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Jean Lévine

# Analysis and Control of Nonlinear Systems

## A Flatness-based Approach

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## A Flatness-based Approach

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*To Martine, Sonia and Benjamin*

# Preface

The present book is a translation and an expansion of lecture notes corresponding to a course of Mathematics of Control delivered during four years at the École Nationale des Ponts et Chaussées (Marne-la-Vallée, France) to Master students. A reduced version of this course has also been given at the Master level at the University of Paris-Sud since eight years. It may therefore serve as lecture notes for teaching at the Master or PhD level but also as a comprehensive introduction to researchers interested in flatness and more generally in the mathematical theory of finite dimensional systems and control.

This book may be seen as an outcome of the applied research policy pioneered by the École des Mines de Paris (now MINES-ParisTech), France, aiming not only at academic excellence, but also at collaborating with industries on specific innovative projects to enhance technological innovation using the most advanced know-how. This influence, though indirectly visible, mainly concerns the originality of some of the topics addressed here which are, in a sense, a theoretic synthesis of the author's applied contributions and viewpoints in the control field, continuously elaborated and modified in contact with the industrial realities. Such a synthesis wouldn't have been made possible without the scientific trust and financial support of many companies during periods ranging from two to ten years. Particular thanks are due to Elf, Shell, Ifremer, Sextant Avionique, Valeo, PSA, IFP and Micro-Controle/Newport, and to all the outstanding engineers of these companies, from which the author could learn so much. The author particularly wishes to express his gratitude to Frédéric Autran and Bernard Rémond (Valeo), Alain Danielo and Roger Desailly (Micro-Controle), and Emmanuel Sedda (PSA).

The largest part of this book, dealing with flatness and applications, is inspired by works in collaboration, successively, with Benoît Charlet and Riccardo Marino, and then with Michel Fliess, Philippe Martin and Pierre Rouchon. The author addresses his warmest thanks to all of them for many fruitful discussions, in particular those in which the notion of differential

flatness could be brought to light. Some of the material used in the Singular Perturbation Chapter has been elaborated with Pierre Rouchon and Yann Creff, starting with a collaboration with Elf on distillation control. Their contributions are warmly acknowledged.

The author is also indebted to all his former PhD students, and particularly Michel Cohen de Lara, Guchuan Zhu, Régis Baron, Jean-Christophe Ponsart, Philippe Müllhaupt, Balint Kiss, Rida Sabri, Thierry Miquel, Thomas Devos and Jérémy Malaizé, in addition to the previously cited ones, Benoît Charlet, Philippe Martin, Pierre Rouchon and Yann Creff, for their skillful help to develop various applications of flatness in particularly interesting directions.

The author also wishes to warmly thank all his colleagues of the Centre Automatique et Systèmes, and more particularly Guy Cohen, Pierre Carpentier and Laurent Praly for their constant scientific trust and friendly encouragements during more than twenty years.

He would also like to especially acknowledge a recent fruitful collaboration with Felix Antritter of the Bundeswehr University, München, on symbolic computation of flatness conditions.

The first part of this manuscript was translated into english when the author visited the Department of Mathematics and Statistics of the University of Kuopio, Finland, from April to June 2006, as an Invited Professor funded by a Marie Curie Host Fellowship for the Transfer of Knowledge (project PARAMCOSYS, MTKD-CT-2004-509223), and was used as lecture notes for a course delivered during this period. The author is not only indebted to Markku Nihtilä, Chairman of this department, for his kind invitation, but also for his stimulating discussions and encouragements without which this book would not yet be finished. Many thanks are also addressed to Petri Kokkonen for his most efficient and enthusiastic help in the exercise sessions and in his careful reading of a draft version of this manuscript.

This manuscript has also been used as lecture notes for a two months intensive course given in March and April 2007 at the School of Electrical Engineering and Computer Science of the University of Newcastle, Australia, at the invitation of Jose De Dona and his group, where the decision to append a second part, dealing with industrial applications, has been taken. The author particularly wishes to express his profound thanks to Jose De Dona, Maria Seron, Jaqui Ramage and Graham Goodwin.

The author is also deeply indebted to Prof. Claus Hillermeier of the Bundeswehr University, München, for his kind invitation to publish this manuscript in the Springer collection he is supervising.

*J. Lévine*

# Contents

# Chapter 1
# Introduction

This book is made of two parts, Theory and Applications.

In the first Part, two major problems of automatic control are addressed: *trajectory generation*, or *motion planning*, and *tracking* of these trajectories.

In order to make this book as self-contained as possible we have included a survey of Differential Geometry and Dynamical System Theory. The viewpoint adopted for these topics has been tailored to prepare the reader to the language and tools of flatness-based control design, that is why we have preferred to place them ahead in Chapters 2 and 3 rather than to release them in an Appendix.

Recalls of linear system theory are also provided in Chapter 4, such as controllability and the corresponding Brunovský canonical form, since they constitute a first solution to the trajectory generation and tracking problems, which are generalized in the next chapters to flat systems, using a different approach, leading to simpler calculations.

The last chapters (from Chapter 5 to Chapter 8), are then devoted to the analysis of Lie-Bäcklund equivalence and flat systems. Note that a large part of Chapter 6 is devoted to the characterization of flat systems. This part, not essential to understand the examples and applications of flatness all along, may be skipped at first reading. However, the reader interested in this essential but difficult theoretical aspect, still full of unsolved questions, may find there a self-contained presentation.

In the second Part, the applications have been selected according to their pedagogical potentials, to illustrate as many control design techniques as possible in various industrial contexts: control of various types of motors, magnetic bearings, cranes and aircraft automatic flight design.

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## 1.1 Trajectory Planning and Tracking

The problems of *trajectory generation*, or *motion planning*, and *tracking* of these trajectories are studied in the context of *finite dimensional nonlinear systems*, namely systems described by a set of nonlinear differential equations, influenced by a finite number of inputs, or control variables.

In practice, a system represents our knowledge of the evolution of some variables with respect to time, and the control variables are often designed as the inputs of the *actuators* driving the system. They may be freely chosen in order to achieve some tasks, or may be subject to constraints resulting from technological restrictions.

Numerous examples of such systems may be found in mechanical systems driven by motors (satellites, aircraft, cars, cranes, machine tools, *etc.*), electric circuits or electronic devices driven by input currents or voltages (converters, electromagnets, motors, *etc.*), thermal machines driven by heat exchangers or resistors, chemical reactors, chemical, biotechnological or food processes driven by input concentrations of some chemical components, or mixtures of these examples.

