. Then, the asymptotic equivalence in the space-time domain between the master equation and the fractional diffusion equation is due to the continuity theorem for sequences of characteristic functions, after the application of the analogous theorem for sequences of Laplace transforms [56]. Therefore, there is convergence in law or weak convergence for the corresponding probability distributions and densities. Here, weak convergence means that the Laplace transform and/or Fourier transform (characteristic function) of the probability density function are pointwise convergent (see ref. [56]).

2.2 Limit Theorems: The Coupled Case

The diffusive limit in the coupled case is discussed by Meerschaert *et al.* [57]. The coupled case is relevant as, in general, log-returns and waiting times are not independent [28]. Based on the results summarized in [42] and discussed in [58, 59], it is possible to prove the following theorem for the coupled case:

Theorem

Let $\varphi(\xi, \tau)$ be the (coupled) joint probability density of a CTRW. If, under the scaling $\xi \to h\xi$ and $\tau \to r\tau$, the Fourier-Laplace transform of $\varphi(\xi, \tau)$ behaves as follows:

$$\widetilde{\widehat{\varphi}}_{h,r}(\kappa, s) = \widetilde{\widehat{\varphi}}(h\kappa, rs) \tag{40}$$

and if, for $h \to 0$ and $r \to 0$, the asymptotic relation holds:

$$\widetilde{\widehat{\varphi}}_{h,r}(\kappa, s) = \widetilde{\widehat{\varphi}}(h\kappa, rs) \sim 1 - \mu|h\kappa|^{\alpha} - \nu(rs)^{\beta}, \tag{41}$$

with $0 < \alpha \leq 2$ and $0 < \beta \leq 1$. Then, under the scaling relation $\mu h^{\alpha} = \nu r^{\beta}$, the solution of the (scaled) coupled CTRW master (integral) equation, eq. (3), $p_{h,r}(x, t)$, weakly converges to the Green function of the fractional diffusion equation, $u(x, t)$, for $h \to 0$ and $r \to 0$.

Proof

The Fourier-Laplace transform of the scaled conditional probability density $p_{h,r}(x, t)$ is given by:

$$\widetilde{\widehat{p}}_{h,r}(\kappa, s) = \frac{1 - \widetilde{\psi}(rs)}{s} \frac{1}{1 - \widetilde{\widehat{\varphi}}(h\kappa, rs)}. \tag{42}$$

Replacing eq. (41) in eq. (42) and observing that $\widetilde{\psi}(s) = \widetilde{\widehat{\varphi}}(0, s)$, one asymptotically gets for small h and r:

$$\widetilde{\widehat{p}}_{h,r}(\kappa, s) \sim \frac{\nu r^{\beta} s^{\beta-1}}{\nu r^{\beta} s^{\beta} + \mu h^{\alpha}|\kappa|^{\alpha}}, \tag{43}$$

which for vanishing h and r, under the hypotheses of the theorem, converges to:

$$\widetilde{\widehat{p}}_{0,0}(\kappa, s) = \widetilde{\widehat{u}}(\kappa, s) = \frac{s^{\beta-1}}{s^{\beta} + |\kappa|^{\alpha}}, \tag{44}$$

where $\widetilde{\widehat{u}}(\kappa, s)$ is the Fourier-Laplace transform of the Green function of the fractional diffusion equation (see eq. (36)). The asymptotic equivalence in the space-time domain, between $p_{0,0}(x, t)$ and $u(x, t)$, the inverse Fourier-Laplace transform of $\widetilde{\widehat{u}}(\kappa, s)$, is again ensured by the continuity theorem for sequences of characteristic functions, after the application of the analogous theorem for sequences of Laplace transforms [56]. There is convergence in law or weak convergence for the corresponding probability distributions and densities.

An important consequence of the above theorem is the following corollary showing that in the case of marginal densities with finite first moment of waiting times and finite second moment of log-returns, the limiting density $u(x, t)$ is the solution of the ordinary diffusion equation (and thus the limiting process is the Wiener process). The corollary can be used to justify the popular Geometric Brownian Motion model of stock prices, here with expected return set to zero. Again, in order to derive this result, no reference is necessary to the Efficient Market Hypothesis [38, 39].

