Werner Schiehlen
Editor

CISM

International Centre
for Mechanical Sciences

Dynamical Analysis of Vehicle Systems

CISM Courses and Lectures, vol. 497

SpringerWienNewYork

SpringerWienNewYork

CISM COURSES AND LECTURES

The series presents lecture notes, monographs, edited works and proceedings in the field of Mechanics, Engineering, Computer Science and Applied Mathematics.
Purpose of the series is to make known in the international scientific and technical community results obtained in some of the activities organized by CISM, the International Centre for Mechanical Sciences.

INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 497

DYNAMICAL ANALYSIS OF VEHICLE SYSTEMS

THEORETICAL FOUNDATIONS AND ADVANCED APPLICATIONS

EDITED BY

WERNER SCHIEHLEN
UNIVERSITY OF STUTTGART, GERMANY

SpringerWienNewYork

This volume contains 226 illustrations

Printed in Italy
SPIN 12068067

All contributions have been typeset by the authors.

ISBN 978-3-211-76665-1 SpringerWienNewYork

PREFACE

This book contains an edited version of the lecture notes used for the Course "Dynamical Analysis of Vehicle Systems - Theoretical Foundations and Advanced Applications" offered at the Centre International des Sciences Mécaniques (CISM) during the Fiszdon Session. The Course took place in CISM's Palazzo del Torso in Udine, Italy, 23 - 27 October 2006. The Course was well attended by engineers from academia and industry, with a total number of 45 persons from twelve countries.

This volume presents an integrated approach to the common fundamentals of rail and road vehicles based on multibody system dynamics, rolling wheel contact and control system design. The mathematical methods presented allow an efficient and reliable analysis of the resulting state equations, and may also be used to review simulation results from commercial vehicle dynamics software.

The book will also provide a better understanding of the basic physical phenomena of vehicle dynamics most important for the engineering practice in research and in industry. Particular attention will be paid to developments of future road and rail vehicles. Again, mechatronic trains and mechatronic cars show many similarities which result in an interdisciplinary stimulation of the design concepts used. The automation of individual vehicle traffic on roads, and on rail, is an important point of issue in the future: Drivers reading the newspaper, watching television, surfing the Internet while their vehicles automatically find their way to the desired destination - rapid and secure. The course features two recent developments. The Railcab is an individual vehicle on existing railway tracks with point-to-point link in a complex controlled network. The Driver Assistance Systems are devices to control distance and keep in lane on existing roads, relieving the driver and leading to a significant improvement in driving safety and comfort.

The Course was originally initiated together with Professor Karl Popp, University of Hanover, Germany who passed away unexpectedly. The Lecturers and the Editor of this volume agreed to continue and, thus, the Course was delivered in memoriam of Professor Karl Popp, too. We thank Professor Giulio Maier, Rector of CISM, for the kind support during the final coordination of the Course. Further, the CISM staff is acknowledged for the excellent organisation. Finally, Professor Paolo Serafini is gratefully acknowledged for his encouragement to publish these lecture notes and his patience while it took longer to complete their editing in book form.

Werner Schiehlen

CONTENTS

Vehicle and Guideway Modelling: Suspensions Systems

Werner Schiehlen
Institute of Engineering and Computational Mechanics, University of Stuttgart, Germany

Abstract Performance, safety and comfort of a vehicle are related to its low frequency motions. The corresponding mechanical models are characterized for all kinds of vehicles by stiff parts represented as rigid bodies and soft components like springs, dampers and actuators. The method of multibody systems is most appropriate for the analysis of vehicle motions and vibrations up to 50 Hz. In this contribution the derivation of the equations of motions of multibody systems is shown step by step up to the computer-aided evaluation of these equations.

Starting with kinematics for rigid body vehicle systems, the foundations of dynamics together with the principles of d'Alembert and Jourdain are used to get the equations of motion. Then, some aspects of multibody dynamics formalisms and computer codes for vehicle dynamics are discussed. Further, models of randomly uneven guideways are presented. Performance criteria for ride comfort and safety are considered. Finally, the analysis of the suspension of a car model is presented in detail.

1 Kinematics

The elements of multibody systems for vehicle modelling, see Figure 1, include rigid bodies which may also degenerate to particles, coupling elements like springs, dampers or force controlled actuators as well as ideal, i.e. rigid kinematical connecting elements like joints, bearings, rails and motion controlled actuators. The coupling and connection elements are generating internal forces and torques between the bodies of the system and external forces with respect to the environment. Both of them are considered as massless elements. The kinematical constraints resulting from the connecting elements may be holonomic or nonholonomic, scleronomic or rheonomic, respectively. Holonomic constraints reduce the motion space of the system while nonholonomic constraints reduce the velocity space in addition. The constraint equations are called rheonomic if they depend explicitly on time, and scleronomic otherwise. Real vehicle systems are subject to holonomic constraints only which may be given by geometrical or integrable kinematical conditions. However, in more simplified models, e.g. rolling of a rigid wheel or wheelset on a rigid plane, nonholonomic constraints may occur. Some configurations of holonomic connecting elements are listed in Table 1 depending on the number of degrees of freedom characterizing the remaining possibilities of motion. Now the motion of vehicle parts will be described mathematically depending on space and time. This is the task of kinematics.

Table 1. Configurations of holonomic connecting elements

Motion	Degrees of Freedom		
	1	2	3
Rotary	Revolute Joint	Universal Joint	Spherical Joint
Linear	Prismatic Joint	Planar Joint	
Mixed	Screw Joint	Cylindric Joint	General Planar Joint

1.1 Frames of Reference for Vehicle Kinematics

A prerequisite for the mathematical description of position, velocity and acceleration of a mechanical system is the definition of appropriate frames of reference. The frames required in vehicle dynamics are shown in Figure 2 with the details summarized in Table 2. There will be used only right-handed Cartesian frames with the unit base vectors $\boldsymbol{e}_\upsilon$, $|\boldsymbol{e}_\upsilon| = 1$ where the Greek indices generally take the integers 1, 2, 3. A basis or frame $\{O, \boldsymbol{e}_\upsilon\}$, respectively, is completely defined by its origin O and its base vectors $\boldsymbol{e}_\upsilon$. For distinction between different frames the upper right index is used if necessary. The inertial frame $\left\{O^I, \boldsymbol{e}_\upsilon^I\right\}$ serves as the general reference frame, in particular for the evaluation of the acceleration. The given trajectory of the vehicle is assumed to be a space curve with the moving frame $\left\{O^B, \boldsymbol{e}_\upsilon^B\right\}$ also known as Frenet frame or moving trihedron. The origin O^B is moving with a given speed tangential to the trajectory. The reference frame $\left\{O^R, \boldsymbol{e}_\upsilon^R\right\}$ is closely related to the moving frame. Its origin and the first unit vector coincide with the moving frame $O^R = O^B$, $\boldsymbol{e}_1^R = \boldsymbol{e}_1^B$. The second base vector $\boldsymbol{e}_2^R$, however, is parallel to the guideway surface considering the bank of the road or the track, respectively, pointing to the right with respect to the direction of motion.

The body-fixed frame $\left\{O^i, \boldsymbol{e}_\upsilon^i\right\}$ is the principal axis frame of the rigid body K_i located in its center of mass C_i. This frame describes uniquely the position in space of the body. Finally, there is defined a local frame $\left\{O^j, \boldsymbol{e}_\upsilon^j\right\}$ to describe constraint elements between bodies. It is oriented according to the local specifications like the direction of a joint axis. In the following a frame is simply identified by its name (upper right index) only.

1.2 Kinematics of a Rigid Body in an Inertial Frame

First of all some definitions and remarks on the nomenclature are presented. The position of a particle P in space is uniquely defined by the position vector $\boldsymbol{x}$ represented in the inertial frame $\left\{O^I, \boldsymbol{e}_\upsilon^I\right\}$ by its coordinates x_υ as

$$\boldsymbol{x} = x_1\boldsymbol{e}_1 + x_2\boldsymbol{e}_2 + x_3\boldsymbol{e}_3 \ . \tag{1.1}$$

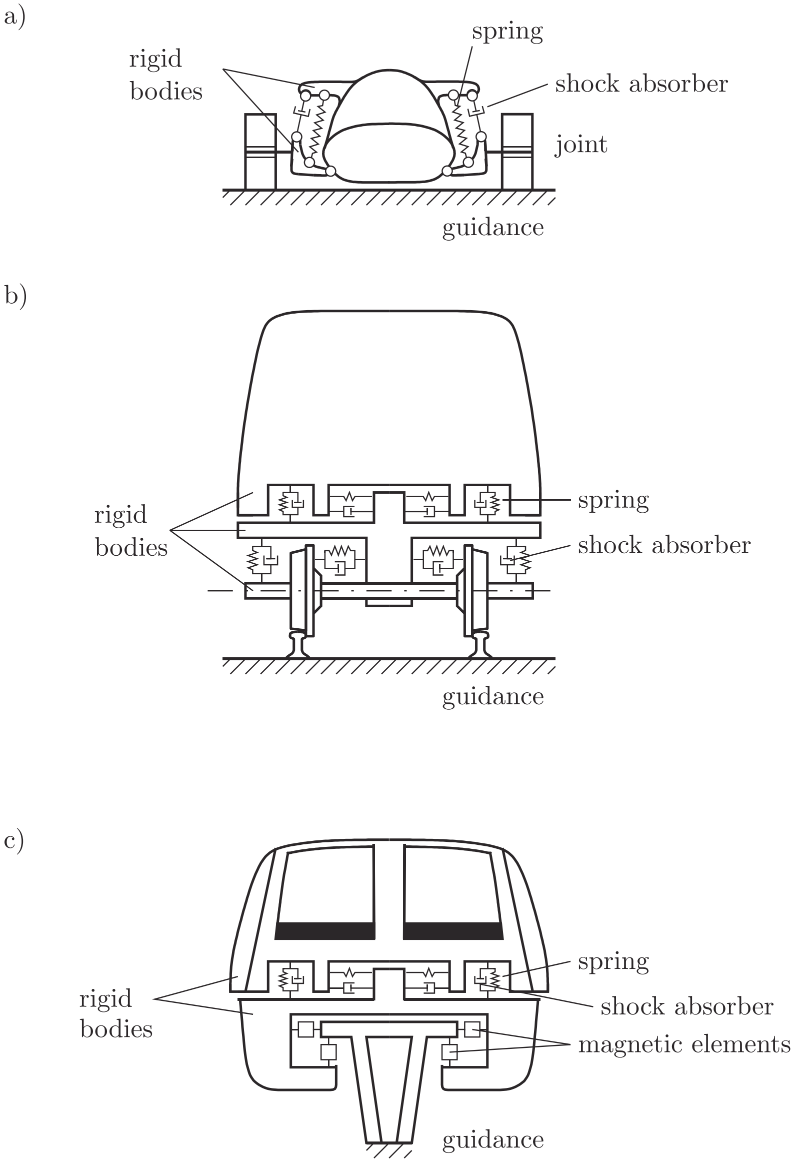

Figure 1. Multibody models of vehicles:
a) road vehicle, b) rail vehicle, c) magnetically levitated vehicle