The notion of trajectory generation, or motion planning, corresponds to what we intuitively mean by preparing a flight plan or a motion plan in advance. More precisely, it consists in the off-line generation of a path, and the associated control actions that generate the path. This path is supposed to relate a prescribed initial point to a prescribed final point, in *open-loop*, *i.e.* based on the knowledge of the system model only, in the ideal case where disturbances are absent, and without taking account of possible measurements of the system state. Such a trajectory is often called *reference* or *nominal trajectory*, and the associated control the *reference* or *nominal control*. This notion is quite natural in the context of controlled mechanical systems such as aircraft, cars, ships, underwater vehicles, cranes, mechatronic systems, machine tools or positioning systems. It is also of interest in many other fields such as chemical, biotechnological or food processes, where we may want to change the concentration of a chemical component from its present value to another one in a fast but smooth way, for energy savings or productivity increase, or some other reason.

The tracking aspect concerns the design of a control law able to follow the reference trajectory even if some unknown disturbances force the system to deviate from it. For this purpose, this control law must take into account additional information, namely on-line measurements, or *observations*, from which the deviations at every time with respect to the reference trajectory can be deduced. In practice, such observations are provided by *sensors*. The class of controls that take into account the system state observations, is generally called *feedback* or *closed-loop* control. Without deviation (*i.e.* without disturbances), the control coincides with its reference, but as soon as a deviation is detected, the closed-loop control law must ensure the convergence of the system to its reference trajectory. The type of convergence (local, global, exponential, polynomial, *etc.* ) that can be guaranteed, its rate, sometimes called *time constant* of the closed-loop system, and other robustness properties versus disturbances, modelling errors, *etc.* , will also be addressed in this book.

These two problems are particularly easy to solve for the class of nonlinear systems called *differentially flat*, or shortly *flat*, systems, introduced by M. Fliess, P. Martin, P. Rouchon and the author (Fliess et al. [1992a,b]) and actively developed since then (see e.g. the surveys and books by Martin et al. [1997], Lévine [1999], Rudolph [2003], Rudolph et al. [2003], Sira-Ramirez and Agrawal [2004], Rudolph [2003], Rudolph et al. [2003], Müllhaupt [2009]).

Most of the examples and applications of differential flatness of this book could have been presented using only elementary and intuitive mathematics. Though insufficiently precise for a mathematician, the mathematical ambiguities may be balanced by their physical evidence. However, if the reader wants to acquire a deeper understanding and/or wishes to solve more advanced problems, a precise mathematical background and a rigorous description of flat systems and their properties are required. Unfortunately, the corresponding mathematics are not easy. Their proper background comes from the theory of manifolds of jets of infinite order Krasil'shchik et al. [1986], Zharinov [1992]. Since at present no self-contained presentation of this theory for control systems is available, we have decided to privilege this aspect in this book, while keeping the mathematical level as accessible as possible. Nevertheless, applications also receive a prominent place in this book (Part II) to present flat systems from every angle.

## 1.2 Equivalence and Flatness

To give an intuitive idea of differential flatness, a flat system is a system whose integral curves (curves that satisfy the system equations) can be mapped in a one-to-one way to ordinary curves (which need not satisfy any differential equation) in a suitable space, whose dimension is possibly different than the one of the original system state space.

This definition can be made rigorous by introducing several notions and tools: we need to work with mappings that are one-to-one between vector spaces or manifolds of different dimension, and infinitely differentiable. According to the well-known *constant rank theorem* (see section 2.3), such mappings don't exist between finite dimensional manifolds. Therefore, it may only become possible if the original manifolds can be embedded in infinite dimensional ones. A classical way to realize this embedding consists in using the natural coordinates together with an infinite sequence of their time derivatives, called *jets of infinite order* (see *e.g.* Krasil'shchik et al. [1986], Zharinov [1992]).

In this framework, if two manifolds of jets of infinite order are mapped in a one-to-one and differentiable way, we say that they are *Lie-Bäcklund equivalent*. More precisely, two systems are said Lie-Bäcklund equivalent if and only if there exists a smooth one-to-one time-preserving mapping between their integral curves (trajectories that are solutions of the system differential equations) which maps tangent vectors to tangent vectors, in order to preserve time differentiation. Going back to our above stated intuitive definition of flatness, a flat system is Lie-Bäcklund equivalent to a system whose integral curves have no differential constraints (ordinary curves), that we call *trivial system*. Thus, finally, a system is flat if and only if it is *Lie-Bäcklund equivalent* to a *trivial system*.

Therefore, it becomes clear that the study of flat systems passes through the study of Lie-Bäcklund equivalence, a notion that plays a central role in this book. In addition, the notion of flatness may be interpreted as a change of coordinates that transforms the system in its "simplest" form, where calculations become elementary since the coordinates and the vector field describing the system are "straightened up". Recall that a transformation *straightens out coordinates, curves, surfaces, vector fields, distributions (families of vector fields), etc.* . if they are changed into lines, planes, constant vector fields, orthonormal frames, *etc.* . In particular, the integration of differential equations or partial differential equations in these coordinates may be done explicitly, as far as the associated straightening out transformations may be obtained.

These considerations indeed strongly suggest that the language of Differential Geometry is particularly well adapted to our context. However, the usual finite dimensional standpoint is too narrow for our purpose and its extension to manifolds of jets of infinite order seems difficult to circumvent. For the sake of completeness, we first introduce the reader to classic finite dimensional tools (Part I, Chapter 2), and then to their extension to jets of infinite order (Part I, Chapter 5).

Other approaches are indeed possible: finite dimensional differential geometric approaches Charlet et al. [1991], Franch [1999], Shadwick [1990], Sluis [1993], differential algebra and related approaches Fliess et al. [1995], Aranda-Bricaire et al. [1995], Jakubczyk [1993], infinite dimensional differential geometry of jets and prolongations Fliess et al. [1999], van Nieuwstadt et al. [1998], Pomet [1993], Pereira da Silva and Filho [2001], Rathinam and Murray [1998].

In the framework of linear finite or infinite dimensional systems, the notions of flatness and *parametrization* coincide as remarked by Pommaret [2001], Pommaret and Quadrat [1999], and in the behavioral approach of Polderman and Willems [1997], flat outputs correspond to *latent variables* of *observable image representations* Trentelman [2004] (see also Fliess [1992] for a module theoretic interpretation of the behavioral approach).