Corollary

If the Fourier-Laplace transform of $\varphi(\xi, \tau)$ is regular for $\kappa = 0$ and $s = 0$, and, moreover, the marginal waiting-time density, $\psi(\tau)$, has finite first moment τ_0 and the marginal jump density, $\lambda(\xi)$, is symmetric with finite second moment σ^2, then the limiting solution of the master (integral) equation for the coupled CTRW is the Green function of the ordinary diffusion equation.

Proof

Due to the hypothesis of regularity in the origin and to the properties of Fourier and Laplace transforms, we have that:

$$\begin{aligned} \widehat{\widetilde{\varphi}}_{h,r}(\kappa, s) &= \widehat{\widetilde{\varphi}}(h\kappa, rs) \sim \widehat{\widetilde{\varphi}}(0,0) + \\ &+ \frac{1}{2}\left(\frac{\partial^2 \widehat{\widetilde{\varphi}}}{\partial \kappa^2}\right)_{(0,0)} h^2\kappa^2 + \left(\frac{\partial \widehat{\widetilde{\varphi}}}{\partial s}\right)_{(0,0)} rs = \\ &= 1 - \frac{\sigma^2}{2} h^2 \kappa^2 - \tau_0 rs, \end{aligned} \tag{45}$$

and as a consequence of the theorem, under the scaling $h^2\sigma^2/2 = \tau_0 r$, one gets, for vanishing h and r:

$$\widehat{\widetilde{p}}_{0,0}(k, s) = \widehat{\widetilde{u}}(k, s) = \frac{1}{s + k^2}, \tag{46}$$

corresponding to the Green function (36) for $\alpha = 2$ and $\beta = 1$, that is the solution of the Cauchy problem for the ordinary diffusion equation.

3 Summary and Outlook

In this paper, a discussion of continuous-time random walks (CTRWs) has been presented as phenomenological models of tick-by-tick market data.

Continuous-time random walks are rather general and they include compound Poisson processes as particular instances. Well-scaled limit theorems have been presented for a rather general class of CTRWs.

It is the hope of this author that this paper will stimulate further research on high-frequency econometrics based on the concepts outlined above. There are several possible developments.

First of all, one can abandon the hypothesis of i.i.d. log-returns and waiting times and consider various forms of dependence. In this case, it is no longer possible to exploit the nice properties of Laplace and Fourier transforms of convolutions, but, still, Monte Carlo simulations can provide hints on the behaviour of these processes in the diffusive limit.

A second possible extension is to include volumes as a third stochastic variable. This extension is straightforward, starting from a three-valued joint probability density.

A third desirable extension is to consider a multivariate rather than univariate model that includes correlations between time series.

The present author is currently involved in these extensions and is eager to know progress in any direction by other independent research groups. He can be contacted at `scalas@unipmn.it`

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Why Macroeconomic Price Indices are Sluggish in Large Economies ?

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Summary. Two new reasons are discussed for sluggish behavior of macroeconomic variables such as price indices.

One is slow spread of the news of microeconomic idiosyncratic shocks in the economy, when the economy is organized into tree structures of heterogeneous subgroups or clusters of agents or goods. Clusters are not symmetrically treated, but the concept of ultrametric distances measure disparities or similarities of clusters.

Another is the effects of uncertainties that affect decision processes, such as about the cost surfaces, or about the shapes of cost landscapes which may have many local minima which are not known precisely. Effectiveness of many search algorithm is reduced in the face of this kind of uncertainty. Flat cost landscapes, called entropic barriers, are discussed as an example.[3]

1 Introduction

The standard approach such as real business cycle theory is based on the premise that the microeconomic behavior of the optimizing agents mimics dynamics of the macroeconomy. This premise is wrong, because we see that macro- and micro-behavior are clearly different in many aspects, such as in their speeds of responses.

Two of the causes for sluggish responses are familiar: complex organizations of macroeconomy, and effects of uncertainty. On complexity of organization, the existing economic literature treats the phenomena of sluggish adjustments or responses of economic variables such as prices, wages, or unemployment rates by treating adjusting variables or agents all on equal footing, that is, without introducing some notions of similarity or closeness between various heterogeneous groups of variables or agents.

