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International Centre for Mechanical Sciences

Reduced-Order Modelling for Flow Control

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 528

REDUCED-ORDER MODELLING FOR FLOW CONTROL

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FOREWORD

Practical interests in flow control have no longer to be demonstrated. Flow control has motivated rapid developments in the past two decades in experiments, flow stability theory and computational fluid dynamics (CFD). Recent advances in experimental studies include applications of more and more sophisticated actuators and sensors. However, up to now, most of the results are predominantly related to open loop, at most, adaptive approaches. Early closed-loop applications of control methods were in noise control based on anti-noise concepts. These studies established the pioneering link between fluid mechanics and control theory. However, in most aerodynamic applications, turbulent flows are encountered. Due to the intrinsic nonlinearities, turbulence gives rise to a large variety of temporal and spatial scales of more or less organized nature. Turbulence has remained one of the last not satisfactorily resolved physical phenomenon of practical importance in engineering sciences. It is obvious that the complexity of these flows is so pronounced that simpler – if this term can be used for turbulent flows – descriptions need to be derived.

The encountered complexity is observed at three levels. First, the characterization of the flow itself is complex and depends on the type of available information (e.g. sensors). Any state information is by nature incomplete or of excessive extent for turbulent flows. Second, the effect of any actuator is by nature 3D and unsteady, thus difficult to characterize. Third, the complete modelling of the flow (CFD), its sensitivity to perturbations, etc. exceeds available computer power by many orders of magnitudes, particularly for online capability in experiment. In the same vein, the predominantly (locally) linear approaches of control theory need to be adapted to the reality of the complex, turbulent flow characteristics. This leads to different levels of fluid mechanics one has to take into account. These levels can start from a detailed fluid mechanics characterization, including the more or less organized nature of the turbulent flows, the so called white-box model and end with entirely black and eventually 'empty-box' models.

The communities of flow control, applied mathematics and turbulence in fluids have then to work altogether in a close manner. Each domain enriches the other for the dedicated goal of controlling different types of flows. No currently available approach can be re-

tained, due to the variability of the physics to be controlled (e.g. flow separation, drag, lift, mixing, noise generation and fluid structure interactions). The present volume is written by leading experts of flow control and represents the state of the art of the different approaches whose complementarity will open new areas by mutual fertilization.

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PREFACE

Active turbulence control is a rapidly evolving new field of fluid dynamics with large industrial importance. Several developments serve as catalyzers. Actuators and sensors have become increasingly more powerful, cheaper and more reliable to be considered for practical applications. In fact, aeronautic, car and other transport-related industries work at active turbulence control solutions for selected demonstrators. Examples are the high-lift configuration of an airfoil or drag reduction of cars. The past stigma of active control as proof of a inferior aerodynamic design is replaced by the realization that active control is a critical enabler for future performance enhancements. Not much phantasy is required to envision a not-too-far future in which active control will be commonly seen on cars, trains, airplanes, helicopters, wind energy plants, air-conditioning systems, and virtually all flow related products. Active turbulence control is having an impact of epic proportions.

Active control requires at minimum parameter adjustments for flow conditions and occasionally in-time response using flow sensors. Hence, active control generally requires — or at least benefits — from a closed-loop scheme for optimal performance. Closed-loop control has clearly been demonstrated to be superior to (blind) open-loop control in many cases. Performance of closed-loop control does not only depend on the chosen actuators and sensors. It critically depends also on the control logic with its underlying model.

Model development and control design for closed-loop flow control is the focus of this book. Wiener (1948) discriminates between black-, grey-, and white-box models. The black-box models identify the dynamics between the input (actuation) and the output (sensing) from data — ignoring any other aspect of the flow. The white boxes represent the full-state representation, here: Navier-Stokes discretizations. And the grey boxes resolve a small yet relevant portion of the full state dynamics, here: the evolution of coherent structures. All models have their relative merits and shortcomings. Black-box models represent the behavior of experiments with accessible accuracy. On the downside, physical understanding of coherent structures and associated nonlinearities is discarded. Navier-Stokes discretizations are accurate representations of the flow but come with a large com-

putational load. This computational expense is a large challenge for control design and too large for any foreseeable operation in experiments. Reduced-order models for the coherent-structures are a good compromise between required resolution and necessary simplicity for online-capability in experiment. Their price is a large experience in model development. For later reference, we add to Wiener's classification model-free approaches (or 'empty boxes') which make only qualitative assumptions about the dynamics.

The authors describe the current state on closed-loop flow control from various, necessarily biased experimental and computational angles. In particular, we have attempted to provide a book with elementary self-consistent descriptions of the main methods. Thus, our book may serve also as guide through the large jungle of myriad of publications in the field. Topics include the complete span of flow control based on white-box models (first two chapters), grey-box models (second two chapters) and black- to empty-box models (final two chapters):

These lecture notes originate from a course held at the Centre International des Sciences Mécqniques (CISM) in Udine, Italy in September 2008. The Editors thank Prof. W. Schneider for the kind invitation to this course. We thank the CISM staff and the Rector Prof. G. Maier for dependable, professional support in all technical aspects of the course. The beautiful city of Udine, the cooperating late-summer weather, and the magnificent Palazzo del Torso provided the perfect forum for many memorable interactions during class-room time, breakfast, lunch and dinner. We thank the authors for their excellent lectures and equally illuminating chapters. Each chapter condenses a long-term research and teaching effort of the corresponding authors. We thank the participants for coming with large curiosity and penetrating questions, making our course a lively worthwhile event.

Poitiers, Poznań and Boston in February 2010
Bernd Noack, Marek Morzyński and Gilead Tadmor
on behalf of the whole co-author team

CONTENTS

Flow control and constrained optimization problems

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Abstract Constrained optimization is presented as a key enabler for answering numerous important questions in the heart of flow control. These problems range from the extraction of Proper Orthogonal Decomposition modes and tools from linear control theory to optimal control which can be applied to any type of non-linear systems. The determination of optimal growth disturbances is presented as a particular case of constrained optimization. The chapter shall provide a complete description for deriving analytically and solving numerically any specific formulation of constrained optimization.

1 Introduction

The objective of this chapter is to present within the unified framework of constrained optimization problems, different numerical tools which change completely our ideas on flow control in the last decade. We will see in particular that reduced-order modeling based on Proper Orthogonal Decomposition modes (see the contribution by B. Noack et al. in this book), as well as classical techniques of linear control (Linear Quadratic Regulator and Linear Quadratic Gaussian methods) and optimal control, have in common the resolution of a constrained optimization problem. Beyond that, we will also show that the concept of optimal disturbances, introduced in stability theory to explain the transition to turbulence of linearly stable flows, can be also formulated as a constrained optimization problem and, if needed, be solved simultaneously to a control problem. Lastly, we will highlight that inverse methods (model identification or parameter estimation) can be interpreted as a particular constrained optimization problem. The objective is to give the possibility to the interested reader of rapidly developing by him/her-self the analytical and numerical solutions to the constrained optimization problem of his/her interest. The choice was thus made to detail as much as possible the different stages.