## 1.3 Equivalence in System Theory

Several equivalence relations have been studied to characterize *system equivalence* by various transformation groups. Traditionally, geometric objects are said to be *intrinsically* defined when their definition is not affected by change of coordinates (diffeomorphism) Boothby [1975], Chern et al. [2000], Choquet-Bruhat [1968], Demazure [2000], Dieudonné [1960], Kobayashi and Nomizu [1996], Olver [1995], Pham [1992]. In other words, two geometric objects are said equivalent if there exists a diffeomorphism mapping the first one into the second and *vice versa*. In the same spirit, *system equivalence by static feedback* has been introduced to deal with the equivalence of systems under static feedback action in an intrinsic way, namely independently of the choice of coordinates where the system and/or the control inputs are expressed. They yield classifications (*i.e.* partition of the set of systems into cosets) and canonical forms ("simplest" system representatives of the cosets) of major interest, such as the ones provided by Brunovský for linear controllable systems Brunovský [1970] (see also Rosenbrock [1970], Wolovich [1974], Tannenbaum [1980], Kailath [1980], Antoulas [1981], Polderman and Willems [1997], Sontag [1998] and, for extensions in the nonlinear case Sommer [1980], Jakubczyk and Respondek [1980], Hunt et al. [1983b], Marino [1986], Charlet et al. [1989, 1991], Gardner and Shadwick [1992], Isidori [1995], Nijmeijer and van der Schaft [1990], Marino and Tomei [1995]). However, equivalence relations which only involve static state feedback appear to be too fine to study flat systems. They are finer than the Lie-Bäcklund one which corresponds to the equivalence under a special class of dynamic feedback called *endogenous dynamic feedback* Martin [1992], Fliess et al. [1995], Martin [1994], van Nieuwstadt et al. [1994], Aranda-Bricaire et al. [1995], Pomet [1993], van Nieuwstadt et al. [1998], Fliess et al. [1999], Lévine [2006], that strictly contains the class of static feedback.

## 1.4 Equivalence and Stability

In the stability analysis of closed-loop systems, the notion of equivalence, though different than the previously discussed ones and called here *topological equivalence*, is also most important: in the introduction to dynamical system theory (Chapter 3), we emphasize on the equivalence between the behavior (stability or instability) of a nonlinear system around an equilibrium point and the one of its tangent linear approximation.

If the latter tangent linear approximation is *hyperbolic* (if it has no eigenvalue on the imaginary axis of the complex plane), the nonlinear system can be proved to be topologically equivalent to its tangent linear approximation. More precisely (Hartman-Grobman's Theorem), hyperbolic systems can be shown to be equivalent to a linear system made up with two decoupled linear subsystems, the first one being stable and the second one being unstable. These subsystems respectively live in locally defined *invariant manifolds* called *stable* and *unstable*, their respective dimensions corresponding to the number of eigenvalues, counted with their multiplicities, of the tangent linear approximation at the equilibrium point with negative and positive real parts.

In the non hyperbolic case, a nonlinear system may be shown to be topologically equivalent to a system made up with a linear stable subsystem and a linear unstable subsystem, obtained as before from the linear tangent approximation, and completed by a nonlinear neutral one, coupled to the previous linear ones. These subsystems respectively live in locally defined *stable*, *unstable* and *centre manifolds* (Shoshitaishvili's Theorem).

Singularly perturbed systems are introduced in this framework in Section 3.3, which extends the previous approach to control systems. We particularly insist on the links between singularly perturbed systems, multiple time scales and hierarchical control.

## 1.5 What is a Nonlinear Control System?

### *1.5.1 Nonlinearity versus Linearity*

Before talking about nonlinearity, let us discuss the definition of linearity. First, linearity is a coordinate dependent property since a linear system might look nonlinear after a nonlinear change of coordinates. Take the following elementary example: $\dot{x} = u$ in a sufficiently small neighborhood of the initial condition $x_0 = 0$, and transform $x$ into $\xi = \sqrt{x+1}$ and $u$ into $v = u$. We have $\dot{\xi} = \frac{\dot{x}}{2\sqrt{x+1}} = \frac{u}{2\xi}$. Therefore, the transformed system, namely $\dot{\xi} = \frac{v}{2\xi}$ is no more linear.

Note that in the previous transformation, $\xi$ doesn't depend on $u$ and is invertible in the sense that $x = \xi^2 - 1$, and $v$ is also invertible as a function of $u$[1]. Clearly, the set of transformations that enjoy these properties forms a group with respect to composition, and the linearity property of the system thus depends on this group[2]. More precisely, a system is said linear if it can be transformed into a linear system by a transformation of this group. The number of linear systems thus depends on the "size" of the group. This is why transformations depending on the input and its successive time derivatives, generating a larger group than the above mentioned one, will be introduced later.

[1] this transformation is actually a local $C^\infty$ diffeomorphism: in addition to its local invertibility, it is of class $C^\infty$ in a neighborhood of $x_0 = 0$, with $C^\infty$ inverse.

[2] indeed, the smoothness of the transformations, which may be $C^k$ for any $k \geq 2$ or analytic, is also part of the group definition.

Linear systems form a distinguished class in the set of nonlinear systems since they enjoy simpler properties as far as controllability, open-loop stability/instability, stabilizability, *etc.* are concerned. Therefore, they should be detected independently of the particular choice of coordinates in which they are expressed.

## *1.5.2 Uncontrolled versus Controlled Nonlinearity*

In order to outline some fundamental differences between linear and nonlinear systems we may start with stability aspects for uncontrolled systems, by considering a linear system perturbed by a small nonlinearity, that significantly modifies the behavior of the original linear system. We next show that, once the system is controlled, what counts is the control efficiency to attenuate or remove the phenomenon created by the open-loop nonlinearity, as presented in the next example.

### An introductory Example

This example is presented in three steps. We first start with a linear non controlled system, a spring with linear stiffness (force exerted by the spring proportional to its length variation), with a nonlinear perturbation, that may physically result from a defect of the spring, and modelled as a small nonlinear perturbation of the stiffness coefficient. It turns out that this small defect creates a big change in the system behavior, that doesn't exist in linear systems. At a second step, we connect the system with a passive device, that may be interpreted as a special case of feedback control, and show how the system behavior is locally modified. Finally, the third step consists in replacing the passive device by an active one to globally transform the original nonlinear system behavior into a linear one that may be tuned as we want.