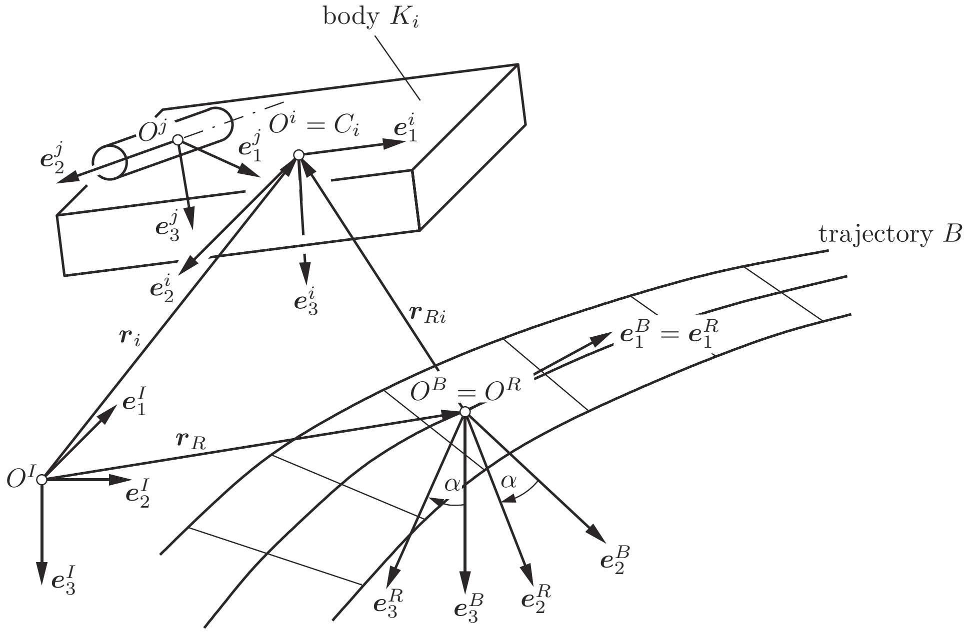

Figure 2. Frames of reference

This set of coordinates may be summarized in a column matrix $\boldsymbol{x}^I$ often simply called a vector, i.e.,

$$\boldsymbol{x}^I = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \equiv \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}} \tag{1.2}$$

where upper right index defines the frame in which the coordinates are measured. This index will often be deleted if there isn't any possibility for a mix-up of frames or if there is used only one frame identified in the text. The goal is to present all vector and tensor quantities in one common frame, e.g. the inertial frame I. Then, it is possible, to integrate subsystems easily into the complete system.

For a particle P moving in time its coordinates are time-dependent, too, and they define a trajectory in space. The mathematical representation results in the vector equation $\boldsymbol{x} = \boldsymbol{x}(t)$ equivalent to three scalar equations according to the three degrees of freedom of the particle in the three-dimensional space. The velocity $\boldsymbol{v}(t)$ and the acceleration $\boldsymbol{a}(t)$ of the particle follow by differentiation with respect to time as

$$\boldsymbol{v}(t) = \frac{\mathrm{d}^I \boldsymbol{x}(t)}{\mathrm{d}t}, \quad \boldsymbol{v}^I(t) = \dot{\boldsymbol{x}}^I(t) = \begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \end{bmatrix}^{\mathrm{T}}, \tag{1.3}$$

$$\boldsymbol{a}(t) = \frac{\mathrm{d}^I \boldsymbol{v}(t)}{\mathrm{d}t}, \quad \boldsymbol{a}^I(t) = \dot{\boldsymbol{v}}^I(t) = \ddot{\boldsymbol{x}}(t) = \begin{bmatrix} \ddot{x}_1 & \ddot{x}_2 & \ddot{x}_3 \end{bmatrix}^{\mathrm{T}}. \tag{1.4}$$

Table 2. Frames of reference

Frame of reference	Origin of frame	Orientation of axis
Inertial frame $\{O^I\,,\,\boldsymbol{e}_v^I\}$	O^I space fixed	$\boldsymbol{e}_1^I$, $\boldsymbol{e}_2^I$ in horizontal plane $\boldsymbol{e}_3^I$
Moving trihedron $\{O^B\,,\,\boldsymbol{e}_v^B\}$	O^B trajectory-fixed	$\boldsymbol{e}_1^B \equiv \boldsymbol{e}_t$ tangential to trajectory $\boldsymbol{e}_2^B \equiv \boldsymbol{e}_n$ normal to trajectory $\boldsymbol{e}_3^B \equiv \boldsymbol{e}_b$ bi-normal to trajectory
Reference frame $\{O^R\,,\,\boldsymbol{e}_v^R\}$	O^R trajectory-fixed	$\boldsymbol{e}_1^R \equiv \boldsymbol{e}_1$ $\boldsymbol{e}_2^R$ in guideway plane $\boldsymbol{e}_3^R$ normal to guideway plane
Body-fixed frame $\{O^i\,,\,\boldsymbol{e}_v^i\}$	$O^i \equiv C_i$ body-fixed in center of mass	$\boldsymbol{e}_v^i$ principle inertia axes
Local frame $\{O^j\,,\,\boldsymbol{e}_v^j\}$	O^i arbitrary	$\boldsymbol{e}_v^j$ locally specified axes

The upper right index refers to the frame of reference in which the operations, in particular the differentiation, have to be executed. In the inertial frame I the differentiation of vectors is just performed by differentiation of the scalar coordinates.

The motion of a particle P on a curvilinear trajectory in space may be shown in the moving frame B, too. The position of the point is uniquely identified by the arc length $s(t)$ as a generalized coordinate, Figure 3. Then, the position vector $\boldsymbol{r}$ is a function of the arc length, $\boldsymbol{r} = \boldsymbol{r}(s)$. For the velocity and acceleration vector it yields, see e.g. Magnus and Müller (1990),

$$
\begin{aligned}
\boldsymbol{v}(t) &= \frac{\mathrm{d}^I\boldsymbol{r}(t)}{\mathrm{d}t} = \frac{\mathrm{d}^I\boldsymbol{r}(s)}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = v\,\boldsymbol{e}_t\;, \quad v = \dot{s}\;, \\
\boldsymbol{v}^B(t) &= \begin{bmatrix} \dot{s} & 0 & 0 \end{bmatrix}^{\mathrm{T}},
\end{aligned} \tag{1.5}
$$

$$
\begin{aligned}
\boldsymbol{a}(t) &= \frac{\mathrm{d}^I\boldsymbol{v}(t)}{\mathrm{d}t} = a_t\boldsymbol{e}_t + a_n\boldsymbol{e}_n = \dot{v}\,\boldsymbol{e}_t + \frac{v^2}{\rho}\boldsymbol{e}_n\;, \\
\boldsymbol{a}^B(t) &= \begin{bmatrix} \ddot{s} & s^2/\rho & 0 \end{bmatrix}^{\mathrm{T}},
\end{aligned} \tag{1.6}
$$

where ρ is the curvature of the trajectory in point P. Further, it is

$v = \dot{s}$ the tangent velocity,
$a_t = \dot{v} = \ddot{s}$ the tangent acceleration, and
$a_n = {v^2}/{\rho} = {\dot{s}^2}/{\rho}$ the normal or centripetal acceleration.

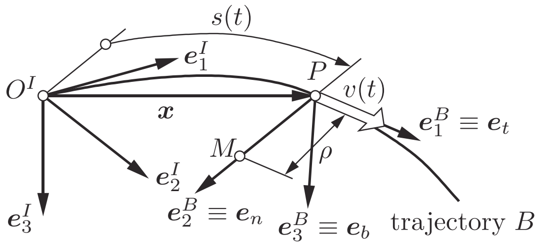

Figure 3. Trajectory of particle P

Special cases of the general motion in space are the motion in a straight line ($\rho \to \infty$), and the motion on a circle ($\rho = \text{const}$). Often the functions $s(t), \dot{s}(t), \ddot{s}(t), \dot{s}(s), \ddot{s}(s), \ddot{s}(\dot{s})$ are depicted in kinematical diagrams for graphical visualization of the motion along a track.

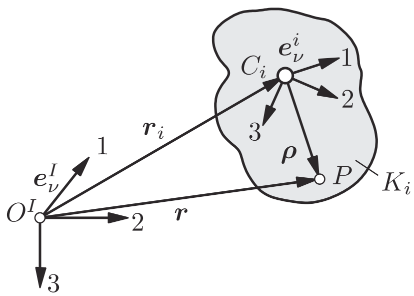

Figure 4. Position of a rigid body K_i in the inertial frame I

The translational motion of a rigid body K_i is completely described by the general relations for a particle applied to a body-fixed point, e.g. the center of mass C_i of the rigid body, and the corresponding position vector $\boldsymbol{r}_i$, see Figure 4. The rotational motion of a rigid body K_i follows from the relative position of two frames where one of them is a body-fixed frame. For coinciding origins the position of the body-fixed frame i relative to the inertial frame I is uniquely defined by three rotation angles according to the three rotational degrees of freedom of a rigid body in space. Both frames are related to each other by three elementary rotations performed successively around different base vectors using three rotation angles. If, for example, the frame i is revolved around the coinciding 3-axes of frame I and i by the angle γ, the relation of the corresponding base vectors is given by the matrix $\boldsymbol{S}^{Ii}$ as shown in Figure 5 and reads as

$$\begin{bmatrix} \boldsymbol{e}_1^I \\ \boldsymbol{e}_2^I \\ \boldsymbol{e}_3^I \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\boldsymbol{S}^{Ii} = \boldsymbol{\gamma}_3} \begin{bmatrix} \boldsymbol{e}_1^i \\ \boldsymbol{e}_2^i \\ \boldsymbol{e}_3^i \end{bmatrix} \tag{1.7}$$

The row ν of the elementary rotation matrix $\boldsymbol{\gamma}_3$ is composed of the coordinates of the

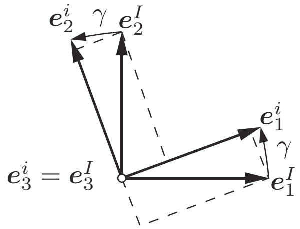

Figure 5. Elementary rotation with angle γ around 3-axes

base vector $\boldsymbol{e}_\upsilon^I$ in frame i. The corresponding matrices for positive rotations around the remaining axes read as

$$\boldsymbol{\alpha}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \ \boldsymbol{\beta}_2 = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{1.8}$$

where the elementary rotation matrices are characterized by the name of the rotation angle while the index defines the axis of rotation. There are numerous possibilities to choose the name of the angle and the sequence of the rotation axes used which is not commutative. In vehicle dynamics the Cardan angles α, β, γ are often used, see Figure 6, which are different from the well-known Euler angles ψ, ϑ, φ.