The current chapter is organized as follows: In section 2.1, we introduce the issues of flow control and present, for facilitating future discussions, the different actors on the control scene. Then we introduce the linearized framework, often used in flow control, and finish by formulating a series of questions related directly to different aspects of flow control. In section 2.2, we give some essential elements of linear control theory and continue in section 2.3 by an introduction of model reduction seen under the specific angle of projection methods. In section 3, we focus on the fundamental aspects of optimal control theory. At this stage, the presentation will remain very similar to what can be found in Gunzburger (1997a) and more recently in Gunzburger (2003). Section 4 considers the case of LQR control for a generic system and shows that the solution of a high-dimensional Riccati differential equation is necessary to determine the feedback control law that minimizes the value of the cost function. Section 5 highlights that the determination of optimal disturbances corresponds to a constrained optimization problem for which the control is the initial condition of the dynamical system. Lastly, sections 6 and 7 consider the case where the constraint corresponds to a time-dependent partial differential equation, linear and nonlinear respectively. Section 7 finishes with some numerical results of optimal control for the Burgers equation.

2 Elements of control theory and model reduction

2.1 Flow control

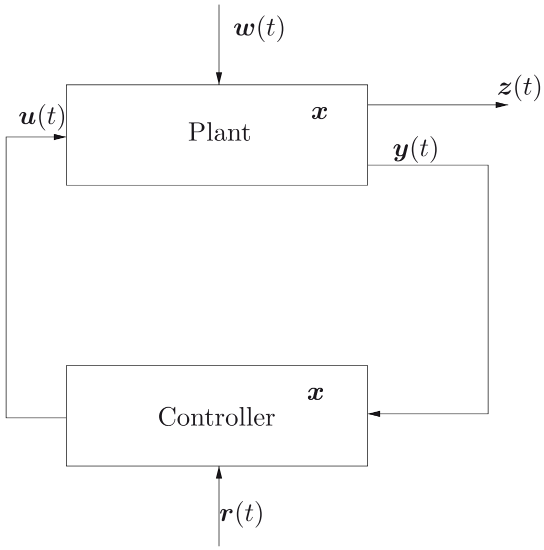
First, in section 2.1.1, we give the scope of flow control and introduce the terminology necessary to present constrained optimization problems as a main topic in modern fluid mechanics. Then, in section 2.1.2, we introduce the linearized framework used in linear control theory. Finally, in section 2.1.3, we list different types of problems which can appear within the framework of flow control while insisting on their similarity.

2.1.1 Scope and objectives of flow control

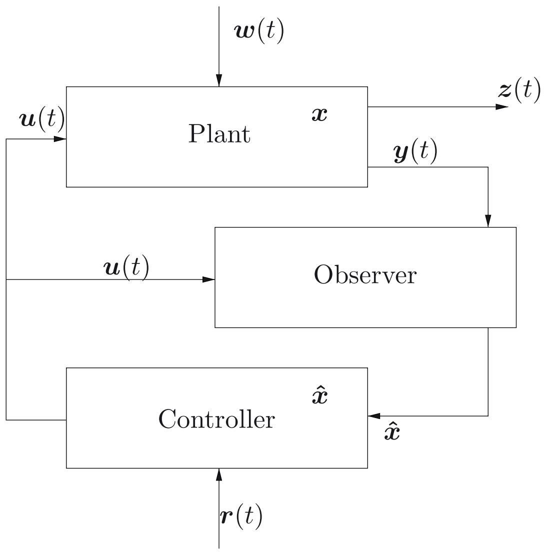
2.1.1.1 General points The goal of a flow control system is to achieve some desired objective by manipulating properly the flow configuration (physical properties, volume forcing or boundary conditions). Based on the type of actuation, either *passive* (no energy expenditure) or *active*, and on the means by which the control evolves in response to changes in the flow, *open-loop* or *closed-loop*, different strategies can be considered (see Gad-el-Hak, 2000, for a discussion on this classification). By nature, passive control strategies are similar to shape optimization. Determining the shape that a surface of revolution must have to offer the least resistance to the motion goes back to Newton (end of 17^{th} century) and involved the invention of the calculus of variations. We will see in section 3.1.1 that this question can be formalized as a constrained optimization problem by simply modifying the space on which the solutions are required. In open-loop, the parameters of the actuators are set once for all at the design stage and remain constant throughout the optimization procedure whatever the changes undergone by the flow. With this type of strategy, the sensitivity of the system to external disturbances or to error modeling (change in the parameters of the system) is then important. In addition, stabilizing an unstable solution - what may sometimes be interesting from a point of view of the performances - becomes difficult. For these reasons, we will consider throughout this chapter the case of closed-loop control or feedback control where there exist sensors for measuring at least partially the effects of the control on the system.

2.1.1.2 Terminology In the control literature[1], the mathematical model of the system to be controlled is called *plant*. In general, this model only approximates the behavior of the physical system. We will go back to this point and to the consequences in terms of optimization in section 2.3. The corresponding *state variables* of the plant is noted $\boldsymbol{x}$. The objective of a control system is to make the *reference output* $\boldsymbol{z}$ behave in a desired way by manipulating the *plant input* $\boldsymbol{u}$ (see Fig. 1). The *reference input* $\boldsymbol{r}$ specifies the desired behavior of the reference output. In feedback flow control the *measured plant output* $\boldsymbol{y}$ is fed back into the controller for determining the control. Compared to $\boldsymbol{u}$, the *disturbance input* $\boldsymbol{w}$ consists of those inputs to the plant that are generated by the environment. It includes one contribution coming from the state disturbances $\boldsymbol{w_1}$ and another contribution coming from the measurement noise $\boldsymbol{w_2}$. In the idealized case called *full-state configuration* (see Fig. 1(a)), the entire state $\boldsymbol{x}$ is assumed to be available for the controller. In the general case called *observer-based*

[1] Here, and in the rest of the chapter, we decide to use the standard notations in textbooks of control theory to familiarize the reader coming from fluid mechanics to these notations. Then, otherwise stated, $\boldsymbol{u}$ denotes the control and not a velocity field. Moreover, quantities expressed in boldface correspond to vector quantities.



(a) Full state configuration.



(b) Observer-based configuration.

Figure 1. Typical block diagrams for feedback control.

configuration (see Fig. 1(b)), the plant states that are not measured directly is estimated by an observer. Thereafter, all the quantities with a hat correspond to estimated variables: for instance, $\hat{\boldsymbol{x}}$ are estimated states.

2.1.1.3 Plant modelling The next stage is the determination of the system of equations for the plant (Fig. 2). Starting from a physical system and some measured data, the modelling phase consists of deriving a set of Partial Differential Equations (PDEs) or Ordinary Differential Equations (ODEs). In the first case, after discretization in space of the PDEs with any numerical method (finite element, finite volume, ...), a set of ODEs is obtained. Sometimes the ODEs are discretized in time as well, yielding discrete-time dynamical systems. Here, to simplify the presentation, we will concentrate on continuous-time systems. Finally, since any dynamical system can be reduced to a first-order system of differential equations by changing the set of variables, we obtain a non-linear *state space model* given by

$$\mathcal{S} : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)), \\ \boldsymbol{z}(t) = \boldsymbol{h}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)), \\ \boldsymbol{y}(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \boldsymbol{u}(t), \boldsymbol{w}(t)), \end{cases}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^{n_x}$, $\boldsymbol{u} \in \mathbb{R}^{n_u}$, $\boldsymbol{w} \in \mathbb{R}^{n_w}$, $\boldsymbol{y}(t) \in \mathbb{R}^{n_y}$ and $\boldsymbol{z} \in \mathbb{R}^{n_z}$. The non-linear functions $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{h}$ are defined accordingly.