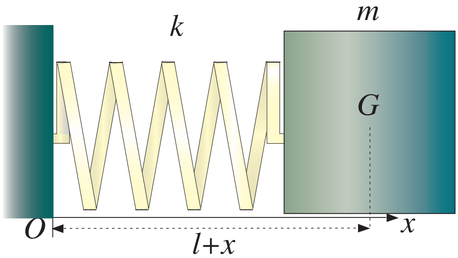


**Fig. 1.1** Spring and mass

### Uncontrolled nonlinear perturbation

Consider a system made of a mass and an undamped spring of pulsation $\omega$ whose position, denoted by $x$, satisfies :

$$\ddot{x} + \omega^2 x = 0 \tag{1.1}$$

with the spring stiffness $k$ related to the pulsation $\omega$ by $\omega = \sqrt{\frac{k}{m}}$, $m$ being the mass of the rigid body attached to the spring.

Setting $\dot{x} = v$, the expression $v^2 + \omega^2 x^2$, proportional to the mechanical energy of the spring, remains constant along any trajectory of (1.1) since $\frac{d}{dt}(v^2 + \omega^2 x^2) = 2\dot{x}(\ddot{x} + \omega^2 x) = 0$. In other words, in the $(x, v)$-plane (phase plane), these trajectories are the ellipses of equation $v^2 + \omega^2 x^2 = C$, where $C$ is an arbitrary positive constant, and thus are closed curves around the origin, whose focuses are determined by the initial conditions $(x_0, v_0)$. We indeed recover the classical interpretation that once the spring is released from its initial position $x_0$ with initial velocity $v_0$, it oscillates forever at the pulsation $\omega$. This motion is neither attenuated nor amplified.

However, if the spring stiffness is not exactly a constant, even very close to it, but if this aspect has been neglected, a very different behavior may be expected.

Assume in fact that the spring stiffness is a linear slowly decreasing function of the length : $\frac{k(x)}{m} = \omega^2 - \varepsilon x$, with $\varepsilon > 0$ small, which means that the pulling force produced by the spring is $k(x)x = \omega^2 x - \varepsilon x^2$. The spring's dynamical equation becomes

$$\ddot{x} + \omega^2 x - \varepsilon x^2 = 0 \tag{1.2}$$

a nonlinear equation because of the $x^2$ term. Setting as before $v = \dot{x}$, we easily check that the expression (the mechanical energy up to a constant)

$$E_\varepsilon(x, v) = v^2 + x^2(\omega^2 - \frac{2}{3}\varepsilon x) \tag{1.3}$$

is such that $\frac{d}{dt}E_\varepsilon(x, v) = 0$ along the integral curves of (1.2), and thus remains constant with respect to time. The perturbed spring trajectories are therefore described by the curves of equation $E_\varepsilon(x, v) = E_\varepsilon(x_0, v_0)$ shown on Figure 1.2. We see that for a small initial length and velocity, the spring's behavior is not significantly changed with respect to the previous linear one. On the contrary, for larger initial length and velocity, the spring becomes too sluggish and thus unstable.

The differences with respect to the original linear system are thus twofold:

1. the only equilibrium point of the linear system (1.1) is the origin $(0, 0)$ whereas system (1.2) has two equilibria $(0, 0)$ and $(\frac{\omega^2}{\varepsilon}, 0)$;

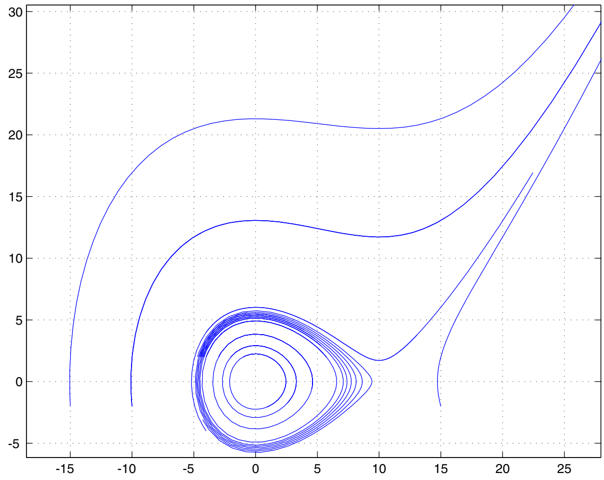


**Fig. 1.2** Destabilization of the spring caused by its nonlinear stiffness

2. the linear system behavior is purely oscillatory, whereas the perturbed nonlinear one is oscillatory near the origin but unstable for larger initial conditions.

### Adding a damper

This phenomenon is well-known on truck's trailers or on train wagon bogies where it is necessary to add a damper to dissipate the energy excess stored in the spring when released. In fact, the appending of a damper may be interpreted as a feedback: in (1.2), a frictional force $Kv$, proportional to the velocity, is added, which amounts to consider that the system is controlled by the force $u = Kv$ :

$$\ddot{x} + \omega^2 x - \varepsilon x^2 + u = \ddot{x} + \omega^2 x - \varepsilon x^2 + Kv = 0. \tag{1.4}$$

Doing the previous calculation again, $\frac{dE_\varepsilon}{dt}$ along an arbitrary trajectory of (1.4), we find that $\frac{dE_\varepsilon}{dt} = -2Kv^2 < 0$, which proves that the function $E_\varepsilon$ is monotonically decreasing along the trajectories of (1.4). It is readily seen that, for $|x| < \frac{\omega^2}{\varepsilon}$, the function $E_\varepsilon$ is strictly convex and admits the origin $x = 0, v = 0$ as unique minimum. Consequently, the decreasing rate of $E_\varepsilon$ along the trajectories such that $|x| < \frac{\omega^2}{\varepsilon}$ implies that the trajectories all converge to the origin, and thus that the system is stabilized thanks to the damper

### Active control

The stability can be improved yet if the damper is replaced by an active hydraulic jack for instance. Indeed, if the force $u$ produced by the damper can be modified at will, it suffices to set

$$u = -\omega^2 x + \varepsilon x^2 + K_1 x + K_2 v$$

with $K_1 > 0$ and $K_2 > 0$, and the equation (1.4) becomes the exponentially stable linear differential equation

$$\ddot{x} = -K_1 x - K_2 \dot{x}.$$

The thread followed in this simple example is quite representative of one of the main orientations of this course: we first analyze the nonlinearities that might influence the non controlled system, and then various feedback loop designs to compensate some or all of the unwanted dynamical responses are studied.

# Part I
# THEORY

# Chapter 2
# Introduction to Differential Geometry

This Chapter aims at introducing the reader to the basic concepts of differential geometry such as *diffeomorphism*, *tangent* and *cotangent space*, *vector field*, *differential form*. Special emphasis is put on the *integrability of a family of vector fields*, or *distribution*[1], according to its role in nonlinear system theory,

For simplicity's sake, we have defined a *manifold* as the solution set to a system of implicit equations expressed in a given coordinate system, according to the *implicit function theorem*. One can then get rid of the coordinate choice thanks to the notion of *diffeomorphism* or *curvilinear coordinates*. Particular interest is given to the notion of *straightening out* coordinates, that allow to express manifolds, vector fields or distributions in a trivial way.