It is true that some rudimentary notion of social distances between different clusters of agents is found in economic literature, such as agents being

[3] For some other aspects of uncertainty, see Aoki, Yoshikawa, and Shimizu (2002).

placed at sites of lattices. However, no explicit notions of similarity, correlations or distances between various groups of variables or agents are examined. No models with more formal notion of distances among different groups of agents apparently exist.

The manner of how news of idiosyncratic disturbances of some types or price changes spreads through the macroeconomy need be analyzed in more systematic manner by introducing some notion of distance between clusters of agents.

In our view, attempts at dealing with groups of agents in the economic literature do not go far enough. We introduce the notion of hierarchically structured clusters of goods or producers as an essential ingredient in models designed to explain sluggish adjustment processes.

This paper analyzes a particular aspect of the macroeconomy, namely adjustment speeds of macroeconomic price indicies. We use the concept of ultrametrics as measure of distances between clusters.

Since the publication of Keynes' General Theory (1936), inflexibility or ridigity of prices has been always a forcal point of macroeconomics. Many economists take inflexibility of prices as a sign of agents' irrationality. They argue that well organized market forces should make prices flexible. In this paper we explain that prices are necessarily slow to change in large economies.

To improve on the existing literature, we need more appropriate notion than correlation to measure relations among agents or variables, since the notion of correlation is not transitive as has been known, for example, in the numerical taxonomy literature. Feigelman and Ioffe (1991) have an example of three patterns: A=(1, 1, 1, 1), B=(1,1,-1,1) and C=(1,1,1,-1). Calculating correlations by $\rho = (1/4)\sum_i x_i y_i$ where xs and ys are the components of the patterns above, we see that $\rho_{A,B} = \rho_{A,C} = 1/2$ but $\rho_{B,C} = 0$.

To avoid this kind of intransitivity of correaltions, which makes correlation unsuitable as a measure of similarity of patterns, we use the notion of ultrametrics as a measure of distance between clusters of agents. The concept of ultrametrics has been in the literature of mathematics and physics, especially in spin glass models. For these, see Schikhof (1984), or Mézard, Parisi and Virasoro (1986), among others. See Aoki (1996, p. 34) for some elementary explanation of the ultrametrics, and some simple economic applications. Aoki (2002), and Aoki and Yoshikawa (2003) have a more complex example of labor market dynamics and Okun's law,where unemployed workers from different sectors of economy or different human capitals or job experiences form separate clusters and ultrametrics are used to measure distances between clusters. This distance is then used to generate probabilities of unemployed being recalled by a given industry when job openings are created. Taylor's well-known analysis of adjustment of wages is different from ours. He treats groups of workers with different wage contracts as *the* source of slow wage/price dynamics. His model and virtually all multi-sector models treat sectors symmetrically with equal distances between any two sectors. These groups are not hierarchically arranged. There is no notion of adjustment speeds as functions of some similarity measures among clusters, Taylor (1980).

We place clusters of agents as leaves of a tree. Distances between clusters are measured by counting the number of levels one must travel towards the root of the tree to find a common node shared by the two leaves. This is called ultrametric distances.[4] News, such as that of the arrival of some idiosyncratic shocks at some sites, spread throughout the tree as stochastic processes governed by the backward Chapman-Kolmogorov equation, called the master equaation in this paper, in which transition rates are functions of ultrametric distances.

2 Tree Models

We use upside down trees to represent hierarchical structures. At the bottom of a tree we have leaves, also called sites, where each leaf represents a cluster of agents or a (price of) goods, as the case may be. Agents in the same cluster are alike in some sense. They may be producers of some close substitute goods, or they may have similar reaction or decision making delays given a disturbance of some kind in signals they use, and so on. A number of the leaves, denoted by m_1, share a common node of a tree. These leaves are connected to nodes located on level 2 of the tree. There are m_2 of the nodes which branch out from a node at level three, and so on.[5] In general we have K levels in a tree. The top of the tree is the root consisting of a single cluster with $N = m_1 m_2 \cdots m_K$ number of clusters or leaves.

Without loss of generality we assume that an exogenous idiosyncratic disturbance occurs at site 1 at time zero. Let this disturbance be felt at site i at time t with probability $P_i(t)$. The initial condition is $P_1(0) = 1$, and $P_i(0)$, $i \neq 1$. These Ps are governed by the master equation where transition rates are functions of ultrametric distances. See Aoki(1996, 2002) for several examples.