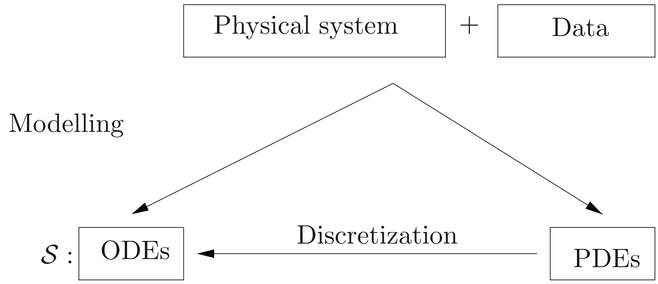


Figure 2. Broad framework of the determination of the plant equations (after Antoulas, 2005).

2.1.2 Linearized framework

Often, in practice, the non-linear system $\boldsymbol{f}$ is linearized around an operating condition of interest. To simplify the future notations, we will assume that the system does not depend explicitly on time and suppress for the moment the dependance on the external disturbance $\boldsymbol{w}$ writing for the plant equations $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t))$. Depending on the applications, this operating condition can be a particular solution of the unsteady dynamical system $\boldsymbol{f}$, that is to say $\boldsymbol{f}(\boldsymbol{x}_e(t), \boldsymbol{u}_e(t)) = \boldsymbol{0}$ or an equilibrium point of $\boldsymbol{f}$ characterized by $\dot{\boldsymbol{x}}_e = \boldsymbol{f}(\boldsymbol{x}_e, \boldsymbol{u}_e) = \boldsymbol{0}$. In the domain of flow instabilities, this equilibrium point corresponds to a steady solution of the Navier-Stokes equations.

We then introduce the first-order perturbations $\tilde{\boldsymbol{x}}(t)$ and $\tilde{\boldsymbol{u}}(t)$ such that

$$\boldsymbol{x}(t) = \boldsymbol{x}_e(t) + \tilde{\boldsymbol{x}}(t) \qquad \text{and} \qquad \boldsymbol{u}(t) = \boldsymbol{u}_e(t) + \tilde{\boldsymbol{u}}(t).$$

Expanding $\boldsymbol{f}$ in a Taylor series about $(\boldsymbol{x}_e, \boldsymbol{u}_e)$, we obtain

$$\begin{aligned}\dot{\boldsymbol{x}}_e(t) + \dot{\tilde{\boldsymbol{x}}}(t) =& \boldsymbol{f}(\boldsymbol{x}_e(t), \boldsymbol{u}_e(t)) + J_x(\boldsymbol{x}_e(t), \boldsymbol{u}_e(t))\tilde{\boldsymbol{x}}(t) + J_u(\boldsymbol{x}_e(t), \boldsymbol{u}_e(t))\tilde{\boldsymbol{u}}(t) \\ &+ \text{higher order terms}\end{aligned}$$

where J_x (respectively J_u) is the Jacobian matrix of $\boldsymbol{f}$ with respect to $\boldsymbol{x}$ (respectively $\boldsymbol{u}$):

$$(J_x)_{ij} = \frac{\partial f_i}{\partial x_j} \quad \text{with} \quad 1 \leq i \leq n_x \quad ; \quad 1 \leq j \leq n_x$$

and

$$(J_u)_{ij} = \frac{\partial f_i}{\partial u_j} \quad \text{with} \quad 1 \leq i \leq n_x \quad ; \quad 1 \leq j \leq n_u.$$

Neglecting the higher order terms and letting

$$A(t) = J_x(\boldsymbol{x}_e(t), \boldsymbol{u}_e(t)) \quad \text{and} \quad B(t) = J_u(\boldsymbol{x}_e(t), \boldsymbol{u}_e(t))$$

we obtain the linearized state space model

$$\dot{\tilde{\boldsymbol{x}}}(t) = A(t)\tilde{\boldsymbol{x}}(t) + B(t)\tilde{\boldsymbol{u}}(t)$$

where $A(t) \in \mathbb{R}^{n_x \times n_x}$ is the state matrix and $B(t) \in \mathbb{R}^{n_x \times n_u}$ is the input matrix.

Similarly, the nonlinear functions $\boldsymbol{z} = \boldsymbol{h}(\boldsymbol{x}, \boldsymbol{u})$ and $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}, \boldsymbol{u})$ may be linearized around the equilibrium point, resulting in a linear, parameter time-varying (LPTV) system given by

$$\mathcal{S}_{LPTV} : \begin{cases} \dot{\boldsymbol{x}}(t) = A(t)\boldsymbol{x}(t) + B(t)\boldsymbol{u}(t), \\ \boldsymbol{z}(t) = C_1(t)\boldsymbol{x}(t) + D_1(t)\boldsymbol{u}(t), \\ \boldsymbol{y}(t) = C_2(t)\boldsymbol{x}(t) + D_2(t)\boldsymbol{u}(t), \end{cases}$$

where for convenience the notation of the fluctuations was removed.

The state model $\mathcal{S}_{LPTV}$ can be further simplified when the system is time-invariant. Adding the linearized contribution from the external disturbances, the system becomes

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B_1\boldsymbol{w}(t) + B_2(t)\boldsymbol{u}(t),\\ \boldsymbol{z}(t) &= C_1\boldsymbol{x}(t) + D_{11}\boldsymbol{w}(t) + D_{12}\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= C_2\boldsymbol{x}(t) + D_{21}\boldsymbol{w}(t) + D_{22}\boldsymbol{u}(t).\end{aligned}$$

This is the more general class of model that we can consider for linear-time invariant (LTI) systems. Throughout this chapter, we will restrict our attention to the simplified[2] linear system

$$\mathcal{S}_{LTI}: \begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B\boldsymbol{u}(t),\\ \boldsymbol{z}(t) = C_1\boldsymbol{x}(t) + D_1\boldsymbol{u}(t),\\ \boldsymbol{y}(t) = C_2\boldsymbol{x}(t) + D_2\boldsymbol{u}(t), \end{cases} \tag{1}$$

where $C_1 \in \mathbb{R}^{n_z \times n_x}$ and $C_2 \in \mathbb{R}^{n_y \times n_x}$ are the output matrices and where $D_1 \in \mathbb{R}^{n_z \times n_u}$ and $D_2 \in \mathbb{R}^{n_y \times n_u}$ are the input to output coupling matrices. A dynamical system with single input ($n_u = 1$) and single output ($n_y = 1$) is called a SISO (single input and single output) system, otherwise it is called MIMO (multiple input and multiple output) system. When this is not necessary, we will not mention the variable $\boldsymbol{z}$ thereafter.