The resulting rotation matrices $\boldsymbol{S}^{Ii}$ which present the relation between frames I and i are obtained by the corresponding matrix multiplications

$$\boldsymbol{S}^{Ii}(\alpha, \beta, \gamma) = \boldsymbol{\alpha}_1 \boldsymbol{\beta}_2 \boldsymbol{\gamma}_3 \ , \quad \boldsymbol{S}^{Ii}(\psi, \vartheta, \varphi) = \boldsymbol{\psi}_3 \boldsymbol{\vartheta}_1 \boldsymbol{\varphi}_3 \ . \tag{1.9}$$

Since matrix products are not commutative, the sequence of the elementary rotations has to be strongly observed. The Cardan and Euler angles, respectively, are defined by successive rotations around the 1-, 2-, 3-axis and 3-, 1-, 3-axis, respectively, starting from the inertial frame I. The sequence of the elementary rotations is uniquely identified by the sequence of the indices of the elementary rotation matrices as shown in (1.9).

The rotation matrices are orthogonal matrices

$$\boldsymbol{S}^{iI} \left(\boldsymbol{S}^{Ii}\right)^{-1} = \left(\boldsymbol{S}^{Ii}\right)^{\mathrm{T}} = \boldsymbol{S}^{iI}, \quad \det \boldsymbol{S} = +1 \tag{1.10}$$

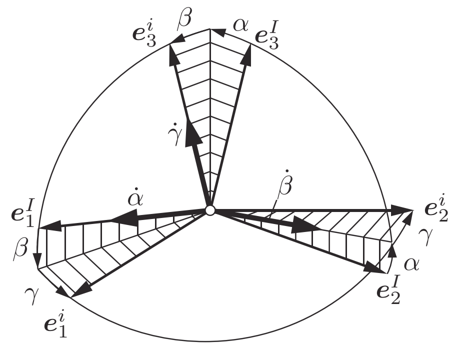

Figure 6. Spatial rotation with Cardan angles α, β, γ

where the inversion is also represented by the exchange of the upper indices. The inverse rotation matrix is simply found by transposition of the original rotation matrix $\boldsymbol{S}^{Ii}$. Using Cardan angles the rotation matrix reads explicitly as

$$\boldsymbol{S}^{Ii}(\alpha,\beta,\gamma) = \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ c\alpha s\gamma + s\alpha s\beta c\gamma & c\alpha c\gamma - s\alpha s\beta s\gamma & -s\alpha c\beta \\ s\alpha s\gamma - c\alpha s\beta c\gamma & s\alpha c\gamma + c\alpha s\beta s\gamma & c\alpha c\beta \end{bmatrix} \tag{1.11}$$

where the abbreviations c and s stands for *cos* and *sin*, respectively. In applications often small rotations are found, $\alpha, \beta, \gamma \ll 1$, resulting in the linearized rotation matrix

$$\boldsymbol{S}^{Ii}(\alpha,\beta,\gamma) = \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix}. \tag{1.12}$$

This result is also obtained if the elementary rotation matrices are linearized and multiplied with each other. Due to the vector property of small rotations the sequence of the multiplications has no longer to be considered. For small rotations the Cardan angles may be assigned directly to the rotational motions around the body-fixed axes, Figure 7. In vehicle engineering the following notations are used,

- α roll motion,
- β pitch motion,
- γ yaw motion.

The coordinates of a vector $\boldsymbol{x}$ read differently for different frames. The relation between the coordinates $\boldsymbol{x}^i$ in frame i and the coordinates $\boldsymbol{x}^I$ in frame I is given by the transformation law for vector coordinates as

$$\boldsymbol{x}^i = \boldsymbol{S}^{iI}\boldsymbol{x}^I \quad \text{and} \quad \boldsymbol{x}^I = \boldsymbol{S}^{Ii}\boldsymbol{x}^i\,, \tag{1.13}$$

respectively, what is easily proven by (1.10). Please observe that the same indices appear in both forms in a neighbouring sequence. This property is often helpful in applications.

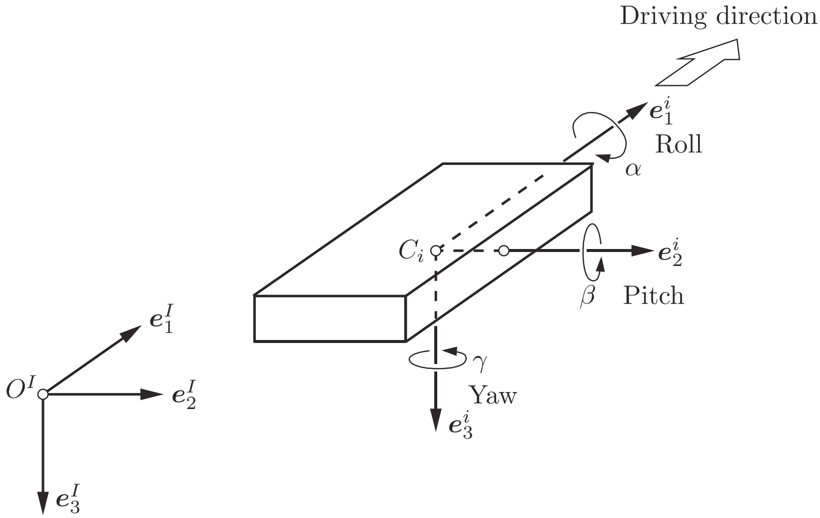

Figure 7. Notation of rotary vehicle motions

Further, it has to be pointed out that the rotation matrix $\boldsymbol{S}$ is a function of time, $\boldsymbol{S} = \boldsymbol{S}(t)$, what has to be considered for time derivations.

The position of a rigid body K_i in the inertial frame is uniquely described by the position quantities $\{\boldsymbol{r}_i, \boldsymbol{S}^{Ii}\}$ which characterize the body-fixed frame $\{C^i, \boldsymbol{e}_v^i\}$. During motion the position quantities are functions of time. Thus, the position coordinates of an arbitrary particle P of the rigid body read in the inertial frame I as

$$\boldsymbol{r}^I(t) = \boldsymbol{r}_i^I(t) + \boldsymbol{\rho}^I(t) \ , \quad \boldsymbol{\rho}^I(t) = \boldsymbol{S}^{Ii}(t)\boldsymbol{\rho}^i \ , \tag{1.14}$$

where in the body-fixed frame it yields $\boldsymbol{\rho}^i = \mathbf{const}$, see also Figure 4.

The motion of a rigid body K_i will be now presented in the inertial frame I, too. The change of the position of its particle P with respect to time relative to frame I is found by differentiation of (1.14) as

$$\dot{\boldsymbol{r}}^I(t) = \dot{\boldsymbol{r}}_i^I(t) + \dot{\boldsymbol{S}}^{Ii}(t)\boldsymbol{\rho}^i = \dot{\boldsymbol{r}}_i^I(t) + \dot{\boldsymbol{S}}^{Ii}(t)\boldsymbol{S}^{iI}(t)\boldsymbol{\rho}^I(t) \ . \tag{1.15}$$

The first term on the right-hand side represents the translational velocity of the origin C_i of the body-fixed frame i. The second term is obviously related to the rotation of the body-fixed frame and represents the body's rotation. This term will now be discussed in more detail. The matrix product $[\dot{\boldsymbol{S}}(t)\boldsymbol{S}^{\mathrm{T}}(t)]$ is screw symmetric, i.e., $[\bullet] = -[\bullet]^{\mathrm{T}}$, what follows immediately from the differentiation of the orthogonality condition $\boldsymbol{S}(\mathrm{t})\boldsymbol{S}^{\mathrm{T}}(t) = \boldsymbol{E}$ according to (1.10):

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\boldsymbol{S}(t)\boldsymbol{S}^{\mathrm{T}}(t)\right] &= \dot{\boldsymbol{S}}(t)\boldsymbol{S}^{\mathrm{T}}(t) + \boldsymbol{S}(t)\dot{\boldsymbol{S}}^{\mathrm{T}}(t) \\ &= \dot{\boldsymbol{S}}(t)\boldsymbol{S}^{\mathrm{T}}(t) + \left[\dot{\boldsymbol{S}}(t)\boldsymbol{S}^{\mathrm{T}}(t)\right]^{\mathrm{T}} = \boldsymbol{0} \ . \end{aligned} \tag{1.16}$$

The matrix product [•] will be abbreviated by the symbol $\tilde{\boldsymbol{\omega}}\,(t)$ and identified by the corresponding three coordinates $\boldsymbol{\omega}_v = \boldsymbol{\omega}_v(t)$ as follows

$$\dot{\boldsymbol{S}}^{Ii}\left(\boldsymbol{S}^{Ii}\right)^{\mathrm{T}} = \dot{\boldsymbol{S}}^{Ii}\boldsymbol{S}^{iI} = \tilde{\boldsymbol{\omega}}^{I}_{Ii}(t) := \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} , \; \boldsymbol{\omega}^{I}_{Ii} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} . \tag{1.17}$$

Both quantities, the screw symmetric tensor $\tilde{\boldsymbol{\omega}}^{I}_{Ii}$ and the corresponding rotational velocity vector $\boldsymbol{\omega}^{I}_{Ii}$, respectively, describe the rotational motion of system i or body K_i, respectively, relative to the inertial frame I. The upper indices indicate that both quantities are represented in the inertial frame I. If there is no chance for mixing up the frames the upper and lower index I is simply deleted. The screw symmetric tensor corresponding with a vector (•) is identified by the symbol ($\tilde{\bullet}$) and it replaces the vector product

$$\tilde{\boldsymbol{\omega}}\boldsymbol{\rho} \equiv \boldsymbol{\omega} \times \boldsymbol{\rho}\ . \tag{1.18}$$

In coordinates, in any frame, one gets accordingly

$$\tilde{\boldsymbol{\omega}}\,\boldsymbol{\rho} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \begin{bmatrix} \rho_1 \\ \rho_2 \\ \rho_3 \end{bmatrix} = \begin{bmatrix} \omega_2\rho_3 - \omega_3\rho_2 \\ \omega_3\rho_1 - \omega_1\rho_3 \\ \omega_1\rho_2 - \omega_2\rho_1 \end{bmatrix} . \tag{1.19}$$

This notation of the vector product is most valuable for numerical computations since the vector product is not defined in matrix calculus.

The rotational velocity vector $\boldsymbol{\omega}^{i}_{Ii}$ in the body-fixed frame i follows from transformation or direct evaluation, respectively. The application of transformation (1.13) to (1.19) results in

$$\tilde{\boldsymbol{\omega}}^{i}_{Ii} = \boldsymbol{S}^{iI}\tilde{\boldsymbol{\omega}}^{I}_{Ii}\boldsymbol{S}^{Ii} = \boldsymbol{S}^{iI}\left(\dot{\boldsymbol{S}}^{Ii}\boldsymbol{S}^{iI}\right)\boldsymbol{S}^{Ii} = \boldsymbol{S}^{iI}\dot{\boldsymbol{S}}^{Ii} = \left(\boldsymbol{S}^{Ii}\right)^{\mathrm{T}}\dot{\boldsymbol{S}}^{Ii} . \tag{1.20}$$

The first and second term of (1.20) represents the transformation law for tensor coordinates where the same indices appear again in a neighbouring sequence, the first and last term show the direct evaluation. The vector corresponding to (1.20) is $\boldsymbol{\omega}^{i}_{Ii} = \boldsymbol{S}^{iI}\boldsymbol{\omega}^{I}_{Ii}$ where the transformation law for vector coordinates has been used again.