We also use another measure to gauge the speed with which news or disturbances travel through the tree. We define the expected distance reached by the disturbance originated at site 1 by time t, i.e.,

$$< d(t) >= \sum_i d(i,1) P_i(t),$$

where $d(i,1)$ is the ultrametric distance between node i and site 1, which is the source of news or disturbance. This averaged distance indicates how far, on the average, the news of disturbance has spread through the model.

[4] See for example, Aoki (1996, p. 31).

[5] Trees need not have symmetric shape or profiles, that is, the number of branches from a node can be different from node to node.

3 Two Types of Lags in Tree Dynamics

3.1 Multiplier Lags

Responses of a macroeconomic price index to shocks to one of its component prices consists of two components. One is the well-known dynamic delays in multipliers or impluse responses. which are familiar in economics and econometrics. The other is called information lag in this paper. It refers to delays in the news or effects of exogenous shocks which originated in one sector of an economy spreading to other sectors stochastically. The multiplier lag is simply illustrated here by dynamic responses of a second order ordinary differential equation to a step input changes. Propagations of shocks are treated in this paper stochastically. This requires solving the master equations for states that are the leaves of trees.

To illustrate this, it is convenient to use Laplace transforms to relate output responses to input changes as

$$H(s)Y(s) = U(s),$$

where s is the Laplace transform variable, $Y(s)$ is the transform of the model output that is response to change in input, and $U(s)$ the Laplace transform of input.

We may write this more directly as

$$Y(s) = \frac{U(s)}{H(s)}.$$

The time domain expression is

$$\frac{d^2y}{dt} + (a+b)\frac{dy}{dt} + aby = u.$$

Here $1/H(s)$ is called the transfer function. This expression shows how the signal at the input side of the model is transferred to the output side of the model. If the input signal to the system at time zero at full constant force, then, without loss of generality, we can think of the input signal $u(t) = 1$ for t positive. Its Laplace transform is $U(s) = 1/s$. On the other hand, if the signal gradually appear to this system, we may have something like $u(t) = 1 - e^{-\mu t}$ with $\mu > 0$,for $t > 0$ say. This input is initially zero and graudally reach its full force after about the time elapse of $4/\mu$. Its Laplace transform is $U(s) = \mu/[s(s+\mu)]$.

As a simple illustration of the difference of these two types of input signals on $y(t)$, we assume that dynamics are described by a second order ordinary constant coefficient differential equation with zero initial conditions; $y(0) = 0$, and $dy(0)/dt = 0$. To be very concrete suppose that $H(s) = (s+a)(s+b)$ with some positive a and b. This is the transfer function of a dynamic system described by a second order differential equation with two stable eigenvalues $-a$, and $-b$.

With the step input, the dynamic response is obtained by taking the inverse Laplace transform of

$$Y(s) = \frac{1}{s(s+a)(s+b)} = \frac{C}{s} + \frac{A}{s+a} + \frac{B}{s+b},$$

where A, B, and $C$are the constants, $C = 1/ab$, $A = -1/[(b-a)a]$, and $B = 1/[(b-a)b]$. The time response of the pair of input and output with this transfer function is given by

$$y(t) = C + Ae^{-at} + Be^{-bt}.$$

This $y(t)$ expression shows the multiplier effects of this block or unit of dynamics with the indicated transfer function. If $a < b$, then after the time span of about $4/a$ units of time, the output nearly settles to a constant, $y(t) \approx 1/ab$.[6] It takes about this much time for the effect of a sudden application of a step signal at the input to settle down at the output of the model.

With the other input with a gradually rising magnitude such as $u(t) = 1 - e^{-\mu t}$, with μ a positive constant much smaller than a and b, then $y(t)$ is approximately equal to

$$y(t) = \frac{\mu}{b-a}[-\frac{e^{-at}}{a-\mu} + \frac{e^{-bt}}{b-\mu}] + \frac{\mu}{(a-\mu)(b-\mu)]e^{-\mu t}}.$$

The first two exponential terms are due to the dynamic multiplier effects, and the third term is due to information transmission delay when $u(t)$ gradually appear at the input terminal of this block or unit with the second-order dynamics.