The advantage of linear systems is that the state, solution of (1), can be found explicitly from the input and the initial conditions (see Zhou et al., 1996):

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{x}(0) + \int_0^t e^{A(t-\tau)}B\boldsymbol{u}(\tau)\ \mathrm{d}\tau$$

where the matrix exponential is defined by the power series:

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots$$

The reference and measured plant outputs are then generated as a function of the initial conditions and the input:

$$\boldsymbol{z}(t) = C_1e^{At}\boldsymbol{x}(0) + \int_0^t C_1e^{A(t-\tau)}B\boldsymbol{u}(\tau)\ \mathrm{d}\tau + D_1\boldsymbol{u}(t)$$

and

$$\boldsymbol{y}(t) = C_2e^{At}\boldsymbol{x}(0) + \int_0^t C_1e^{A(t-\tau)}B\boldsymbol{u}(\tau)\ \mathrm{d}\tau + D_2\boldsymbol{u}(t).$$

We will see in section 2.2.2 the consequences in terms of observability and controllability of the system $\mathcal{S}$.

[2] $B_1 = D_{11} = D_{21} = 0$, $B \triangleq B_2$, $D_1 \triangleq D_{12}$ and $D_2 \triangleq D_{22}$.

2.1.3 Different types of problems

Within the general framework of flow control, various types of problems can be considered:

Problem 1: How to determine the control law $\boldsymbol{u}$ to apply to the dynamical system $\mathcal{S}$ to minimize a given norm[3] of $\boldsymbol{z}$?

In lack of particular assumption on the model, this problem is designated as optimal control. The model $\boldsymbol{f}$ can then be a Direct Numerical Simulation (Bewley et al., 2001), a Large Eddy Simulation (El Shrif, 2008) or a reduced-order model (see section 2.3) obtained by Proper Orthogonal Decomposition (Bergmann et al., 2005; Bergmann and Cordier, 2008).

Problem 2: Now let us assume that the control system design corresponds to state feedback *i.e.* $\boldsymbol{u} = K\boldsymbol{x}$ for the full state configuration or $\boldsymbol{u} = K\hat{\boldsymbol{x}}$ for the observer-based configuration. Then how to determine the control law $\boldsymbol{u}$, or equivalently the gain matrix K, to apply to $\mathcal{S}$ to minimize a given norm of $\boldsymbol{z}$?

If the system $\mathcal{S}$ is Linear Time Invariant (LTI) then the problem is called Linear Quadratic Regulator or LQR, see section 4 or in Burl (1999).

Problem 3: Let $\hat{\boldsymbol{y}}$ be the estimated value of the output based on the estimated state $\hat{\boldsymbol{x}}$. For an LTI system $\mathcal{S}$, the state space system for the observer is

$$\begin{aligned} \dot{\hat{\boldsymbol{x}}}(t) &= A\hat{\boldsymbol{x}}(t) + B_2\boldsymbol{u}(t) + L\left(\boldsymbol{y}(t) - \hat{\boldsymbol{y}}(t)\right), \\ \hat{\boldsymbol{y}}(t) &= C_2\hat{\boldsymbol{x}}(t) \end{aligned} \tag{2}$$

where L is the observer gain matrix.

Then how to determine the gain matrix L so that $\hat{\boldsymbol{x}}$ is roughly equal to $\boldsymbol{x}$? This question corresponds to the observer design. It can be shown (see section 2.2.2) that this problem is dual to the control problem described at the previous item.

Problem 4: How to determine one or more parameters of the system $\mathcal{S}$ knowing the input $\boldsymbol{x}$ and the corresponding output $\boldsymbol{y}$?

Depending on the authors, this question corresponds to the estimation of physical parameters or data inversion (Tarantola, 2005), to systems' identification (Juang and Phan, 2001) or to model calibration (see Cordier et al., 2010, for an application to reduced-order models derived by POD).

[3] An exact definition will be given in section 3.1.2.

Problem 5: The model $\mathcal{S}$ being known, how to determine the input $\boldsymbol{u}$ to apply to $\mathcal{S}$ to obtain given output $\boldsymbol{y}$?

This question, which is very similar to that of the first item, corresponds to a problem of data inversion.

Problem 6: How to determine the initial condition $\boldsymbol{x_0}$ which maximizes the energetic amplification of the dynamical system $\mathcal{S}$?

With this question, we can introduce the concept of optimal disturbances and optimal growth (Schmid and Henningson, 2001). We will see an application in section 5 for the linearized channel flow.

All these problems are sufficiently generals to appear in many scientific disciplines sometimes very distant from each other (engineering, medical or social sciences, ...). In addition, these problems clearly all involve at a different level the resolution of a constrained optimization problem (minimization for the great majority, maximization for the problem of optimal disturbances). The solution of constrained optimization problems will thus be the object of a detailed description in section 3.

2.2 Input-output framework

In section 2.1.2 we learned how, starting from a nonlinear model of dynamics $\mathcal{S}$ resulting from any physical modeling, to determine a linear-time invariant system. Is this step sufficient for control? On one hand, the answer is affirmative because there exist many methods of control dedicated to the linearized systems. On the other hand, we will now see that in general it is necessary to be much more careful since the mapping of measurements $\boldsymbol{y}$ (output) to the control $\boldsymbol{u}$ (input) is crucial to have a chance of success for the control.

2.2.1 Similarity transformations

The objective of this section is to demonstrate that the equations of the state-space system are not unique. Starting from the state-space system (1), reproduced here for convenience:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= A\boldsymbol{x}(t) + B_2\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= C_2\boldsymbol{x}(t) + D_2\boldsymbol{u}(t)\end{aligned}$$

we consider a new state vector

$$\tilde{\boldsymbol{x}}(t) = \mathbb{T}^{-1}\boldsymbol{x}(t)$$

where $\mathbb{T}$ is a constant, invertible transformation matrix. Since $\mathbb{T}$ is invertible, we have $\boldsymbol{x}(t) = \mathbb{T}\tilde{\boldsymbol{x}}(t)$ and $\dot{\boldsymbol{x}}(t) = \mathbb{T}\dot{\tilde{\boldsymbol{x}}}(t)$ ($\mathbb{T}$ independent of time). We then obtain immediately a new state-space system defined in terms of the state $\tilde{\boldsymbol{x}}$:

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) &= \left(\mathbb{T}^{-1}A\mathbb{T}\right)\tilde{\boldsymbol{x}}(t) + \left(\mathbb{T}^{-1}B_2\right)\boldsymbol{u}(t),\\ \boldsymbol{y}(t) &= \left(C_2\mathbb{T}\right)\tilde{\boldsymbol{x}}(t) + D_2\boldsymbol{u}(t)\end{aligned}$$

In summary, the new state-space model is generated by using the following similarity transformations:

$$A \longrightarrow \mathbb{T}^{-1}A\mathbb{T} \quad ; \quad B_2 \longrightarrow \mathbb{T}^{-1}B_2 \quad ; \quad C_2 \longrightarrow C_2\mathbb{T} \quad ; \quad D_2 \longrightarrow D_2.$$

Since there exists an infinite number of state representations for a given system, a natural question is then how we can determine the transformation $\mathbb{T}$ best adapted to control?

2.2.2 Controllability and observability

This section addresses the following fundamental questions:

1. Can we always control a flow?
2. Can the state of a system be estimated from the measurements?

In practice, the answers to these questions provide a guide to the selection of actuators and sensors, and are also useful for developing controllers and observers.