From (1.15) and (1.17) it follows for rigid body K_i

$$\boldsymbol{v}^{I}(t) = \boldsymbol{v}^{I}_{i}(t) + \tilde{\boldsymbol{\omega}}^{I}_{Ii}(t)\boldsymbol{\rho}^{I}(t)\ . \tag{1.21}$$

Considering (1.18), this is the relation for rigid body kinematics well-known from each mechanics textbook as

$$\boldsymbol{v}(t) = \boldsymbol{v}_i(t) + \boldsymbol{\omega}_{Ii}(t) \times \boldsymbol{\rho}(t). \tag{1.22}$$

The relations (1.21) or (1.22), respectively, represent the motion of a rigid body composed by an absolute translational velocity $\boldsymbol{v}_i$ of the body-fixed reference point $O^i = C^i$, and a rotation with the angular velocity $\boldsymbol{\omega}_{Ii}$. The fundamental kinematical quantities $\{\boldsymbol{v}_i, \boldsymbol{\omega}_{Ii}\}$ are also denoted as twist characterizing uniquely the motion of a rigid body.

1.3 Kinematics of a Rigid Body in a Moving Reference Frame

From a mathematical point of view the kinematical description of the motion of a rigid body is most convenient in the inertial frame I resulting in a more simple representation of the fundamental laws of mechanics. In engineering applications, however, a moving reference frame R related to the vehicle or the guideway, respectively, turns out to be more adequate. The frame R allows a problem-oriented choice of the coordinates and an efficient description of the forces and torques acting on the system. Moving reference frames are also useful in experiments since many measurement data are not related to the inertial frame. The choice of the reference frame R depends on the problem under consideration. In many cases the frame R characterizes the large nonlinear reference motion of a vehicle while the small deviation from the reference motion results in linear kinematical relations even for rotations.

In the following the motion of a rigid body is represented in a moving reference frame R the motion of which is known in the inertial frame I by the position vector $\boldsymbol{r}_R(t)$ of its origin, and rotation matrix $\boldsymbol{S}^{IR}(t)$, see Figure 8. This means that the translational and angular guidance velocities are also known, $\boldsymbol{v}_R^I = \dot{\boldsymbol{r}}_R^I$, $\tilde{\boldsymbol{\omega}}_{IR}^I = \dot{\boldsymbol{S}}^{IR}\boldsymbol{S}^{RI}$ according to (1.3) and (1.17). Considering Figure 8, the absolute position quantities $\{\boldsymbol{r}_i\,, \boldsymbol{S}^{Ii}\}$ of the rigid body reads as

$$\boldsymbol{r}_i^I(t) = \boldsymbol{r}_R^I(t) + \boldsymbol{S}^{IR}(t)\boldsymbol{r}_{Ri}^R(t)\ , \tag{1.23}$$

$$\boldsymbol{S}^{Ii}(t) = \boldsymbol{S}^{IR}(t)\boldsymbol{S}^{Ri}(t)\ . \tag{1.24}$$

The absolute motion $\{\boldsymbol{v}_i, \boldsymbol{\omega}_{Ii}\}$ of the rigid body K_i is now found by formal differentiation

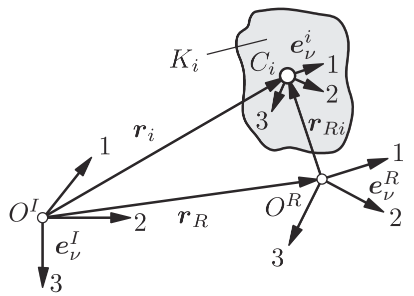

Figure 8. Position of a rigid body in the reference frame R

in frame I and subsequent transformation in frame R as

$$\boldsymbol{v}_i^R(t) = \boldsymbol{v}_R^R(t) + \tilde{\boldsymbol{\omega}}_{IR}^R(t)\,\boldsymbol{r}_{Ri}^R(t) + \dot{\boldsymbol{r}}_{Ri}^R(t)\ , \tag{1.25}$$

$$\boldsymbol{\omega}_{Ii}^R = \boldsymbol{\omega}_{IR}^R + \boldsymbol{\omega}_{Ri}^R\ . \tag{1.26}$$

Due to the rotation of the frame R the guidance motion and the relative motion characterized by the indices IR and Ri , respectively, are not simply added but there appears an additional term in (1.25). This implies the well-known law of differentiation in a rotating frame

$$\frac{\mathrm{d}^I}{\mathrm{d}t}\boldsymbol{r}(t) = \frac{\mathrm{d}^R}{\mathrm{d}t}\boldsymbol{r}(t) + \boldsymbol{\omega}_{IR}(t) \times \boldsymbol{r}(t) \; . \tag{1.27}$$

By formal differentiation or application of (1.27) to (1.25) and (1.26) one gets finally the absolute translational and rotational acceleration of the rigid body K_i again written in the reference frame R as

$$\boldsymbol{a}_i^R(t) = \boldsymbol{a}_R^R + \left(\dot{\tilde{\boldsymbol{\omega}}}_{IR}^R + \tilde{\boldsymbol{\omega}}_{IR}^R \tilde{\boldsymbol{\omega}}_{IR}^R\right) \boldsymbol{r}_{Ri}^R + 2\tilde{\boldsymbol{\omega}}_{IR}^R \dot{\boldsymbol{r}}_{Ri}^R + \ddot{\boldsymbol{r}}_{Ri}^R \; , \tag{1.28}$$

$$\boldsymbol{\alpha}_{Ii}^R(t) = \dot{\boldsymbol{\omega}}_{IR}^R + \tilde{\boldsymbol{\omega}}_{IR}^R \boldsymbol{\omega}_{Ri}^R + \dot{\boldsymbol{\omega}}_{Ri}^R \; . \tag{1.29}$$

Thus, in addition to the guidance and relative acceleration the Coriolis acceleration with the characteristic factor 2 is found for translations.

1.4 Kinematics of Multibody Systems

So far only one free rigid body K_i was considered the position of which is uniquely described in the inertial frame I as

$$\boldsymbol{r}_i^I = [r_{i1}\, r_{i2}\, r_{i3}]^{\mathrm{T}} \; , \quad \boldsymbol{S}^{Ii} \equiv \boldsymbol{S}_i = \boldsymbol{S}_i(\alpha_i, \beta_i, \gamma_i) \; . \tag{1.30}$$

There are six position coordinates which are summarized in a 6×1- column matrix, simply called local position vector, as

$$\boldsymbol{x}_i(t) = [r_{i1}\, r_{i2}\, r_{i3}\, \alpha_i\, \beta_i\, \gamma_i]^{\mathrm{T}} \; . \tag{1.31}$$

For a free multibody system consisting of p disassembled rigid bodies K_i, $i = 1(1)p$, there remain $6p$ position coordinates resulting in a $6p \times 1$ global position vector of an unconstrained system

$$\boldsymbol{x}(t) = [\boldsymbol{x}_1^{\mathrm{T}} \ldots\ldots \boldsymbol{x}_p^{\mathrm{T}}]^{\mathrm{T}} \; . \tag{1.32}$$

Assembling the free system there appear constraints between the position coordinates and their derivatives. In realistic models of vehicles only holonomic constraints are found restricting the motion of the position coordinates by geometric or integrable kinematic constraints. These constraints are implicitly described by algebraical equations which may be time-dependent (rheonomic), too,

$$\varphi_j(\boldsymbol{x}, t) = 0 \; , \quad j = 1(1)q \; . \tag{1.33}$$

Due to q constraints there remain f linear independent position coordinates characterizing $f = 6p - q$ degrees of freedom. The f independent position coordinates are also called generalized coordinates and may be summarized in a $f \times 1$- column matrix as global position vector of the constraint system

$$\boldsymbol{y}(t) = [\, y_1 \ldots\ldots y_f]^{\mathrm{T}} \; . \tag{1.34}$$

By (1.33) and (1.34) the vector $\boldsymbol{x}$ is an explicit function of the f generalized coordinates representing the constraints explicitly,

$$\boldsymbol{x} = \boldsymbol{x}(y, t) \; . \tag{1.35}$$

The choice of generalized coordinates is not unique. E.g., some of the local position coordinates (absolute coordinates) or differences between local coordinates (relative coordinates) may be chosen as generalized coordinates. However, there exists a unique relation between different sets of generalized coordinates represented by a regular, time-invariant $f \times f$ -matrix $\boldsymbol{T}$ resulting in the transformation

$$\boldsymbol{y}\,(t) = \boldsymbol{T}\,\overline{\boldsymbol{y}}\,(t) \;\; , \tag{1.36}$$

where $\boldsymbol{y}$ and $\overline{\boldsymbol{y}}$ are the corresponding global position vectors. The position variables (1.30) may be rewritten for the whole system as

$$\boldsymbol{r}_i(t) = \boldsymbol{r}_i(y,t) \; , \quad \boldsymbol{S}^{Ii}(t) \equiv \boldsymbol{S}_i(t) = \boldsymbol{S}_i(y,t) \; , \quad i = 1(1)p \; . \tag{1.37}$$

The corresponding velocity variables $\{\boldsymbol{v}_i, \boldsymbol{\omega}_i\}$, $\boldsymbol{\omega}_i = \boldsymbol{\omega}_{Ii}$ are obtained by differentiation as

$$\boldsymbol{v}_i(t) = \dot{\boldsymbol{r}}_i(t) = \frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{y}^T}\dot{\boldsymbol{y}} + \frac{\partial \boldsymbol{r}_i}{\partial t} = \boldsymbol{J}_{Ti}(\boldsymbol{y},t)\dot{\boldsymbol{y}} + \overline{\boldsymbol{v}}_i(\boldsymbol{y},t) \; , \tag{1.38}$$

$$\boldsymbol{\omega}_i(t) = \dot{\boldsymbol{s}}_i(t) = \frac{\partial \boldsymbol{s}_i}{\partial \boldsymbol{y}^T}\dot{\boldsymbol{y}} + \frac{\partial \boldsymbol{s}_i}{\partial t} = \boldsymbol{J}_{Ri}(\boldsymbol{y},t)\dot{\boldsymbol{y}} + \overline{\boldsymbol{\omega}}_i(y,t) \; , \tag{1.39}$$

where $\partial \boldsymbol{s}_i$ describes the 3×1-vector of the infinitesimal rotation following from the rotation matrix analogously to the rotational velocity (1.17) as

$$\partial \tilde{\boldsymbol{s}}_i = \partial \boldsymbol{S}_i \boldsymbol{S}_i^{\mathrm{T}} := \begin{bmatrix} 0 & -\partial s_{i3} & \partial s_{i2} \\ \partial s_{i3} & 0 & -\partial s_{i1} \\ -\partial s_{i2} & \partial s_{i1} & 0 \end{bmatrix} , \quad \partial \boldsymbol{s}_i = \begin{bmatrix} \partial s_{i1} \\ \partial s_{i2} \\ \partial s_{i3} \end{bmatrix} . \tag{1.40}$$