This expression is approximately equal to the last term above when μ is much smaller than a or b. The signal $y(t) \approx (1/ab)(1 - e^{-\lambda t})$, which takes a long time to reach its steady state value. In this case it is the behavior of input, not the dynamics, that causes the sluggish output.

3.2 Stochastic Spread of News in Trees: An example

We next turn to the second type of lags that exist in trees with several levels of nodes.

To illustrate our idea simply, we consider two simple economies with four sectors which are organized in two different ways. One is organized as a one level tree, and the other as a two level tree. Two-level trees are generally more sluggish in response than one-level trees. More generally, the larger the number of levels, the slower the dynamics.

[6] If a is larger than b, then the dynamic lag is about $4/b$, that is, with two basic lag structure, it is $4/min(a, b)$ is the multiplier lag.

Without loss of generality we assume that exogenous disturnaves are felt at site 1 at time zero. This disturbance is felt at site i at time t with probability $P_i(t)$. The initial condition is $P_1(0) = 1$, and $P_i(0) = 0$, $i \neq 1$. We pay more attention to the transient behavior than the steady state values of these probabilities becasue the delays are determined by the transient time constants.

The probability at site i is changing over time as the difference of the influx and outflux of probabilities. Denote the transition rate between site i and j by $w(i,j)$. The master equation which describes the dynamics of the probabilities is

$$\frac{dP_i(t)}{dt} = I_i(t) - O_i(t),$$

where the influx to site i is

$$I_i(t) = \sum_{j \neq i} P_j(t) w(j,i),$$

and

$$O_i(t) = P_i(t) \sum_{j \neq i} w(i,j).$$

For the one level tree with four sites

$$I_1(t) = P_2(t)w(2,1) + P_3(t)w(3,1) + P_4(t)w(4,1),$$

and

$$O_1(t) = P_1(t)[w(1,2) + w(1,3) + w(1,4)],$$

and with similar expressions for the other Is and Os. There are similar expressions for the outflows and inflows at other sites as well. We also assumne that $w(i,j) = w(j,i)$ for all i and j.

Next, we posit that the transition rates $w(i,j)$ depends only on the ultrametric distance. Exogenous disturbances is felt first by agents or goods in the same cluster, and then the news or effects will gradually and stochastically propagate to other leaves, that is, to other clusters of the trees. Therefore we speak of the expected changes in price indicies as the results of such shocks to one cluster.

Thus for the one-level tree

$$w(i,j) = q < 1,$$

$i \neq j$, where $q = \exp(-\gamma d(i,j)) = \exp(-\gamma)$, for all i and j between 1 and 4, because the ultrametric distance between any pair of sites is the same, and where γ is some constant. Later we identify it with the inverse of *economic temperature*.

For the two-level tree we have[7]

[7] This model is similar to the one in Aoki (1996, p. 38). Details of analysis differ somewhat.

$$w(1,2) = w(3,4) = q,$$

and

$$w(1,3) = w(1,4) = q^2$$

because $d(1,3) = 2$, hence $w(1,3) = \exp(-2\gamma) = q^2$.

The master equation for the probability vector $\mathbf{P}(t)$of the one level tree consists of probabilities at the four leaves

$$\frac{d\mathbf{P}(t)}{dt} = W\mathbf{P}(t),$$

with

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2 & W_1 \end{bmatrix},$$

where

$$W_1 = \begin{pmatrix} -3q & q \\ q & -3q \end{pmatrix},$$

$$W_2 = qe_2e_2',$$

where $e_2 = [1 \;\; 1]'$.

This matrix W has eigenvalue 0 with eigenvector $(1\,1\,1\,1)'$, and triple repeated eigenvalue $-4q$ with three independent eigenvectors $(1\,1\,-1\,-1)'$, $(1\,-1\,0\,0)'$, and $(0\,0\,1\,-1)'$.