Controllability describes the ability of the control $\boldsymbol{u}$ to influence the state $\boldsymbol{x}$. Conversely, observability describes the ability to reconstruct the state $\boldsymbol{x}$ based on available measurements $\boldsymbol{y}$. To simplify the description, consider $\mathcal{S}_{LTI}$ given by (1) with $D_2 = 0$. In this case, the output $\boldsymbol{y}$ is given (see section 2.1.2) by:

$$\boldsymbol{y}(t) = \underbrace{\int_0^t C_2 e^{A(t-\tau)} B_2 \boldsymbol{u}(\tau)\, \mathrm{d}\tau}_{T_1} + \underbrace{C_2 e^{At}\boldsymbol{x}(0)}_{T_2}.$$

The term T_1 defines a mapping from the space of the control $\boldsymbol{u}$ to the space of the state $\boldsymbol{x}$. Since this map is linear, the image is a subspace of the state-space $\mathbb{R}^{n_x}$ called the controllability subspace. This subspace depends only on the matrices A and B_2, and is denoted $S_{\mathcal{C}}$. Similarly, the term T_2 defines a mapping from the space of the state $\boldsymbol{x}$ to the space of measurement $\boldsymbol{y}$. Since this map is also linear, the image is a subspace of the state-space $\mathbb{R}^{n_y}$ called the observability subspace. This subspace depends only on the matrices C_2 and A, and is denoted by $S_{\mathcal{O}}$. The kernel of this linear map forms a subspace, called the unobservable subspace. Since for these states,

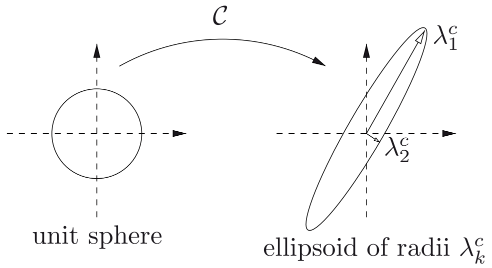


Figure 3. Geometric interpretation of the controllability operator: mapping of unit sphere onto ellipsoid. The direction corresponding to λ_1^c is more controllable than the direction corresponding to λ_2^c.

$\boldsymbol{y} = \boldsymbol{0}$, it means that the elements of the kernel[4] may be added to any another initial state without changing the output.

2.2.2.1 Controllability Suppose the system defined in (1) is stable. Then, for $\boldsymbol{x}(-\infty) = \boldsymbol{0}$, the state at time zero $\boldsymbol{x}(0) = \boldsymbol{x_0}$ is given by

$$\boldsymbol{x_0} = \int_{-\infty}^{0} e^{-A\tau} B_2 \boldsymbol{u}(\tau) \, \mathrm{d}\tau.$$

This defines the controllability operator $\mathcal{C}$ by $\boldsymbol{x_0} = \mathcal{C}\boldsymbol{u}$. In geometric terms analogous to the moment of inertia tensor, $\mathcal{C}$ defines a controllability ellipsoid in the state space, with the longest principal axes along the most controllable directions (see Fig. 3).

The controllability gramian is an $n_x \times n_x$ matrix whose eigenvectors span the controllability subspace. It is defined[5] for the system (1) as

$$W_c(t) = \mathcal{C}\mathcal{C}^H = \int_0^t e^{A\tau} \, B_2 \, B_2^H \, e^{A^H \tau} \, \mathrm{d}\tau \tag{3}$$

where the exponent H denotes the transconjugate operator (transpose conjugate).

[4]The kernel or null space of a linear transformation is the set of vectors that map to zero. If we associate a matrix $\mathcal{A}$ to the linear transformation, the null space of $\mathcal{A}$ is the set of all vectors x for which $\mathcal{A}x = 0$.

[5]The controllability gramian and later the observability gramian (section 2.2.2.2) can be defined in a more general way by considering a weighted inner product (see appendix A or Ilak 2009).

If the system (1) is stable, we can consider the infinite horizon Gramian ($t \longrightarrow +\infty$) and forget the dependance on time. Since W_c is clearly self-adjoint, it admits a set of real, non-negative eigenvalues λ_k^c and orthonormal eigenvectors $\boldsymbol{x_k^c}$. The eigenvalues are a measure of the amount of control energy required to obtain the corresponding eigenvectors. For two states, $\boldsymbol{x_1^c}$ and $\boldsymbol{x_2^c}$ with $\|\boldsymbol{x_1^c}\|_2 = \|\boldsymbol{x_2^c}\|_2$ where $\|\cdot\|_2$ denote the classical L_2 norm ($\|\boldsymbol{x}\|_2^2 = \boldsymbol{x}^H\boldsymbol{x}$) then if

$$\lambda_1^c = \left(\boldsymbol{x_1^c}\right)^H W_c \boldsymbol{x_1^c} = \|\boldsymbol{x_1^c}\|_{W_c}^2 > \|\boldsymbol{x_2^c}\|_{W_c}^2 = \left(\boldsymbol{x_2^c}\right)^H W_c \boldsymbol{x_2^c} = \lambda_2^c$$

it means that $\boldsymbol{x_1^c}$ is more controllable than $\boldsymbol{x_2^c}$.

When the size of the system $\mathcal{S}$ is not too high, the controllability gramian can be determined[6] directly as the solution of a Lyapunov[7] equation given by:

$$AW_c + W_c A^H + B_2 B_2^H = 0.$$

By definition, the dynamical system (1), or equivalently the pair (A, B_2) is said to be state controllable if and only if, for any initial state $\boldsymbol{x}(0) = \boldsymbol{x_0}$ and any final state $\boldsymbol{x_f}$, there exists an input $\boldsymbol{u}(t)$ such that $\boldsymbol{x}(t_f) = \boldsymbol{x_f}$ for $t_f - t_0 < +\infty$. Unfortunately, this criterion is not very usable. In practice, the controllability of a system will be verified using one or the other of the following equivalent criteria[8] (Lewis and Syrmos, 1995; Zhou et al., 1996; Skogestad and Postlethwaite, 2005):

1. Kalman criterion

$$rank\left(\begin{bmatrix} B_2 & AB_2 & A^2B_2 & \cdots A^{n_x-1}B_2 \end{bmatrix}\right) = n_x.$$

2. $W_c > 0$.
3. W_c is full-rank.
4. $\mathrm{Im}(\mathcal{C}) = \mathbb{R}^{n_x}$.

Finally, let

$$E_u \triangleq \int_{-\infty}^{0} \|\boldsymbol{u}\|_2^2 \, \mathrm{d}t = \int_{-\infty}^{0} \boldsymbol{u}^H(t)\boldsymbol{u}(t) \, \mathrm{d}t,$$

[6] The proof is based on the time differentiation of (3). It can be found in section A7 of Burl (1999).

[7] A common way to solve continuous-time Lyapunov equation is with the function `lyap` of Matlab or with the Slicot library that can be found in `http://www.slicot.net`.

[8] We remind that the rank of a matrix $\mathcal{A}$ corresponds to the maximal number of linearly independent rows or columns of $\mathcal{A}$. Moreover, a symmetric matrix $\mathcal{A}$ is said positive definite (simply denoted $\mathcal{A} > 0$) if $\boldsymbol{x}^H\mathcal{A}\boldsymbol{x} > 0$ for all non-zero vectors $\boldsymbol{x}$. Finally, $Im(f)$ denotes the image of the operator f. If f is a mapping from E to F, then $Im(f) = \{\boldsymbol{y} \in F \text{ such that } f(\boldsymbol{x}) = \boldsymbol{y}, \text{ for some } \boldsymbol{x} \in E\}$.