The $3 \times f$-functional or Jacobian matrices $\boldsymbol{J}_{Ti}$, $\boldsymbol{J}_{Ri}$ of translation and rotation, respectively, identify the relation between the local and the generalized or global coordinates. The formation of these matrices is defined using the rules of matrix multiplication as shown for the translation matrix

$$\frac{\partial \boldsymbol{r}_i}{\partial \boldsymbol{y}^{\mathrm{T}}} = \partial \boldsymbol{r}_i \left(\frac{1}{\partial \boldsymbol{y}^{\mathrm{T}}} \right) = \boldsymbol{J}_{Ti} = \begin{bmatrix} \dfrac{\partial r_{i1}}{\partial y_1} & \dfrac{\partial r_{i1}}{\partial y_2} & \cdots & \dfrac{\partial r_{i1}}{\partial y_f} \\ \dfrac{\partial r_{i2}}{\partial y_1} & \dfrac{\partial r_{i2}}{\partial y_2} & \cdots & \dfrac{\partial r_{i2}}{\partial y_f} \\ \dfrac{\partial r_{i3}}{\partial y_1} & \dfrac{\partial r_{i3}}{\partial y_2} & \cdots & \dfrac{\partial r_{i3}}{\partial y_f} \end{bmatrix} . \tag{1.41}$$

From (1.38) and (1.39) one obtains by a second differentiation the acceleration variables $\{\boldsymbol{a}_i, \boldsymbol{\alpha}_i\}$ depending on the position vector $\boldsymbol{y}$ and its derivatives,

$$\boldsymbol{a}_i(t) = \dot{\boldsymbol{v}}_i(t) = \boldsymbol{J}_{Ti}(\boldsymbol{y},t)\ddot{\boldsymbol{y}} + \frac{\partial \boldsymbol{v}_i}{\partial \boldsymbol{y}^T}\dot{\boldsymbol{y}} + \frac{\partial \boldsymbol{v}_i}{\partial t} \; , \tag{1.42}$$

$$\boldsymbol{\alpha}_i(t) = \dot{\boldsymbol{\omega}}_i(t) = \boldsymbol{J}_{Ri}(\boldsymbol{y}, t)\ddot{\boldsymbol{y}} + \frac{\partial \boldsymbol{\omega}_i}{\partial \boldsymbol{y}^T}\dot{\boldsymbol{y}} + \frac{\partial \boldsymbol{\omega}_i}{\partial t} \ . \tag{1.43}$$

For scleronomic, time-invariant constraints the partial time derivatives in (1.38), (1.39), (1.42) and (1.43) are vanishing.

In addition to the real motions, the virtual motions are required in the next chapter dealing with dynamics. A virtual motion is defined as an arbitrary, infinitesimally small variation of the position completely compatible with the constraints at any time. Rheonomic constraints are considered to be frozen at the time under consideration. The symbol δ of the virtual motion has the properties

$$\delta \boldsymbol{r} \neq \boldsymbol{0} \ , \quad \delta t \equiv 0 \ . \tag{1.44}$$

The symbol δ follows the rules of calculus, i.e., it yields

$$\delta(c\boldsymbol{r}) = c\delta\boldsymbol{r} \ , \quad \delta(\boldsymbol{r}_1 + \boldsymbol{r}_2) = \delta\boldsymbol{r}_1 + \delta\boldsymbol{r}_2 \ , \quad \delta\boldsymbol{r}(\boldsymbol{y}) = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{y}^{\mathrm{T}}}\delta\boldsymbol{y} \ . \tag{1.45}$$

Thus, the virtual motion of a multibody system reads as

$$\delta\boldsymbol{r}_i = \boldsymbol{J}_{Ti}\delta\boldsymbol{y} \ , \quad \delta\boldsymbol{s}_i = \boldsymbol{J}_{Ri}\delta\boldsymbol{y} \ , \quad i = 1(1)p \ . \tag{1.46}$$

This completes the kinematics for rigid body vehicle systems.

2 Dynamics

For the generation of the equations of motion of multibody systems, in addition to kinematics, the inertia of the bodies and the acting forces have to be considered. The Newton-Euler approach, also called the synthetic method, uses the free body diagram resulting in full set of local equations which may be reduced by the principles of d'Alembert and Jourdain to the equations of motion. The Lagrangian approach, representing the analytical method, is based on energy considerations and the equations of motions are found directly but without any information on the reaction forces.

2.1 Inertia Properties

The inertia of a rigid body K_i is characterized by its mass m_i and its inertia tensor $\boldsymbol{I}_{Ci}$. The coordinates of the inertia tensor read in the body-fixed frame $\{C_i, \boldsymbol{e}_v^{\mathrm{i}}\}$, see Figure 4, as

$$\boldsymbol{I}_{Ci}^i = \int_{m_i} (\boldsymbol{\rho}^{\mathrm{T}}\boldsymbol{\rho}\boldsymbol{E} - \boldsymbol{\rho}\boldsymbol{\rho}^{\mathrm{T}})\mathrm{d}m = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}_{C_i} = \mathbf{const} \ . \tag{2.1}$$

The vector $\boldsymbol{\rho} \equiv \boldsymbol{\rho}^i = [\rho_1 \ \rho_2 \ \rho_3]^{\mathrm{T}}$ describes a material point with mass $\mathrm{d}m$ with respect to the center of mass C_i and $\boldsymbol{E}$ means the 3×3-identity matrix. The inertia tensor $\boldsymbol{I}_{Ci}^i$ is symmetric and positive definite, and constant in the body-fixed frame.

The coordinates of the inertia tensor depend on the mass distribution and on the choice of the reference frame. For a parallel displacement of the body-fixed frame from the center of mass C_i to an arbitrary body-fixed point O_i characterized by the vector $\boldsymbol{s}$ one gets

$$\boldsymbol{I}^i_{Oi} = \boldsymbol{I}^i_{Ci} + (\boldsymbol{s}^{\mathrm{T}}\boldsymbol{s}\boldsymbol{E} - \boldsymbol{s}\boldsymbol{s}^{\mathrm{T}})m_i \; . \tag{2.2}$$

Thus, the diagonal elements of an inertia tensor are minimal for the center of mass.

For a homogeneous, purely rotational displacement by the rotation matrix $\boldsymbol{S}^{ii'}$ from frame $\boldsymbol{e}^i_\upsilon$ to $\boldsymbol{e}^{i'}_\upsilon$ around the center of mass the transformation law for tensors applies as

$$\boldsymbol{I}^i_{Ci} = \boldsymbol{S}^{ii'}\boldsymbol{I}^{i'}_{Ci}\boldsymbol{S}^{i'i} \quad \text{or} \quad \boldsymbol{I}^{i'}_{Ci} = \boldsymbol{S}^{i'i}\boldsymbol{I}^{i}_{Ci}\boldsymbol{S}^{ii'} \; . \tag{2.3}$$

Please note that the inertia tensor may be time-variant if the frame $\{C_i, \boldsymbol{e}^{i'}_\upsilon\}$ is not body-fixed. This is especially true if the inertial frame is chosen, $i' \equiv I$, due to $\boldsymbol{S}^{iI} = \boldsymbol{S}^{iI}(t)$.

For all reference points there exists a special body-fixed frame in which the off-diagonal elements of the inertia tensor are vanishing, e.g.,

$$\boldsymbol{I}_{Ci} = \mathbf{diag}\,[I_1 \; I_2 \; I_3] = \mathbf{const} \; . \tag{2.4}$$

The remaining diagonal elements I_υ are called principal moments of inertia with reference to C_i and the corresponding axes are the principal inertia axes. Both quantities follow from the eigenvalue problem

$$(I_\upsilon \boldsymbol{E} - \boldsymbol{I}^i_{Ci})\boldsymbol{x}_\upsilon = \boldsymbol{0} \; . \tag{2.5}$$

Thus, the principal moments of inertia are the eigenvalues of the matrix $\boldsymbol{I}^i_{Ci}$ and the eigenvectors $\boldsymbol{x}_\upsilon = \boldsymbol{e}^i_\upsilon$ define the principal inertia axes which have to be unit vectors $\boldsymbol{x}^{\mathrm{T}}_\upsilon \boldsymbol{x}_\upsilon = 1$.

2.2 Newton-Euler Equations

The synthetic method is based on the laws of Newton (1687) and Euler (1758) relating the translational motion represented by the momentum $\boldsymbol{p}$ of a body K to the sum of the external forces $\boldsymbol{f}$ and the rotational motion represented by the moment of momentum $\boldsymbol{h}_O$ to the sum of the external torques $\boldsymbol{l}_O$,

$$\frac{\mathrm{d}^I}{\mathrm{d}t}\boldsymbol{p} = \boldsymbol{f}, \quad \frac{\mathrm{d}^I}{\mathrm{d}t}\boldsymbol{h}_O = \boldsymbol{l}_O \; . \tag{2.6}$$

The time derivatives of the momentum $\boldsymbol{p}$ and the moment of momentum $\boldsymbol{h}_O$ have to be evaluated in the inertial frame I. The common reference point O of the moment of momentum and the resulting external torque may be an inertially fixed point like the origin of the inertial frame, $O \equiv O_I$, or the moving center of mass of the body, $O \equiv C$.

The fundamental laws (2.6) will now be applied to the rigid body K_i, $i = 1(1)p$, of a multibody system and appropriate frames are chosen. First of all, the bodies K_i are dismantled and the constraints are replaced by reaction forces acting then externally on the bodies involved in the same amount but with opposite sign according to the counteraction principle (action = reaction). Further, the center of mass is used as reference point for all bodies, $O \equiv C_i$.

In the inertial frame I momentum and moment of momentum for a rigid body K_i using the inertia properties m_i ,$\boldsymbol{I}_{Ci}$ read as

$$\boldsymbol{p}^I_i = m_i \boldsymbol{v}^I_{Ci}, \quad m_i = \text{const} \; , \tag{2.7}$$

$$\boldsymbol{h}_{Ci}^I = \boldsymbol{I}_{Ci}^I \boldsymbol{\omega}_i^I , \quad \boldsymbol{I}_{Ci}^I = \boldsymbol{I}_{Ci}^I(t) \ . \tag{2.8}$$

where $\boldsymbol{v}_{Ci}^I$ and $\boldsymbol{\omega}_i^I$ mean absolute velocities. Introducing (2.7) and (2.8) in (2.6) and omitting the index C one finally gets Newton's and Euler's equations

$$m_i \dot{\boldsymbol{v}}_i^I \qquad = \boldsymbol{f}_i^I , \qquad m_i = \text{const} \ , \tag{2.9}$$

$$\boldsymbol{I}_i^I \dot{\boldsymbol{\omega}}_i^I + \tilde{\boldsymbol{\omega}}_i^I \boldsymbol{I}_i^I \boldsymbol{\omega}_i^I \quad = \boldsymbol{l}_i^I \ , \qquad \boldsymbol{I}_i^I = \boldsymbol{I}_i^I(t) \ . \tag{2.10}$$

In a second step these equations are transformed in a body-fixed frame resulting in

$$m_i \dot{\boldsymbol{v}}_i^i \qquad = \boldsymbol{f}_i^i , \qquad m_i = \text{const} \ , \tag{2.11}$$

$$\boldsymbol{I}_i^i \dot{\boldsymbol{\omega}}_i^i + \tilde{\boldsymbol{\omega}}_i^i \boldsymbol{I}_i^i \boldsymbol{\omega}_i^i \quad = \boldsymbol{l}_i^i \ , \qquad \boldsymbol{I}_i^i = \textbf{const} \ . \tag{2.12}$$

Equations (2.9) and (2.10), and (2.11) and (2.12) look completely identical. If there is only one body like in gyrodynamics, then, (2.12) is preferable due to the time-invariance of the inertia tensor. In multibody dynamics, however, this advantage is fading.