The probabilities evolve with time according to

$$P_1(t) = 1/4 + (3/4)e^{-4qt},$$

and

$$P_2(t) = P_3(t) = P_4(t) = (1/4) - (3/4)e^{-4qt}.$$

Approximately after time span of $1/q$, the probabilities are all about $1/4$.[8] It takes about this time span for the initial shock to propagate to all the sectors. Hence this is the time lag for the shock initialted at sector 1 to spread probabilistically to all the other sectors, i. e., for macroeconomic price index to fully reflect the price shock to one of its sectors.[9]

In the other case, the matrix W is given by

$$W = \begin{bmatrix} W_1 & W_2 \\ W_2 & W_1 \end{bmatrix},$$

where

---

[8] Note that $e^{-1} = 0.018$.

[9] All probabilities eventually stop changing and reach some constants, such as $1/4$ here. This means that the news of disturbance at site 1 has reached all four sites equally by then.

$$W_1 = \begin{pmatrix} -(q+2q^2) & q \\ q & -(q+2q^2) \end{pmatrix},$$

$$W_2 = q^2 e_2 e_2',$$

where $e_2 = [1\ \ 1]'$.

This matrix W has eienvalues 0, with eigenvector (1 1 1 1), eigenvalue $\lambda_1 = -4q^2$, with eigenvector $(1\,1\ -1\ -1)'$, and double repeated eigenvalue $\lambda_2 = -2(q+q^2)$, with eigenvectors $(1\ -1\,0\,0)'$, and $(0\,0\,1\ -1)'$. Note that the magnitude of λ_1 is less than that of λ_2 because q is less than one. The associated with eigenvalue λ_1 is faster than that associated with eigenvalue λ_2. It represents the escape rate of probability from site 1 to site 2.

The probabilities evolve with time as

$$P_1(t) = (1/4 + (1/4)e^{-\lambda_1 t} + (1/2)e^{-\lambda_2 t},$$

$$P_2(t) = (1/4 + (1/4)e^{-\lambda_1 t} - (1/2)e^{-\lambda_2 t},$$

$$P_3(t) = P_4(t) = 1/4 - (1/4)e^{-\lambda_1 t}.$$

After time span of $2/q(1+q)$, the term $e^{-\lambda_2 t}$ is approximately zero. After time span of $1/q^2$ all probabilies are approximately equal to 1/4. Note however, that the time span $1/q^2$ is much longer than that of $1/q$, that is, the two-level tree is much more sluggish than the one-level tree.

To compare dynamic behavior of this model with the one-level tree, we can aggregate the tree by defining a two-dimensional state vector with components $S_1(t) = P_1(t) + P_2(t)$, and $S_2(t) = P_3(t) + P_4(t)$ by defining

$$\mathbf{Q}(t) = S\mathbf{P}(t),$$

where the aggregation matrix S is given by

$$S = \begin{bmatrix} 1\,1\,0\,0 \\ 0\,0\,1\,1 \end{bmatrix}.$$

The dynamic matrix V for this aggregated vector is given by $V = SWS'(SS')^{-1}$ which has eigenvalues 0 and $-4q^2$.

The vector $Q(t)$ has two components $0.5 + 0.5e^{-\mu t}$, and $0.5 - 0.5e^{-\mu t}$ with $\mu = 4q^2$.

The dynamics of the one-level tree is much simpler. It has eigenvalues 0 and -4q, the latter with multiplicity 3. We can similarly aggregate the first two sites and the second two sites separately to produce a two node tree. The eigenvalue are still 0 and -4q. In other words, after the lapse of time of the order $1/q$, the one-level tree has approximately reached its equilibrium state, while dynamics of the two-level tree has not.

This fact remains true when one-level trees of K sites is compared with k level trees with $K = 2^k$. Suppose that we group l of K sites into one cluster, and the remaining $K - l$ sites into another. The eigenvalues are 0 and $-Kq$, repeated $K - 1$ times, while those of k level tree are 0 and $-(2q)^k$.

We can show that the larger the number of hierarchies the slower the process of disturbance propagation, and response of macro-price index to shocks to one of the sectors. Ogielski and Stein (1985),[10] among several others, have shown that in the limit of the number of hierarchy going to infinity, the response becomes power-law, not exponential decay. See also Paladin et al. (1985)

3.3 Inflexible Macroeconomic Prices: An Example

We present an example of slow adjustments of some macroeconomic price (index) composed of prices of goods of several sectors of economy. To be simple suppose that a price index P_I is the weighted average of two sectoral output prices, Q_A and Q_B. We outline how Sector 1 price Q_A is affected by an exogenous shock to site 1 price, since effects on $Q_B(t)$ are similarly anlalyzed.