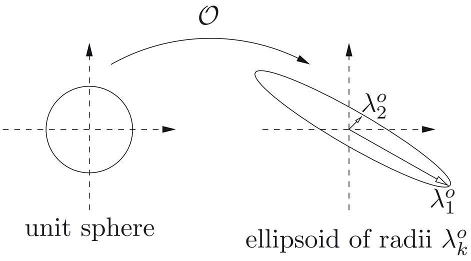


Figure 4. Geometric interpretation of the observability operator: mapping of unit sphere onto ellipsoid. The direction corresponding to λ_1^o is more observable than the direction corresponding to λ_2^o.

with $\boldsymbol{u}(t)$ defined for $t \in]-\infty; 0]$, be the past input energy, it can be shown (Mehrmann and Stykel, 2005) that:

$$E_{u_{min}} = \min_{\boldsymbol{u}} E_u = \boldsymbol{x_0}^H W_c^{-1} \boldsymbol{x_0}.$$

2.2.2.2 Observability We now consider the similar notions as in the previous section but for the output. We will thus follow a similar structure of presentation.

Suppose the system (1) is in some initial state $\boldsymbol{x}(0) = \boldsymbol{x_0}$ and $\boldsymbol{u}(t) = \boldsymbol{0}$ for $t \in [0; +\infty[$. Integrating the dynamics (1), it yields:

$$y(t) = C_2 e^{At} \boldsymbol{x}(0) \tag{4}$$

which defines the observability operator $\mathcal{O}$ by $y(t) = \mathcal{O}\boldsymbol{x_0}$. Similarly to what we have made in section 2.2.2.1 for the controllability, we can analyze this operator in geometric terms (see Fig. 4). Here, $\mathcal{O}$ defines an observability ellipsoid in the state space, with the longest principal axes along the most observable directions.

The observability gramian is an $n_x \times n_x$ matrix whose eigenvectors span the observability subspace. It is defined for the system (1) as

$$W_o(t) = \mathcal{O}^H \mathcal{O} = \int_0^t e^{A^H \tau} C_2^H C_2 e^{A\tau} \, \mathrm{d}\tau. \tag{5}$$

For a stable system, observability can be characterized only by the infinite horizon Gramian ($t \longrightarrow +\infty$) and we can forget the explicit dependance on time in W_o. The eigenvalues λ_k^o of W_o are a measure of the amount of state energy required to obtain the corresponding eigenvectors $\boldsymbol{x_k^o}$. Obviously, we have the result that for two states, $\boldsymbol{x_1^o}$ and $\boldsymbol{x_2^o}$ with $\|\boldsymbol{x_1^o}\|_2 = \|\boldsymbol{x_2^o}\|_2$ then if

$$\lambda_1^o = (\boldsymbol{x_1^o})^H W_o \boldsymbol{x_1^o} = \|\boldsymbol{x_1^o}\|_{W_o}^2 > \|\boldsymbol{x_2^o}\|_{W_o}^2 = (\boldsymbol{x_2^o})^H W_o \boldsymbol{x_2^o} = \lambda_2^o$$

it means that $\boldsymbol{x_1^o}$ is more observable than $\boldsymbol{x_2^o}$.

When the dimension of $\mathcal{S}$ is not too high, a common way of determining the observability gramian W_o is to solve the following Lyapunov equation:

$$A^H W_o + W_o A + C_2^H C_2 = 0.$$

By definition, the dynamical system (1), or equivalently the pair (A, C_2) is said to be state observable if and only if, for any time $t_f > 0$, the initial state $\boldsymbol{x}(0) = \boldsymbol{x_0}$ can be determined from knowledge of the input $\boldsymbol{u}(t)$ and output $\boldsymbol{y}(t)$ in the interval $[0; t_f]$. In practice, the observability of a system is verified through one of the following equivalent criteria (Lewis and Syrmos, 1995; Zhou et al., 1996; Skogestad and Postlethwaite, 2005):

1. Kalman criterion

$$rank\left(\begin{bmatrix} C_2 \\ C_2 A \\ \vdots \\ C_2 A^{n_x - 1} \end{bmatrix}\right) = n_x.$$

2. $W_o > 0$.
3. W_o is full-rank.
4. $\ker(\mathcal{O}) = \mathbf{0}$.

To conclude this section, let

$$E_y = \int_0^{+\infty} \|\boldsymbol{y}\|_2^2 \; \mathrm{d}t = \int_0^{+\infty} \boldsymbol{y}^H(t)\boldsymbol{y}(t) \; \mathrm{d}t,$$

with $\boldsymbol{y}(t)$ defined for $t \in [0; +\infty[$, be the future output energy, it can be shown easily by substituting (4) in E_y that

$$E_y = \boldsymbol{x_0}^H W_o \boldsymbol{x_0}.$$

2.2.2.3 Duality Duality is an important concept in linear control theory because, used advisedly, it can saved a considerable time in the derivation of properties for the systems under investigation. To go further, we will initially admit that for any primal system defined by (1), that is to say

$$\mathcal{S} : \begin{cases} \dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B_2\boldsymbol{u}(t), \\ \boldsymbol{y}(t) = C_2\boldsymbol{x}(t) \end{cases}$$

we can associate another state-space system, known as dual system, and given by

$$\mathcal{S}_{\text{dual}} : \begin{cases} \dot{\boldsymbol{\xi}}(t) = A^H \boldsymbol{\xi}(t) + C_2^H \boldsymbol{\zeta}(t), \\ \boldsymbol{\eta}(t) = B_2^H \boldsymbol{\xi}(t). \end{cases}$$

Here, $\boldsymbol{\xi}$ is the dual state vector, and $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ contain the dual inputs and outputs. Comparing $\mathcal{S}$ and $\mathcal{S}_{\text{dual}}$ it can be seen that we can deduce the dual system from the knowledge of the primal system with the transformations:

$$A \longrightarrow A^H \qquad \text{and} \qquad B_2 \longrightarrow C_2^H. \tag{6}$$

Duality of controllability and observability From the transformations (6) and the definitions (3) and (5), it is evident that the controllability gramian of the primal system is equal to the observability gramian of the dual system, and vice versa. As a consequence, the following results hold:

1. $\mathcal{S}\left(A, B_2\right)$ is controllable if and only if $\mathcal{S}_{\text{dual}}\left(A^H, B_2^H\right)$ is observable,
2. $\mathcal{S}_{\text{dual}}\left(A^H, C_2^H\right)$ is controllable if and only if $\mathcal{S}\left(A, C_2\right)$ is observable.