Equation (2.12) is also known as Euler's equation of gyrodynamics. It can be found with the moment of momentum given in the body-fixed frame from (2.6) directly using the law of differentiation in a rotating frame (1.27),

$$\frac{\mathrm{d}^I}{\mathrm{d}t} \boldsymbol{h}_i^i = \dot{\boldsymbol{h}}_i^i + \tilde{\boldsymbol{\omega}}_i^i \boldsymbol{h}_i^i = \boldsymbol{l}_i^i \ , \quad \boldsymbol{h}_i^i = \boldsymbol{I}_i^i \boldsymbol{\omega}_i^i \ . \tag{2.13}$$

Finally, in an arbitrarily moving reference frame R Newton's and Euler's equations are also available, see e.g. Schiehlen and Eberhard (2004),

$$m_i \ddot{\boldsymbol{r}}_{Ri} + m_i \left[\overset{**}{\boldsymbol{r}}_R + \dot{\tilde{\boldsymbol{\omega}}}_R + \tilde{\boldsymbol{\omega}}_R \tilde{\boldsymbol{\omega}}_R) \boldsymbol{r}_{Ri} + 2\tilde{\boldsymbol{\omega}}_R \dot{\boldsymbol{r}}_{Ri} \right] = \boldsymbol{f}_i \ , \tag{2.14}$$

$$\begin{aligned} &\boldsymbol{I}_i \dot{\boldsymbol{\omega}}_{Ri} + \tilde{\boldsymbol{\omega}}_{Ri} \boldsymbol{I}_i \boldsymbol{\omega}_{Ri} \\ &+ [\boldsymbol{I}_i \boldsymbol{\omega}_R + \tilde{\boldsymbol{\omega}}_R \boldsymbol{I}_i \boldsymbol{\omega}_R + \tilde{\boldsymbol{\omega}}_R \boldsymbol{\omega}_{Ri} sp \boldsymbol{I}_i + 2\tilde{\boldsymbol{\omega}}_{Ri} \boldsymbol{I}_i \boldsymbol{\omega}_R] = \boldsymbol{l}_i \end{aligned} \tag{2.15}$$

Now, the coordinates of all vectors and tensors are related to the reference frame R where $\overset{**}{\boldsymbol{r}}_R$ means that the second time derivation has to be made before in the inertial frame. As a matter of fact, a large number of additional inertia forces and torques appear due to the relative motion.

The Newton-Euler equations represent a set of $6p$ scalar equations for $6p$ unknowns which are composed of unknown velocity and position variables and unknown reaction forces and torques. In an unconstrained system reactions do not exit, i.e., there are $6p$ ordinary differential equations (ODEs) to be solved. In a completely constrained system motion does not occur at all, i.e., altogether $6p$ algebraical equations have to be solved. In vehicle dynamics, due to a certain number of constraints between the bodies, motions and reactions appear featuring a set of differential-algebraical equations (DAEs). However, by the principles of dynamics, a minimal set of f ODEs can be found facilitating the solution and simulation of the problem.

2.3 Principles of d'Alembert and Jourdain

Equations of motion represent a minimal set of ordinary differential equations (ODEs). They can be found from the Newton-Euler equations by elimination of the reaction forces and torques. This is achieved computationally efficient by the principles of dynamics considering the virtual work of a constrained multibody system. For this purpose the external forces acting on the dismantled bodies of the system are subdivided into applied forces $\boldsymbol{f}_i^{(a)}$ and torques $\boldsymbol{l}_i^{(a)}$ as well as reaction forces $\boldsymbol{f}_i^{(r)}$ and torques $\boldsymbol{l}_i^{(r)}$. The latter ones do not contribute to the virtual work of the system

$$\delta W^r = \sum_{i=1}^{p} (\boldsymbol{f}_i^{(r)\mathrm{T}} \delta \boldsymbol{r}_i + \boldsymbol{l}_i^{(r)\mathrm{T}} \delta \boldsymbol{s}_i) = 0 \, , \tag{2.16}$$

where the virtual motions $\delta \boldsymbol{r}_i \, , \delta \boldsymbol{s}_i$ are known from (1.46). Equation (2.16) can be interpreted as a generalized orthogonality condition. For this purpose, in addition to the generalized coordinates $\boldsymbol{y}$, generalized reaction forces $g_j \, , \; j = 1\,(1)\,q$, are introduced and summarized in a $q \times 1$-vector as

$$\boldsymbol{g}(t) = [g_1 \ldots\ldots\ldots g_q]^{\mathrm{T}} \; . \tag{2.17}$$

The number of generalized constraint forces is determined by the number q of constraints. The local constraint forces and torques follow from the implicit constraint equations (1.33) as

$$\boldsymbol{f}_i^{(r)\mathrm{T}} = \sum_{j=1}^{q} g_j \frac{\partial \varphi_j}{\partial \boldsymbol{r}_i^{\mathrm{T}}} = \sum_{j=1}^{q} g_j \frac{\partial \varphi_j}{\partial \boldsymbol{x}^{\mathrm{T}}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}_i^{\mathrm{T}}} = \boldsymbol{g}^{\mathrm{T}} \boldsymbol{F}_{Ti}^{\mathrm{T}} \, , \tag{2.18}$$

$$\boldsymbol{l}_i^{(r)\mathrm{T}} = \sum_{j=1}^{q} g_j \frac{\partial \varphi_j}{\partial \boldsymbol{s}_i^{\mathrm{T}}} = \sum_{j=1}^{q} g_j \frac{\partial \varphi_j}{\partial \boldsymbol{x}^{\mathrm{T}}} \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{s}_i^{\mathrm{T}}} = \boldsymbol{g}^{\mathrm{T}} \boldsymbol{F}_{Ri}^{\mathrm{T}} \, , \quad i = 1(1)p \; . \tag{2.19}$$

In matrix notation the $3 \times q$- Jacobians $\boldsymbol{F}_{Ti} \, , \boldsymbol{F}_{Ri}$ are found from (2.18) and (2.19), and the condition (2.16) is rewritten as

$$\begin{aligned} \delta W^r &= \boldsymbol{g}^{\mathrm{T}} \sum_{i=1}^{p} (\boldsymbol{F}_{Ti}^{\mathrm{T}} \, \delta \boldsymbol{r}_i + \boldsymbol{F}_{Ri}^{\mathrm{T}} \, \delta \boldsymbol{s}_i) \\ &= \boldsymbol{g}^{\mathrm{T}} \sum_{i=1}^{p} (\boldsymbol{F}_{Ti}^{\mathrm{T}} \boldsymbol{J}_{Ti} + \boldsymbol{F}_{Ri}^{\mathrm{T}} \boldsymbol{J}_{Ri}) \, \delta \boldsymbol{y} = 0 \; . \end{aligned} \tag{2.20}$$

Finally, the global $6p \times q$- distribution matrix $\overline{\boldsymbol{Q}}$, and the global $6p \times f$- Jabobian matrix $\overline{\boldsymbol{J}}$ are introduced

$$\begin{aligned} \overline{\boldsymbol{Q}} &= [\boldsymbol{F}_{T1}^{\mathrm{T}} \ldots\ldots\ldots \boldsymbol{F}_{Tp}^{\mathrm{T}} \, \boldsymbol{F}_{R1}^{\mathrm{T}} \ldots\ldots\ldots \boldsymbol{F}_{Rp}^{\mathrm{T}}]^{\mathrm{T}} \, , \\ \overline{\boldsymbol{J}} &= [\boldsymbol{J}_{T1}^{\mathrm{T}} \ldots\ldots\ldots \boldsymbol{J}_{Tp}^{\mathrm{T}} \, \boldsymbol{J}_{R1}^{\mathrm{T}} \ldots\ldots\ldots \boldsymbol{J}_{Rp}^{\mathrm{T}}]^{\mathrm{T}} \; . \end{aligned} \tag{2.21}$$

Then, one gets from (2.20) simply

$$\overline{\boldsymbol{Q}}^{\mathrm{T}} \overline{\boldsymbol{J}} = \overline{\boldsymbol{J}}^{\mathrm{T}} \overline{\boldsymbol{Q}} = \boldsymbol{0} \, , \tag{2.22}$$

what clearly shows the generalized orthogonality between motion and constraint. The orthogonality condition or the vanishing virtual work, respectively, is independent from the coordinates chosen, and it is valid for all constrained mechanical systems.

D'Alembert's principle (1743) follows now from the Newton-Euler equations (2.9) and (2.10) after subdividing the external forces

$$\boldsymbol{f}_i = \boldsymbol{f}_i^{(e)} + \boldsymbol{f}_i^{(r)}, \quad \boldsymbol{l}_i = \boldsymbol{l}_i^{(e)} + \boldsymbol{l}_i^{(r)}, \tag{2.23}$$

and considering the orthogonality (2.16) as

$$\sum_{i=1}^{p} [(m_i \dot{\boldsymbol{v}}_{\boldsymbol{i}} - \boldsymbol{f}_i^{(e)})^{\mathrm{T}} \delta \boldsymbol{r}_i + (\boldsymbol{I}_i \dot{\boldsymbol{\omega}}_i + \tilde{\boldsymbol{\omega}}_i I_i \boldsymbol{\omega}_i - \boldsymbol{l}_i^{(e)})^{\mathrm{T}} \delta \boldsymbol{s}_i] = 0 \; . \tag{2.24}$$

Obviously, the reaction forces are eliminated in (2.24).

Analogously Jourdain's principle (1908) can be stated which is based on the fact that the virtual power of the reaction forces is vanishing, too,

$$\delta P^r = \sum_{i=1}^{p} [\boldsymbol{f}_i^{(r)\mathrm{T}} \delta' \boldsymbol{v}_i + \boldsymbol{l}_i^{(r)\mathrm{T}} \delta' \boldsymbol{\omega}_i] = 0 \; . \tag{2.25}$$

The virtual velocities $\delta' \boldsymbol{v}_i \, , \delta' \boldsymbol{\omega}_i$ are arbitrary, infinitesimal small variations of the velocities completely compatible with the constraints at any time and at any position. Thus, it yields

$$\delta' \boldsymbol{v}_i \neq 0 \; , \quad \delta' \boldsymbol{\omega}_i \neq 0 \; , \quad \delta' \boldsymbol{r}_i \equiv 0 \; , \quad \delta' \boldsymbol{s}_i \equiv 0 \; , \quad \delta' t \equiv 0 \; . \tag{2.26}$$

Moreover, the symbol δ' follows the rules of calculus. Then, it remains Jourdain's principle as

$$\sum_{i=1}^{p} [(m_i \dot{\boldsymbol{v}}_i - \boldsymbol{f}_i^{(e)})^{\mathrm{T}} \delta' \boldsymbol{v}_i + (\boldsymbol{I}_i \dot{\boldsymbol{\omega}}_i + \tilde{\boldsymbol{\omega}}_i \boldsymbol{I}_i \boldsymbol{\omega}_i - \boldsymbol{l}_i^{(e)})^{\mathrm{T}} \delta' \boldsymbol{\omega}_i] = 0 \; . \tag{2.27}$$

Similar to d'Alembert's principle all the reactions disappeared. However, the virtual displacements are replaced by the virtual velocities and the sometimes cumbersome evaluation of the virtual rotations is dropped. Further, Jourdain's principle handles nonlinear and nonholonomic constraints, too, which may appear in controlled vehicle systems.