The interested reader may find a more axiomatic presentation e.g. in Anosov and Arnold [1980], Arnold [1974, 1980], Boothby [1975], Chevalley [1946], Choquet-Bruhat [1968], Demazure [2000], Godbillon [1969], Pham [1992]. The implicit function theorem, the constant rank theorem and the existence and uniqueness of integral curves of a differential equation, which are part of the foundations of analysis, are given without proof. Excellent proofs may be found in Arnold [1974], Cartan [1967], Dieudonné [1960], Marino [1986], Pham [1992], Pontriaguine [1975].

Some applications of these methods to Mechanics may be found in Abraham and Marsden [1978], Godbillon [1969] and, in Isidori [1995], Khalil [1996], Nijmeijer and van der Schaft [1990], Sastry [1999], Slotine and Li [1991], Vidyasagar [1993], other approaches and developments of the theory of control of nonlinear systems.

[1] a geometric object not to be confused with the functional analytic notion of distribution developed by L. Schwartz.

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## 2.1 Manifold, Diffeomorphism

Recall that, given a coordinate system $(x_1, \dots, x_n)$ and a $k$-times continuously differentiable mapping $\Phi$ from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^{n-p}$ with $0 \leq p < n$, the tangent linear mapping $D\Phi(x)$, also called Jacobian matrix of $\Phi$, is the matrix whose entry of row $i$ and column $j$ is $\frac{\partial \Phi_i}{\partial x_j}(x)$.

We start with the following fundamental theorem:

**Theorem 2.1.** ***(Implicit Function Theorem)*** *Let $\Phi$ be a $k$-times continuously differentiable mapping, with $k \geq 1$, from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^{n-p}$ with $0 \leq p < n$. We assume that there exists at least an $x_0 \in U$ such that $\Phi(x_0) = 0$. If for every $x$ in $U$ the tangent linear mapping $D\Phi(x)$ has full rank (equal to $n - p$), there exists a neighborhood $V = V_1 \times V_2 \subset U$ of $x_0$ in $\mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}$, with $V_1 \in \mathbb{R}^p$ and $V_2 \in \mathbb{R}^{n-p}$, and a $k$-times continuously differentiable mapping $\psi$ from $V_1$ to $V_2$ such that the two sets $\{x \in V_1 \times V_2 \; |\Phi(x) = 0\}$ and $\{(x_1, x_2) \in V_1 \times V_2 \; |x_2 = \psi(x_1)\}$ are equal.*

In other words, the function $\psi$ locally satisfies $\Phi(x_1, \psi(x_1)) = 0$ and the "dependent variable" $x_2 = \psi(x_1)$ is described by the $p$ (locally) independent variables $x_1$.

*Example 2.1.* Consider the function $\Phi$ from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $\Phi(x_1, x_2) = x_1^2 + x_2^2 - R^2$ where $R$ is a positive real. Clearly, a solution to the equation $\Phi = 0$ is given by $x_1 = \pm\sqrt{R^2 - x_2^2}$ for $|x_2| \leq R$. The implicit function Theorem confirms the existence of a local solution around a point $(x_{1,0}, x_{2,0})$ such that $x_{1,0}^2 + x_{2,0}^2 = R^2$ (e.g. $x_{1,0} = R$, $x_{2,0} = 0$), since the tangent linear mapping of $\Phi$ at such point is: $D\Phi(x_{1,0}, x_{2,0}) = (2x_{1,0}, 2x_{2,0}) \neq (0, 0)$, and has rank 1.

Note that there are two local solutions according to whether we consider the point $(x_{1,0}, x_{2,0})$ equal to $\left(\sqrt{R^2 - x_{2,0}^2}, x_{2,0}\right)$ or to $\left(-\sqrt{R^2 - x_{2,0}^2}, x_{2,0}\right)$.

The notion of manifold is a direct consequence of Theorem 2.1:

**Definition 2.1.** Given a differentiable mapping $\Phi$ from $\mathbb{R}^n$ to $\mathbb{R}^{n-p}$ $(0 \leq p < n)$, we assume that there exists at least an $x_0$ satisfying $\Phi(x_0) = 0$ and that the tangent linear mapping $D\Phi(x)$ has full rank $(n - p)$ in a neighborhood $V$ of $x_0$. The set $X$ defined by the implicit equation $\Phi(x) = 0$, is called *differentiable manifold of dimension p*. Otherwise stated:

$$X = \{x \in V | \Phi(x) = 0\}. \tag{2.1}$$

The fact that this set is non empty is a direct consequence of Theorem 2.1. If in addition $\Phi$ is $k$-times differentiable (resp. analytic), we say that $X$ is a $C^k$ differentiable manifold, $k = 1, \dots, \infty$ (resp. analytic –or $C^\omega$–).

If non ambiguous, we simply say *manifold*.

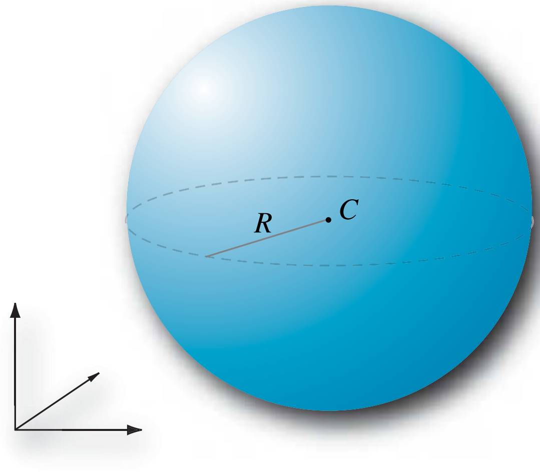


**Fig. 2.1** The sphere of $\mathbb{R}^3$

*Example 2.2.* The affine (analytic) manifold: $\{x \in \mathbb{R}^n | Ax - b = 0\}$ has dimension $p$ if rank $(A) = n - p$ and $b \in \mathrm{Im}A$.

*Example 2.3.* The sphere of $\mathbb{R}^3$ centered at $C$, of coordinates $(x_C, y_C, z_C)$, and of radius $R$, given by $\{(x, y, z) \in \mathbb{R}^3 | (x-x_C)^2+(y-y_C)^2+(z-z_C)^2-R^2 = 0\}$, is a 2-dimensional analytic manifold (see Fig. 2.1).