Duality of the control problem and the observer design If we now consider the cost function

$$\mathcal{J}_{\boldsymbol{y}} = \int_0^T \|\boldsymbol{y}\|_2^2 \; \mathrm{d}t$$

and the corresponding cost function

$$\mathcal{J}_{\boldsymbol{\eta}} = \int_0^T \|\boldsymbol{\eta}\|_2^2 \; \mathrm{d}t$$

based on the dual system, it can easily be proved[9] that $\mathcal{J}_{\boldsymbol{y}} = \mathcal{J}_{\boldsymbol{\eta}}$. This property is fundamental in control theory since it can be employed to determine the observer gain matrix L for the observer design (see problem 2.1.3 in section 2.1.3) based on the solution of the dual control problem. Indeed, let $\boldsymbol{x_e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$ be the state error, the main purpose of state observer design is to minimize $\mathcal{J} = \displaystyle\int_0^T \|\boldsymbol{x_e}\|_2^2 \; \mathrm{d}t$ where $\hat{\boldsymbol{x}}$ is given by

[9] Essentially, the proof is based on two results:

1. the transformations (6), and
2. the following equalities

$$\mathcal{J}_{\boldsymbol{y}} = trace\left(C_2 W_c C_2^H\right) = trace\left(B_2^H W_o B_2\right) \quad \text{(see Burl, 1999, p. 113).}$$

(2). An elegant method of determination of the observer gain matrix then consists in minimizing the same functional $\mathcal{J}$ but by introducing the dual problem of the initial system (Huerre, 2006). We then arrive at a Linear Quadratic Regulator problem whose solution is already known (see section 4). Consequently, we will not detail thereafter the observer design (see classical textbooks Zhou et al., 1996; Burl, 1999; Skogestad and Postlethwaite, 2005, for instance) and we will concentrate on the control problem.

2.2.2.4 Balanced truncation The notions of controllability and observability, as defined respectively in sections 2.2.2.1 and 2.2.2.2, give us a means of deciding whether a state affects the system's input-output map: if a state is unobservable, it does not affect the output, and if a state is uncontrollable, it is unaffected by the input. In terms of model reduction dedicated to control (see section 2.3), in opposition to model reduction for physical understanding, it is then capital to preserve controllable and observable modes, but in which proportion? A simple answer was given by Moore (1981) for stable, linear, input-output systems. This method called balanced truncation consists in transforming the state space system into a balanced form whose controllability and observability Gramians become diagonal and equal (balanced realization), together with a truncation of those states that are both difficult to reach and to observe.

Starting from the similarity transformations given in section 2.2.1, it can be easily shown that the controllability and observability gramians become:

$$W_c \longrightarrow \mathbb{T}^{-1} W_c \left(\mathbb{T}^{-1}\right)^H \quad \text{and} \quad W_o \longrightarrow \mathbb{T}^H W_o \mathbb{T}.$$

In the system of coordinates defined by $\mathbb{T}$, we thus have for a balanced realization:

$$\mathbb{T}^{-1} W_c \left(\mathbb{T}^{-1}\right)^H = \mathbb{T}^H W_o \mathbb{T} = \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_{n_x} \end{bmatrix}$$

where the Hankel singular values σ_i are real, positive and ordered by convention from largest to smallest. An equivalent way of finding the balancing transformation $\mathbb{T}$ is to compute the eigendecomposition of $W_c W_o$ ($W_c W_o = \mathbb{T}\Sigma^2\mathbb{T}^{-1}$). It can be shown (Burl, 1999) that a balanced realization exists whenever the system is stable and minimal[10]. A geometric interpretation of the balanced truncation is given in Fig. 5.

[10] A state space system is minimal if and only if the system is controllable and observable (Zhou et al., 1996). Moreover, a minimal realization of the system is associated with a matrix A of smallest possible dimension.

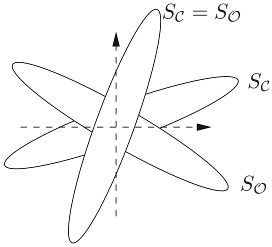


Figure 5. Geometric interpretation of the balanced truncation. $S_\mathcal{C}$ and $S_\mathcal{O}$ are respectively the controllability and observability subspaces.

An attractive feature of balanced truncation is that there exists a priori error bounds that are close to the lower bound achievable by any reduced-order model (Zhou et al., 1996, for instance). Let G denote the transfer function[11] of the LTI system (1) and G_r the corresponding transfer function of a reduced-order model of order r. It can be proved that, in any reduced-order model, the lower bound for the H_∞ error[12] is

$$\|G - G_r\|_\infty \geq \sigma_{r+1}$$

and that the upper bound for the error obtained by balanced truncation is given by

$$\|G - G_r\|_\infty \leq 2 \sum_{j=r+1}^{n_x} \sigma_j .$$

If the Hankel singular values are decreasing sufficiently fast, it means that the error norm of the reduced-order model of order r is very close to the lowest possible value.

[11] For a SISO system, the transfer function G from $\boldsymbol{u}$ to $\boldsymbol{y}$ is defined as

$$G(s) = Y(s)/U(s)$$

where $U(s)$ and $Y(s)$ are the Laplace transform of $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$. Moreover, it can be demonstrated that for an LTI system, we have

$$G(s) = C_2 \left(sI - A\right)^{-1} B_2 + D_2$$

where I is the identity matrix.

[12] The H_∞ norm of the system is defined in terms of the transfer function G as:

$$\|G\|_\infty = \sup_\omega \sigma_1 \left(G(j\omega)\right)$$

where $\sigma_1 \left(\mathcal{A}\right)$ corresponds to the maximum singular value of the matrix $\mathcal{A}$ and ω represents frequency.

The procedure of balanced truncation is very attractive in terms of control but the determination of the controllability and observability gramians via the solution of Lyapunov equations is not computationally tractable for very large systems. In addition, the original method suggested by Moore (1981) is limited to the linear systems. These limitations were raised recently by Lall et al. (2002) and then by Rowley (2005) who introduced approximation methods of gramians based only on snapshots of the primal and dual systems (see section 2.2.2.3). The initial method suggested by Lall et al. (2002) was to first estimate the two gramians, and then in a second time to perform the balanced truncation. The main contribution presented in Rowley (2005) is a specific algorithm that can be used to determine the balanced truncation directly from snapshots of the system *i.e.* without needing to compute the gramians themselves. This method is called Balanced POD for deep connections that it shares with POD. The reader will find all the details of the numerical setting in Rowley (2005).

2.3 Model reduction

In section 2.2.2.4, model reduction was already evoked when the least controllable and observable modes of the system were truncated based on the decrease of the Hankel singular values. In this section, we will first justify the interest of reduced-order modeling for flow control (section 2.3.1), and then present in a general way the current methods of model reduction while giving an emphasis on projection-based methods (section 2.3.2).

2.3.1 Need for reduced-order modeling

For a wing considered at cruising flight conditions *i.e.* for a Reynolds number of about 10^7, Spalart et al. (1997) considered that to obtain numerically a converged solution, it is necessary to integrate the Navier-Stokes equations during about $5\ 10^6$ time steps on about 10^{11} grid points. Then, in spite of the recent and considerable progresses of computers, it remains difficult to solve numerically problems where

- either, a great number of resolution of the state equations is necessary (continuation methods, parametric studies, optimization problems or optimal control,. . .),
- either a solution in real time is searched (active control in closed-loop control for instance).