In the American literature Jourdain's principle is referred to as Kane's equations and the virtual velocities are denoted as partial velocities, see Kane and Levinson (1985). Applying these principles for the generation of the equations of motion, the reactions have not to be considered at all. Therefore, the principles may be also classified as an analytical approach.

2.4 Energy Considerations and Lagrange's Equations

An alternative for the generation of the equations of motion is the analytical method by Lagrange (1788) based on energy considerations. The kinetic energy T of a rigid body reads as

$$T = \frac{1}{2} m \boldsymbol{v}_C^2 + \frac{1}{2} \boldsymbol{\omega} \, \boldsymbol{I}_C \, \boldsymbol{\omega} \; , \tag{2.28}$$

where the inertia properties $\{m, \boldsymbol{I}_C\}$ and the velocity properties $\{\boldsymbol{v}_C, \boldsymbol{\omega}\}$ are related to the center of mass. The kinetic energy is composed by the translational and rotational energy of the body, it is a scalar quantity which may be computed in different frames, too.

The kinetic energy of a multibody system consisting of the bodies K_i, $i = 1(1)p$, comprises the kinetic energy of all bodies as

$$T = \frac{1}{2}\sum_{i=1}^{p}[(\boldsymbol{v}_i^I)^{\mathrm{T}} m_i \boldsymbol{v}_i^I + (\boldsymbol{\omega}_i^I)^{\mathrm{T}} \boldsymbol{I}_i^I \boldsymbol{\omega}_i^I] \,, \tag{2.29}$$

written consistently in the inertial frame I and related to the center of mass C_i of each body K_i . If the work of the applied forces is independent of the path, then the forces have a potential U and it yields

$$\boldsymbol{f}^{(e)} = -\mathrm{grad}\, U \,, \tag{2.30}$$

where U is a scalar function of the position. Forces satisfying (2.30) are called conservative, they do not change the total energy of the system. In contrary, non-conservative forces change the total energy, they are called dissipative since the total energy is decreasing. Conservative forces may be due to gravity, $f_G = mg$, or elasticity, $f_F = -ks$. The corresponding potentials read as

$$U_G = m\,g\,z \,, \quad U_F = \frac{1}{2}ks^2 \,, \tag{2.31}$$

where z represents the vertical displacement of the center of mass of a body with mass m in the direction of gravity with acceleration g, and s means the displacement of an elastic spring with coefficient k. To the potentials a constant may be added, i.e., the origin of a potential can be arbitrarily chosen. The potential energy U of a multibody system is given by the sum of the body potentials $U = \sum U_j$. Multibody systems subject to conservative forces only are called conservative systems. For such systems it yields the energy conservation law

$$T + U = T_0 + U_0 = \mathrm{const} \,. \tag{2.32}$$

The energy conservation law may be derived from Newton's and Euler's law, i.e., it does not contain no new information. Its application is advantageous for conservative systems with one degree of freedom to evaluate a relation between the position and velocity variable. If there are two different positions known, the unknown velocity can be found from (2.32),

Based on energy expressions, the equations of motion of multibody systems may be found, too. This will be shown for multibody systems with holonomic constraints. In contrary to the synthetic method, the body of the system have not to be dismantled, the system is considered as a whole. For this purpose the generalized coordinates $\boldsymbol{y}$ are defined, and the position and the velocity variables (1.37), (1.38) and (1.39) are evaluated. As a result the kinetic energy is available as a function of $y_k\,(t)$ and $\dot{y}_k\,(t)$, $k = 1(1)f$,

$$T = T(y_k, \dot{y}_k) \,. \tag{2.33}$$

The applied forces and torques are projected in the direction of the generalized coordinates and composed to the generalized forces

$$q_k = \sum_{i=1}^{p} \left[\left(\frac{\partial \boldsymbol{r}_i^I}{\partial y_k} \right)^{\mathrm{T}} \boldsymbol{f}_i^{(e)I} + \left(\frac{\partial \boldsymbol{s}_i^I}{\partial y_k} \right)^{\mathrm{T}} \boldsymbol{l}_i^{(e)I} \right] , \quad k = 1(1)f , \tag{2.34}$$

where the rows of the Jacobian matrices (1.41) are used. The generalized forces may be also found by decomposition of the total work of the applied forces and torques

$$\delta W^e = \sum_{i=1}^{p} [\boldsymbol{f}_i^{(e)\mathrm{T}} \delta \boldsymbol{r}_i + \boldsymbol{l}_i^{(e)\mathrm{T}} \delta \boldsymbol{s}_i] = \sum_{k=1}^{f} q_k \delta y_k . \tag{2.35}$$

In any case the reaction forces and torques do not appear.

Now the Lagrangian equations of the second kind read as, see e.g. Magnus and Müller (1990),

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{y}_k} \right) - \frac{\partial T}{\partial y_k} = q_k , \quad k = 1(1)f . \tag{2.36}$$

For the evaluation of the equations of motion two partial and one total differentiations have to be performed with respect to one scalar function $T\,(y_k, \dot{y}_k)$. As a result the minimal number f of equations of motion is found. However, the reaction forces are completely lost and cannot be regained.

For conservative systems the generalized forces follow immediately from the potential

$$q_k = -\frac{\partial U}{\partial y_k} . \tag{2.37}$$

From (2.36) and (2.37) it remains

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{y}_k} \right) - \frac{\partial T}{\partial y_k} + \frac{\partial U}{\partial y_k} = 0 , \quad k = 1(1)f . \tag{2.38}$$

Introducing the Lagrange function $L = T - U$ also called the kinetic potential, then (2.38) is even more simplified and reads

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{y}_k} \right) - \frac{\partial L}{\partial y_k} = 0 , \quad L = T - U , \quad k = 1(1)f . \tag{2.39}$$

For some engineering applications it is advantageous to use surplus coordinates $\overline{y}_j$, $j = 1(1)f + r$, in addition to the f generalized coordinates y_k , $k = 1(1)f$. Then, there exist r geometric constraints between the surplus coordinates

$$\varphi_n = \varphi_n(\overline{y}_j) = 0 , \quad n = 1(1)r . \tag{2.40}$$

The equations of motion are now extended by r Lagrangian multipliers λ_n , $n = 1(1)r$, representing generalized constraint forces

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial T}{\partial \dot{\overline{y}}_j} \right) - \frac{\partial T}{\partial \overline{y}_j} = \overline{q}_j + \sum_{n=1}^{r} \lambda_n \frac{\partial \varphi_n}{\partial \overline{y}_j} , \quad j = 1(1)f + r . \tag{2.41}$$

Thus, a set of differential-algebraical equations remain.

3 Equations of Motion

In chapter 2 there has been presented two methods for the generation of the equations of motion, the synthetic method by Newton-Euler, and the analytic method by d'Alembert, Jourdain or Lagrange, respectively. The principal steps in the generation process by Newton-Euler and Lagrange are shown in Figure 9. Common starting point is the mechanical model of the vehicle composed by the elements of multibody systems. Common result are the equations of motion, they are identical with both methods if the same generalized coordinates are used. However, the effort is different. During the generation of the equations of motion using Lagrange's equations there appear terms in $\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{y}_i}\right)$ which are afterwards eliminated by $\frac{\partial T}{\partial y_i}$ according to (2.36). This means an useless computational effort which is not required with the Newton-Euler approach, see e.g. Schiehlen and Eberhard (2004). On the other hand in the Newton-Euler equations the reactions have to be eliminated. Thus, both of these approaches have disadvantages which are avoided by a combination of the Newton-Euler equations with the principles presented in Section 2.3. The resulting equations of motion are always ordinary differential equations (ODEs). However, their form depends on the type of the multibody system. There are ideal and non-ideal systems, the first ones are characterized by applied forces and torques independent from any reaction while the second ones show a such dependency. E.g., gravitational forces, spring and damper forces are independent from any reactions while sliding friction forces and slip dependent contact forces, regularly found with tires in vehicle dynamics, are a function of the normal or reaction forces, respectively.

Within the class of ideal systems, ordinary and general multibody systems are distinguished. Ordinary multibody systems are due to holonomic constraints and applied forces depending only on position and velocity quantities, they can be always represented by a system of differential equations of the second order. For nonholonomic constraints and/or general force laws one gets general multibody systems.

The equations of ordinary multibody systems read as

$$\boldsymbol{M}\left(\boldsymbol{y},t\right)\ddot{\boldsymbol{y}}\left(t\right)+\boldsymbol{k}\left(\boldsymbol{y},\dot{\boldsymbol{y}},t\right)=\boldsymbol{q}\left(\boldsymbol{y},\dot{\boldsymbol{y}},t\right)\ , \tag{3.1}$$

where $\boldsymbol{y}$ is the $f\times 1$-position vector of the generalized coordinates, $\boldsymbol{M}$ is the $f\times f$-symmetric inertia matrix, $\boldsymbol{k}$ is a $f\times 1$-vector of generalized gyroscopic forces including the Coriolis- and centrifugal forces as well as the gyroscopic torques, and the $f\times 1$-vector $\boldsymbol{q}$ represents generalized applied forces. The equations of motion resulting from the analytical method have always the form (3.1) while the synthetic method often requires some calculations to get a symmetric inertia matrix.