For concreteness suppose that node A is composed of a two level tree with two more nodes a and b with two branches each. There are thus four more basic prices at sites 1 through 4, such as factor prices, prices of intermediate goods and so on. The two-level hierarchical tree traces out the relations among these prices. As shown in the previous section, the tree generates spill-over probabilities of an exogenous shock to one of the basic prices.

The Laplace transform of $\mathbf{P}(t)$, dropping subscript b, is

$$\hat{\mathbf{P}}(s) = \frac{1}{4s}u_0 + \frac{1}{2(s+\lambda_2)}u_2 + \frac{1}{4(s+\lambda_1)}u_1,$$

where s is the Laplace transform variable, us are the column vectors shown above.

Consequently we can write down the explicit expression for the expected values of changes in Q_A, denoted by $E[\delta\hat{Q}_A(s)]$. Assuming that transmission lags of the transfer function $h_a(s), h_b(s), h_i(s)$, $i = 1, \ldots 4$ are not as large as $1/q^2$, we can extract the slowest decaying term out of this as

$$E(\delta Q_A(t)) \approx \frac{1}{4}h_a(-\lambda_1)\{h_1(-\lambda_1) + h_2(-\lambda_1)\} - h_b(-\lambda_1)\{h_3(-\lambda_1 + h_4(-\lambda_1)\}e^{-\lambda_1 t} + \cdots,$$

where the slowest term is extracted.

In the case where $h_a(s) = 1/(s+a)$, $h_b(s) = 1/(s+b)$, $h_i(s) = 1/(s+\alpha_i$, $i = 1, \ldots 4)$, then a sufficient condition that this term is present is

Proposition When q^2 is negligibly small compared with a, b, α_i, $i = 1, \ldots 4$, price Q_A will exhibit sluggish response to an exogenous price change at site 1 if

[10] See Aoki (1996, p. 200) also.

$$\frac{1}{a}(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}) \neq \frac{1}{b}(\frac{1}{\alpha_3}+\frac{1}{\alpha_4}).$$

This condition rules out that effects of exogenous shocks coming from two subtree branches cancel out.

More generally, structures of interconnections between the basic prices and Q_A are conveniently expressed in terms of the Laplace transforms as

$$E[\delta\hat{Q}_A(s)] = \sum_{i=1}^{4} \hat{H}_i(s) E[\delta\hat{q}_i(s)],$$

with $\hat{H}_i(s)$ being the Laplace transform of the transfer function from site i to the price Q_A. The symbol E is the expectation operator, that is the expected values of changes in the basic price q_i with the probabilities of spill-over.

For example, $\hat{H}_i(s)$ could be a simple first order transfer function such as $(s+c_i)$ with some positive constant c_i, or more complex second or third order transfer functions, possibly with complex as well as real roots.

The example of this section illustrates the effects of spill-over delays due to hierarchical tree structure, in addition to the usual delays due to dynamics of transmission which are present in the transfer functions.

The macroeconomic price indices thus have two sources of sluggishness; one is the usual dynamic lags of transfer functions , and the other information spread or spill-over lags, that is, the lags for the signal to arrive as inputs to some transfer function that are connected with the price index expressions.

4 Effects of Uncertainty

4.1 Rugged Landscape Problems

Thus far, we have focussed on the tree structure of the economy that is responsible for producing slow dynamic behavior. This result is generic since it does not depend on any specific assumptions on the model. We now turn to another reason for sluggish macroeconomy that has to do uncertainty.

The standard analysis in economics assumes explicitly or implicitly that agents know the global shape of objective functions, which are smooth and well-behaved, and constraints. In reality, agents have only local knowledge at best, and must try to improve their performance by guessing the right directions to adjust their decision varibles. To do so they face complicated and often hard optimization problems.

Agents are thus often stuck at some local optimal point or in basins associated with local optima, and they may not know of the existence of better local optima or global ones. "Rugged or flat landscapes" are the words often used to indicate that agents do not know which directions they should adjust their decision variables.