The concept of *local diffeomorphism* is essential to describe manifolds in an intrinsic way, namely independently of the choice of coordinates in which the implicit equation $\Phi(x) = 0$ is stated).

**Definition 2.2.** Given a mapping $\varphi$ from an open subset $U \subset \mathbb{R}^p$ to an open subset $V \subset \mathbb{R}^p$, of class $C^k$, $k \geq 1$ (resp. analytic), we say that $\varphi$ is a *local diffeomorphism* of class $C^k$ (resp. analytic) in the neighborhood $U(x_0)$ of a point $x_0$ of $U$ if $\varphi$ is invertible from $U(x_0)$ to a neighborhood $V(\varphi(x_0))$ of $\varphi(x_0)$ of $V$ and if its inverse $\varphi^{-1}$ is also $C^k$ (resp. analytic).

Indeed, if we consider the manifold $X$ defined by (2.1), and if we introduce the change of coordinates $x = (x_1, x_2) = \varphi(z) = (\varphi_1(z_1), \varphi_2(z_2))$ where $\varphi = (\varphi_1, \varphi_2)$ is a local diffeomorphism of $\mathbb{R}^n$, with $\varphi_1$ (resp. $\varphi_2$) local diffeomorphism of $\mathbb{R}^p$ (resp. $\mathbb{R}^{n-p}$), the expression $x_2 = \psi(x_1)$ becomes $\varphi_2(z_2) = \psi(\varphi_1(z_1))$, or $z_2 = \left(\varphi_2^{-1} \circ \psi \circ \varphi_1\right)(z_1)$, which means that the same manifold can be equivalently represented by $x_2 = \psi(x_1)$, in the $x$-coordinates, or by $z_2 = \tilde{\psi}(z_1)$, with $\tilde{\psi} = \varphi_2^{-1} \circ \psi \circ \varphi_1$, in the $z$-coordinates. It results that the notion of manifold doesn't depend on the choice of coordinates, if the coordinate changes are diffeomorphisms.

We also introduce the slightly weaker notion of *local homeomorphism*. We say that $\varphi$ is a local $C^k$ (resp. analytic) homeomorphism if $\varphi$ is of class $C^k$ (resp. analytic), locally invertible and if its inverse is continuous.

Local diffeomorphisms are characterized by the following classical result:

**Theorem 2.2.** ***(of local inversion)*** *A necessary and sufficient condition for $\varphi$ to be a local $C^k$ diffeomorphism ($k \geq 1$) in a neighborhood of $x_0$ is that its tangent linear mapping $D\varphi(x_0)$ is one-to-one.*

We also recall:

**Theorem 2.3.** ***(constant rank)*** *Let $\varphi$ be a $C^k$ mapping ($k \geq 1$) from a $m$-dimensional $C^k$ manifold $X$ to a $r$-dimensional $C^k$ manifold $Y$.*

*(i) for every $y \in \varphi(U) \subset Y$, $\varphi^{-1}(\{y\})$ is a $m-q$-dimensional $C^k$ submanifold of $X$;*
*(ii) $\varphi(U)$ is a $q$-dimensional $C^k$ submanifold of $Y$.*

*In particular,*

*(i)' if $m \leq r$, $\varphi$ is injective from $U$ to $Y$ if and only if* $\operatorname{rank}(D\varphi(x)) = m$ *for every $x \in U$ (thus $\varphi$ is a homeomorphism from $U$ to $\varphi(U)$).*
*(ii)' if $m \geq r$, $\varphi$ is onto from $U$ to $V$, an open subset of $Y$, if and only if* $\operatorname{rank}(D\varphi(x)) = r$.

The notion of curvilinear coordinates provide a remarkable geometric interpretation of a diffeomorphism. In particular, one can find (locally) an adapted system of curvilinear coordinates in which the manifold $X$ given by (2.1) is expressed as a vector subspace of $\mathbb{R}^p$. It suffices, indeed, to introduce the curvilinear coordinates:

$$y_1 = \Phi_1(x), \ldots, y_{n-p} = \Phi_{n-p}(x), y_{n-p+1} = \Psi_1(x), \ldots, y_n = \Psi_p(x),$$

the independent functions $\Psi_1, \ldots, \Psi_p$ being chosen such that the mapping

$$x \mapsto (\Phi_1(x), \ldots, \Phi_{n-p}(x), \Psi_1(x), \ldots, \Psi_p(x))$$

is a local diffeomorphism. In that case, we say that we have (locally) "straightened out" the coordinates of $X$ since

$$X = \{y | y_1 = \cdots = y_{n-p} = 0\}.$$

*Example 2.4.* We go back to the sphere of example 2.3 and introduce the polar coordinates $(\rho, \theta, \varphi)$ corresponding to the transformation $\Gamma$ from $\mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ to $\mathbb{R}^3$, given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Gamma(\rho, \theta, \varphi) = \begin{pmatrix} x_C + \rho \cos\varphi \cos\theta \\ y_C + \rho \cos\varphi \sin\theta \\ z_C + \rho \sin\varphi \end{pmatrix}.$$

Clearly $\Gamma$ is invertible in any of the two open sets defined by the intersection of $\mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ with $\{\cos\varphi > 0\}$ or $\{\cos\varphi < 0\}$, and where the closed subset $\{\rho\cos\varphi = 0\}$, whose image by $\Gamma$ is the pair of points of cartesian coordinates $x = x_C, y = y_C, z = z_C \pm \rho$, is excluded. $\Gamma$ is of class $C^\infty$, and its local inverse is given (e.g. for $\cos\varphi > 0$) by

$$\begin{pmatrix}\rho\\ \theta\\ \varphi\end{pmatrix} = \Gamma^{-1}(x,y,z) = \begin{pmatrix}\sqrt{(x-x_C)^2+(y-y_C)^2+(z-z_C)^2}\\ \arctan\left(\dfrac{y-y_C}{x-x_C}\right)\\ \arctan\left(\dfrac{z-z_C}{\sqrt{(x-x_C)^2+(y-y_C)^2}}\right)\end{pmatrix}.$$

$\Theta$ is also of class $C^\infty$ in the open set

$$\Gamma(\mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z} \cap \{\cos\varphi > 0\}) = \mathbb{R}^2 \times \{z > z_C\} - \{(x_C, y_C, z_C + \rho)\}$$

and thus $\Gamma$ is a local diffeomorphism.

In polar coordinates, the implicit equation defining the sphere becomes $\rho - R = 0$. Therefore, the sphere of $\mathbb{R}^3$ is locally equal to the set $\{\rho = R\}$.