In vehicle dynamics the deviations $\tilde{\boldsymbol{y}}\left(t\right)$ from a reference motion $\boldsymbol{y}=\boldsymbol{y}_R\left(t\right)$ are often small,

$$\boldsymbol{y}(t)=\boldsymbol{y}_R(t)+\tilde{\boldsymbol{y}}(t)\ . \tag{3.2}$$

Then, it follows by a Taylor series expansion under assumption of differentiable vector functions, and skipping of the second and higher order terms from (3.1) the linearized

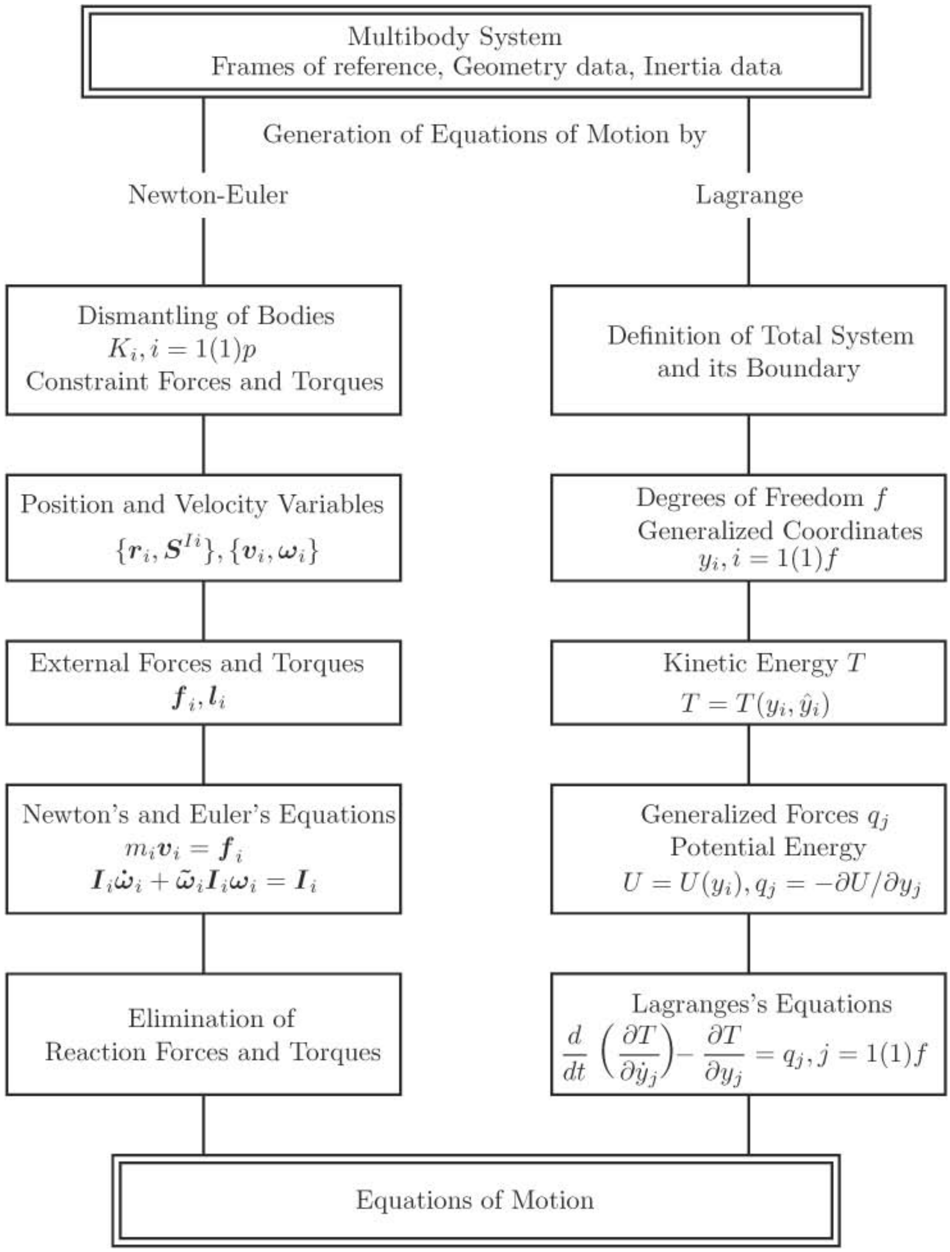

Figure 9. Generation of equations of motion by the methods of Newton-Euler and Lagrange

equations of motion, see also Müller and Schiehlen (1985),

$$\boldsymbol{M}(t)\ddot{\tilde{\boldsymbol{y}}}(t) + \boldsymbol{P}(t)\dot{\tilde{\boldsymbol{y}}}(t) + \boldsymbol{Q}(t)\tilde{\boldsymbol{y}}(t) = \boldsymbol{h}(t) \;, \tag{3.3}$$

where $\boldsymbol{M}(t)$ is the symmetric, positive definite inertia matrix while $\boldsymbol{P}(t)$ and $\boldsymbol{Q}(t)$ characterize the velocity and position dependent forces and the vector $\boldsymbol{h}(t)$ represents the external excitations. If all these matrices are time-invariant and subdivided in a symmetrical and skewsymmetrical part, then the equations of motion of a linear ordinary and time-invariant multibody system are found reading as

$$\boldsymbol{M}\ddot{\boldsymbol{y}}(t) + (\boldsymbol{D}+\boldsymbol{G})\dot{\boldsymbol{y}}(t) + (\boldsymbol{K}+\boldsymbol{N})\boldsymbol{y}(t) = \boldsymbol{h}(t) \;, \tag{3.4}$$

where $\tilde{\boldsymbol{y}}$ was simply replaced by $\boldsymbol{y}$ and the $f \times f$-matrices have the properties

$$\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}} > 0 \;, \quad \boldsymbol{D} = \boldsymbol{D}^{\mathrm{T}} \;, \quad \boldsymbol{G} = -\boldsymbol{G}^{\mathrm{T}} \;, \quad \boldsymbol{K} = \boldsymbol{K}^{\mathrm{T}} \;, \quad \boldsymbol{N} = -\boldsymbol{N}^{\mathrm{T}} \;. \tag{3.5}$$

These matrices have a physical meaning which can be identified after premultiplication of (3.5) from the left by $\dot{\boldsymbol{y}}^{\mathrm{T}}$ resulting in the total time derivative of an energy expression

$$\underbrace{\dot{\boldsymbol{y}}^{\mathrm{T}}\boldsymbol{M}\ddot{\boldsymbol{y}}} + \underbrace{\dot{\boldsymbol{y}}^{\mathrm{T}}\boldsymbol{D}\dot{\boldsymbol{y}}} + \underbrace{\dot{\boldsymbol{y}}^{\mathrm{T}}\boldsymbol{G}\dot{\boldsymbol{y}}} + \underbrace{\dot{\boldsymbol{y}}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{y}} + \underbrace{\dot{\boldsymbol{y}}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{y}} = \underbrace{\dot{\boldsymbol{y}}^{\mathrm{T}}\boldsymbol{h}} \;, \tag{3.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}T \;+\; 2R \;+\; 0 \;+\; \frac{\mathrm{d}}{\mathrm{d}t}U \;+\; 2S \;=\; P \;. \tag{3.7}$$

The inertia matrix $\boldsymbol{M}$ determines the kinetic energy $T = \frac{1}{2}\dot{\boldsymbol{y}}^{\mathrm{T}}\boldsymbol{M}\,\dot{\boldsymbol{y}}$ and therefore the inertia forces, from $T > 0$ it follows again the positive definiteness of the inertia matrix. The damping matrix $\boldsymbol{D}$ defines via Rayleigh's dissipation function $R = \frac{1}{2}\dot{\boldsymbol{y}}^{\mathrm{T}}\boldsymbol{D}\,\dot{\boldsymbol{y}}$ the damping forces while the gyro matrix $\boldsymbol{G}$ describes the gyroscopic forces which do not change the total energy of the system. The stiffness matrix determines the potential energy $U = \frac{1}{2}\boldsymbol{y}^{\mathrm{T}}\boldsymbol{K}\,\boldsymbol{y}$ and, therefore, the conservative position forces while the matrix $\boldsymbol{N}$ identifies the circulatory forces also known as nonconservative position forces. Furthermore, P describes the power of the external excitation forces. For $\boldsymbol{D} = \boldsymbol{0}$, $\boldsymbol{N} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$ the multibody system is conservative, i.e., the total energy is constant for all motions,

$$T + U = \text{const} \;. \tag{3.8}$$

The matrix properties (3.5) allow often to check the equations of motion with respect to the physical phenomena involved.

The equations of motion of ordinary multibody systems in nonlinear or linear form, Eqs. (3.1) or (3.3), respectively, are systems of differential equations of second order. For solution they have to be supplemented by the initial conditions for position and velocity,

$$\boldsymbol{y}(0) = \boldsymbol{y}_0 \;, \quad \dot{\boldsymbol{y}}(0) = \dot{\boldsymbol{y}}_0 \tag{3.9}$$

The state vector of the vehicle is defined by the position vector and the velocity vector as

$$\boldsymbol{x}_F(t) = \begin{bmatrix} \boldsymbol{y}(t) \\ \dot{\boldsymbol{y}}(t) \end{bmatrix} \;, \tag{3.10}$$

where $\boldsymbol{x}$ is the $n \times 1$ -vector of the state variables. For ordinary multibody systems, and therefore for vehicles, too, it holds $n = 2f$ where f means the number of degrees of freedom.

With the state vector (3.10) the equations of motion can be easily transfered into the corresponding state equations. In the nonlinear case it yields

$$\begin{aligned} \underbrace{\begin{matrix} \dot{\boldsymbol{y}}(t) \\ \ddot{\boldsymbol{y}}(t) \end{matrix}}_{\dot{\boldsymbol{x}}_F(t)} \begin{matrix} = \\ = \end{matrix} \underbrace{\begin{matrix} \dot{\boldsymbol{y}}(t) \\ \boldsymbol{M}^{-1}(\boldsymbol{y},\, t)\,[\boldsymbol{q}(\boldsymbol{y},\, \dot{\boldsymbol{y}},\, t) - \boldsymbol{k}(\boldsymbol{y},\, \dot{\boldsymbol{y}},\, t)] \end{matrix}}_{\boldsymbol{a}_F(\boldsymbol{x}_F,\, t)\,,} \end{aligned} \tag{3.11}$$

where a nonlinear $n \times 1$-vector function $\boldsymbol{a}_F$ appears. For vehicles with small linear motions it remains

$$\dot{\boldsymbol{x}}_F(t) = \boldsymbol{A}_F \boldsymbol{x}_F(t) + \boldsymbol{B}_F \boldsymbol{u}_F(t)\ . \tag{3.12}$$

where

$$\boldsymbol{A}_F = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{E} \\ -\boldsymbol{M}^{-1}(\boldsymbol{K} + \boldsymbol{N}) & -\boldsymbol{M}^{-1}(\boldsymbol{D} + \boldsymbol{G}) \end{bmatrix} \tag{3.13}$$

is the $n \times n$-system matrix, $\boldsymbol{B}_F$ the $n \times r$-input matrix and $\boldsymbol{u}_F$ the $r \times 1$-input vector of the excitations acting on the vehicle.

For general multibody systems exist a broader variety of standard representations which will not be discussed in detail. General multibody systems can be uniquely represented by the state equations (3.11) or (3.12), too. However the special form of the system matrix A is no longer found.

4 Formalisms for Multibody Systems

The generation of equations of motion for large multibody systems is a nontrivial task requiring numerous steps during the evaluation of the fundamental relations. Beginning with the space age in the middle of the 1960s the generation of equations of motion was more formalized. The resulting formalisms were used for the development of computer codes for multibody systems, they are the basis of computational multibody dynamics. Twenty-five years later, in 1990, there were known 20 formalisms described in the Multibody System Handbook (Schiehlen, 1990). Many of them are used today.

Multibody system formalisms are based on Newton-Euler equations or Lagrange's equations, respectively, as described in Chapter 2 and 3. Regarding the computational procedure, numerical and symbolical formalisms are distinguished. Numerical formalisms supply the elements of the matrices as numbers in the case of linear time-invariant multibody systems (3.4). In the case of linear time-variant systems (3.3) and nonlinear systems (3.1) a numerical formalism provides the numbers in the equations of motion necessary for each time step required by the simulation programme. In contrary, symbolical formalisms generate the equations of motion only once with the computer how it is done with paper and pencil. The advantage is that variations of the system parameters and, for time-variant systems, the current time have to be inserted in the symbolical equations of motion only. Symbolical formalisms are especially helpful for optimizations and control design.