One can check that the tangent linear mapping of $\Gamma$ is given by

$$D\Gamma = \begin{pmatrix}\cos\varphi\cos\theta & -\rho\cos\varphi\sin\theta & -\rho\sin\varphi\cos\theta\\ \cos\varphi\sin\theta & \rho\cos\varphi\cos\theta & -\rho\sin\varphi\sin\theta\\ \sin\varphi & 0 & \rho\cos\varphi\end{pmatrix}$$

and that $\det(D\Gamma) = \rho^2\cos\varphi$, which precisely vanishes on the closed subset $\{\rho\cos\varphi = 0\}$ where $\Gamma$ is not injective, in accordance with Theorem 2.2.

## 2.2 Vector Fields

### *2.2.1 Tangent space, Vector Field*

Assume, as before, that we are given a differentiable mapping $\Phi$ from $\mathbb{R}^n$ to $\mathbb{R}^{n-p}$ $(0 \le p < n)$, with at least an $x_0$ satisfying $\Phi(x_0) = 0$. The tangent linear mapping $D\Phi(x)$ of $\Phi$ at $x$, expressed in the local coordinate system $(x_1, \ldots, x_n)$, is thus the matrix $\left(\frac{\partial\Phi_j}{\partial x_i}(x)\right)_{1\le i\le n,\, 1\le j\le n-p}$.
It is also assumed that $D\Phi(x)$ has full rank $(n-p)$ in a neighborhood $V$ of $x_0$, so that the implicit equation $\Phi(x) = 0$ defines a $p$-dimensional manifold denoted by $X$.

We easily check that a normal vector at the point $x$ to the manifold $X$ is "carried" by $D\Phi(x)$, or more precisely, is a linear combination of the rows

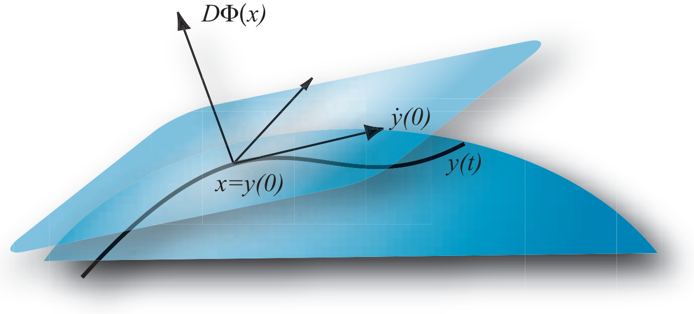


**Fig. 2.2** Tangent and normal spaces to a manifold at a point.

of $D\Phi(x)$. Indeed, let $y(t)$ be a differentiable curve contained in $X$ for all $t \in [0, \tau[$,with $\tau > 0$ sufficiently small, such that $y(0) = x$ (the existence of such a curve results from the implicit function Theorem). We therefore have $\Phi(y(t)) = 0$ for all $t \in [0, \tau[$ and thus $\frac{\Phi(y(t))-\Phi(x)}{t} = 0$. Letting $t$ converge to 0, we get $D\Phi(x).\dot{y}(0) = 0$, where $\dot{y}(0) \stackrel{\text{def}}{=} \frac{dy}{dt}_{|t=0}$ (see Fig.2.2), which proves that the vector $\dot{y}(0)$, tangent to $X$ at the point $x$, belongs to the kernel of $D\Phi(x)$. Doing the same for every curve contained in $X$ and passing through $x$, it immediately results that every element of the range of $D\Phi(x)$ is orthogonal to every tangent vector to $X$ at the point $x$, *Q.E.D.*

This motivates the following:

**Definition 2.3.** The *tangent space* to $X$ at the point $x \in X$ is the vector space

$$\mathrm{T}_x X = \ker D\Phi(x).$$

The *tangent bundle* $\mathrm{T}X$ is the set $\mathrm{T}X = \bigcup_{x \in X} \mathrm{T}_x X$.

Taking into account the fact that $D\Phi(x)$ has rank $n - p$ in $V$,

$$\dim \mathrm{T}_x X = \dim \ker D\Phi(x) = p \ , \forall x \in V.$$

*Example 2.5.* Going back to example 2.3, the tangent space to the sphere of $\mathbb{R}^3$ at the point $(x, y, z) \neq (x_C, y_C, z_C \pm R)$ is

$$\ker \left( (x - x_C) \ (y - y_C) \ (z - z_C) \right) = \operatorname{span} \left\{ \begin{pmatrix} (y - y_C) \\ -(x - x_C) \\ 0 \end{pmatrix}, \begin{pmatrix} (z - z_C) \\ 0 \\ -(x - x_C) \end{pmatrix} \right\}$$

and is clearly 2-dimensional.

**Definition 2.4.** A *vector field* $f$ (of class $C^k$, analytic) on $X$ is a mapping (of class $C^k$, analytic) that maps every $x \in X$ to the vector $f(x) \in \mathrm{T}_x X$.

**Definition 2.5.** An *integral curve* of the vector field $f$ is a local solution of the differential equation $\dot{x} = f(x)$.

The local existence and uniqueness of integral curves of $f$ results from the fact that $f$ is of class $C^k$, $k \geq 1$, and thus locally Lipschitzian[2].

### *2.2.2 Flow, Phase Portrait*

We denote by $X_t(x)$ the point of the integral curve of the vector field $f$ at time $t$, starting from the initial state $x$ at time 0. Recall that if $f$ is of class $C^k$ (resp. $C^\infty$, analytic) there exists a unique *maximal* integral curve $t \mapsto X_t(x)$ of class $C^{k+1}$ (resp. $C^\infty$, analytic) for every initial condition $x$ in a given neighborhood, in the sense that the interval of time $I$ on which it is defined is maximal.

As a consequence of existence and uniqueness, the mapping $x \mapsto X_t(x)$, noted $X_t$, is a local diffeomorphism for every $t$ for which it is defined: $X_t(X_{-t}(x)) = x$ for every $x$ and $t$ in a suitable neighborhood $U \times I$ of $X \times \mathbb{R}$, and thus $X_t^{-1}\big|_U = X_{-t}\big|_U$, where we have denoted by $\varphi\big|_U$ the restriction of a function $\varphi$ to $U$.

When the integral curves of $f$ are globally defined on $\mathbb{R}$, we say that the vector field $f$ is *complete*. In this case, $X_t$ exists for all $t \in \mathbb{R}$, and defines a one-parameter group of local diffeomorphisms, namely:

1. the mapping $t \mapsto X_t$ is $C^\infty$,
2. $X_t \circ X_s = X_{t+s}$ for all $t, s \in \mathbb{R}$ and $X_0 = Id_X$.

As already remarked, the items 1 and 2 imply that $X_t$ is a local diffeomorphism for all $t$.

The mapping $t \mapsto X_t$ is called the *flow associated to the vector field* $f$. It is also often called the *flow associated to the differential equation* $\dot{x} = f(x)$.

It is straightforward to verify that the flow satisfies the differential equation

$$\frac{d}{dt} X_t(x) = f(X_t(x)) \tag{2.2}$$

for all $t$ and every initial condition $x$ such that $X_t(x)$ is defined.

In the time-varying case, namely for a system

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[2] Recall that a function $f$ from $\mathbb{R}^p$ to $\mathbb{R}^p$ is locally Lipschitzian if and only if for every open set $U$ of $\mathbb{R}^p$ and every $x_1, x_2$ in $U$, there exists a real $K$ such that $\|f(x_1) - f(x_2)\| \leq K\|x_1 - x_2\|$.
The differential equation $\dot{x} = f(x)$, with $f$ locally Lipschitzian, admits, in a neighborhood of every point $x_0$, an integral curve passing through $x_0$ at $t = 0$, *i.e.* a mapping $t \mapsto x(t)$ satisfying $\dot{x}(t) = f(x(t))$ and $x(0) = x_0$ for all $t \in I$, $I$ being an open interval of $\mathbb{R}$ containing 0.

$$\dot{x} = f(t, x) \tag{2.3}$$

the corresponding notion of flow is deduced from what precedes by adding a new differential equation describing the time evolution $\dot{t} = 1$, and augmenting the state $\tilde{x} = (x, t)$, which amounts to work with the new vector field $\tilde{f}(\tilde{x}) = (f(t, x), 1)$, which is now a stationary one on the augmented manifold $X \times \mathbb{R}$ of dimension $p + 1$.

We call *orbit* of the vector field $f$ an equivalence class for the equivalence relation "$x_1 \sim x_2$ *if and only if there exists* $t$ *such that* $X_t(x_1) = x_2$ *or* $X_t(x_2) = x_1$".

In other words, $x_1 \sim x_2$ if and only if $x_1$ and $x_2$ belong to the same maximal integral curve of $f$. We also call *orbit of a point* the maximal integral curve passing through this point and its *oriented orbit* the orbit of this point along with its sense of motion.

The *phase portrait* of the vector field $f$ is defined as the partition of the manifold $X$ into oriented orbits.

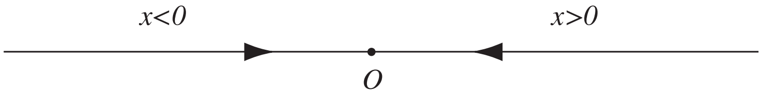


**Fig. 2.3** The 3 orbits of system (2.4).

*Example 2.6.* The flow of the differential equation on $\mathbb{R}$

$$\dot{x} = -x \tag{2.4}$$

is $X_t(x_0) = e^{-t}x_0$. Since $e^{-t}$ is positive for all $t$, two arbitrary points of $\mathbb{R}$ belong to the same integral curve if and only if they belong to the same half-line ($\mathbb{R}_+$ or $\mathbb{R}_-$) or they are both 0, *i.e.* $x_1 \sim x_2$ is equivalent to $\operatorname{sign}(x_1) = \operatorname{sign}(x_2)$ or $x_1 = x_2 = 0$. The system (2.4) thus admits 3 orbits: $\mathbb{R}_+$, $\mathbb{R}_-$ and $\{0\}$, as indicated on Fig. 2.3.

The same conclusion holds for the system $\dot{x} = +x$, the only difference being the orientation of the orbits, opposite to the one of (2.4).

Indeed, the flow and phase portrait do not depend on the choice of coordinates of $X$: if $\varphi$ is a local diffeomorphism and if we note

$$z = \varphi(x)$$

we have

$$\dot{z} = \frac{\partial \varphi}{\partial x} f(\varphi^{-1}(z)). \tag{2.5}$$

Thus, denoting by $g$ the vector field on $\varphi(X) \subset X$ defined by

$$g(z) = \frac{\partial \varphi}{\partial x} f(\varphi^{-1}(z))$$

and $Z_t$ the local flow associated to $g$, one immediately sees that $Z_t$ is deduced from the flow $X_t$ by the formula $Z_t(\varphi(x)) = \varphi(X_t(x))$, or:

$$Z_t \circ \varphi = \varphi \circ X_t. \tag{2.6}$$

It results that if $x_1 \sim x_2$, then $z_1 = \varphi(x_1) \sim z_2 = \varphi(x_2)$, which proves that the orbits of $g$ are the orbits of $f$ transformed by $\varphi$ and the same for their respective phase portraits.

### 2.2.3 Lie Derivative

Consider a system of local coordinates $(x_1, \ldots, x_p)$ in an open set $U \subset \mathbb{R}^p$. The components of the vector field $f$ in these coordinates are denoted by $(f_1, \ldots, f_p)^T$. We now show that to $f$ one can associate in a one-to-one way a first order differential operator called *Lie derivative along* $f$.

Denote, as before, by $t \longmapsto X_t(x)$ the integral curve of $f$ in $U$ passing through $x$ at $t = 0$.

**Definition 2.6.** Let $h$ be a function of class $C^1$ from $\mathbb{R}^p$ to $\mathbb{R}$ and $x \in U$. We call *Lie derivative of* $h$ *along* $f$ *at* $x$, noted $L_f h(x)$, the time derivative, at $t = 0$, of $h(X_t(x))$, *i.e.*:

$$L_f h(x) = \frac{d}{dt} h(X_t(x))_{|t=0} = \sum_{i=1}^{p} f_i(x) \frac{\partial h}{\partial x_i}(x) \ .$$

We also call *Lie derivative of* $h$ *along* $f$, denoted by $L_f h$, the mapping $x \mapsto L_f h(x)$ from $U$ to $\mathbb{R}$.

According to this formula, every vector field $f$ may be identified to the linear differential operator of the first order

$$L_f = \sum_{i=1}^{p} f_i(x) \frac{\partial}{\partial x_i}.$$

It results that, in local coordinates, we can use indifferently the component-wise or the differential operator expression of $f$, namely

$$f = (f_1, \ldots, f_p)^T \sim \sum_{i=1}^{p} f_i(x) \frac{\partial}{\partial x_i}$$