Not surprisingly, the reduction of the costs of solving nonlinear state equations became a major issue in many scientific disciplines ranging from linear algebra to computer graphics. Sometimes, as it is the case in fluid mechanics/turbulence, model reduction has a long tradition but the objective is more centered on the improvement of the understanding of the physical mechanisms. Let us quote for example[13]:

- Prandtl boundary layer equations (Schlichting and Gersten, 2003),
- Reynolds-Averaged Navier-Stokes models (Chassaing, 2000),
- Large Eddy Simulation (Sagaut, 2005),
- Low-order dynamical system based on Proper Orthogonal Decomposition (Aubry et al., 1988),
- Reduced-order models based on global modes (Åkervik et al., 2007),

to name a few. Since less than ten years, the methods of model reduction are mainly considered in fluid mechanics for flow control. Lately, these methods progressed considerably under the efforts of the applied mathematicians who were interested in flow control. It is this specific point of view that is retained in the following presentation of the model reduction methods.

2.3.2 Overview of model-reduction methods

Broadly speaking, model order reduction techniques fall into two major categories:

1. projection-based methods,
2. non-projection based methods.

The first group corresponds to the methods that are currently the most used in fluid mechanics. Therefore, this approach will be detailed in section 2.3.2.1. The second group consists mainly of such methods as Hankel optimal model reduction and state-residualization. More information can be found for these methods in Antoulas (2005).

2.3.2.1 Projection-based methods The projection-based methods can be used for dynamical models going from general nonlinear systems given[14] by

$$\mathcal{S} : \begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \\ \boldsymbol{y}(t) = \boldsymbol{g}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)), \end{cases}$$

to LTI models

$$\mathcal{S}_{LTI} : \begin{cases} E\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + B_2\boldsymbol{u}(t), \\ \boldsymbol{y}(t) = C_2\boldsymbol{x}(t) + D_2\boldsymbol{u}(t), \end{cases}$$

[13] The traditional numerical methods used to solve partial derivative equations (finite difference, finite volume, finite element, spectral method,...) can also be classified in the framework of reduced-order models since these methods consist in reducing an infinite-dimensional problem to a finite-dimensional one (discretized problems).

[14] To simplify the formulations, we did not consider in this section the contribution of the disturbances $\boldsymbol{w}$ to the models.

written here in the so-called descriptor form. The matrix E is not necessarily invertible but, when it is the case, the traditional LTI formulation is found. For these two systems, the state variables $\boldsymbol{x}$ and output variables $\boldsymbol{y}$ are respectively of size n_x and n_y.

The objective of reduced-order modeling is to determine for $\mathcal{S}$ and $\mathcal{S}_{LTI}$ the corresponding simplified models

$$\widehat{\mathcal{S}} : \begin{cases} \dot{\widehat{\boldsymbol{x}}}(t) = \widehat{\boldsymbol{f}}(t, \widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)), \\ \widehat{\boldsymbol{y}}(t) = \widehat{\boldsymbol{g}}(t, \widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)), \end{cases}$$

and

$$\widehat{\mathcal{S}}_{LTI} : \begin{cases} \widehat{E}\dot{\widehat{\boldsymbol{x}}}(t) = \widehat{A}\widehat{\boldsymbol{x}}(t) + \widehat{B_2}\boldsymbol{u}(t), \\ \widehat{\boldsymbol{y}}(t) = \widehat{C_2}\widehat{\boldsymbol{x}}(t) + \widehat{D_2}\boldsymbol{u}(t) \end{cases}$$

where the control $\boldsymbol{u}$ is unchanged. These simplified models are now called reduced-order models since $\widehat{\boldsymbol{x}} \in \mathbb{R}^r$ with $r \ll n_x$ and $\boldsymbol{y} \simeq \widehat{\boldsymbol{y}} \in \mathbb{R}^{n_y}$. A simplified description of model reduction is given in Fig. 6.

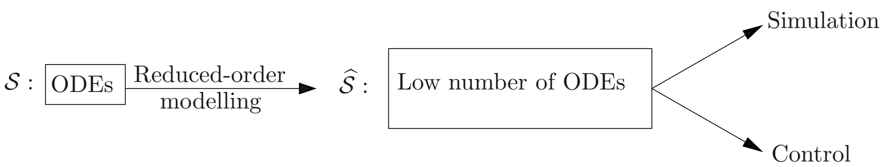


Figure 6. Broad framework of reduced-order modelling (after Antoulas, 2005).

If we want that these reduced-order models can be really usable for the applications concerned, it is necessary that the methods used to derive these simplified models satisfy various constraints:

1. Small approximation error for all admissible input signals $\boldsymbol{u}$ *i.e.*

$$\|\boldsymbol{y} - \widehat{\boldsymbol{y}}\| < \epsilon \times \|\boldsymbol{u}\| \qquad \text{with } \epsilon \text{ a tolerance.}$$

   It means that we need to have estimates of computable error bounds.
2. Stability and passivity (no generation of energy) preserved.
3. Procedure of model reduction numerically stable and efficient.
4. If possible, automatic generation of models.

In what follows we will describe an algorithm that can be used to derive a reduced-order model of any dynamical system. This algorithm, called Petrov-Galerkin projection, is based on a general bi-orthogonal projection basis. Let V and W be two[15] bi-orthogonal matrices of size $\mathbb{C}^{n_x \times r}$, and $Q \in \mathbb{C}^{n_x \times n_x}$ be the weight matrix such that

$$W^H Q V = I_r$$

where I_r is the identity matrix of size r. In the first step of the algorithm, $\boldsymbol{x}$ is projected on the space spanned by the columns of V *i.e.* $\boldsymbol{x} = V\widehat{\boldsymbol{x}}$. In the second step, this projection is inserted in the dynamical system where we have introduced the residual $\mathcal{R}$ of the state equations. At this stage, we obtain for $\mathcal{S}$

$$\begin{cases} \boldsymbol{\mathcal{R}} = V\dot{\widehat{\boldsymbol{x}}}(t) - \boldsymbol{f}(t, V\widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)), \\ \widehat{\boldsymbol{y}}(t) = \boldsymbol{g}(t, V\widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)), \end{cases}$$

and for $\mathcal{S}_{LTI}$

$$\begin{cases} \boldsymbol{\mathcal{R}} = EV\dot{\widehat{\boldsymbol{x}}}(t) - AV\widehat{\boldsymbol{x}}(t) - B_2\boldsymbol{u}(t), \\ \widehat{\boldsymbol{y}}(t) = C_2 V\widehat{\boldsymbol{x}}(t) + D_2\boldsymbol{u}(t). \end{cases}$$

The last step corresponds to a weak projection of the residual on the space spanned by the columns of W *i.e.* $W^H Q\,\boldsymbol{\mathcal{R}} = \mathbf{0}_r$. Finally, we obtain the reduced-order model $\widehat{\mathcal{S}}$ where

$$\widehat{\mathcal{S}} : \begin{cases} \widehat{\boldsymbol{f}}(t, \widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)) = W^H Q\, \widehat{\boldsymbol{f}}(t, V\widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)), \\ \widehat{\boldsymbol{g}}(t, \widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)) = \boldsymbol{g}(t, V\widehat{\boldsymbol{x}}(t), \boldsymbol{u}(t)), \end{cases}$$

and the reduced-order model $\widehat{\mathcal{S}}_{LTI}$ where

$$\begin{aligned} \widehat{A} &= W^H Q A V, & \widehat{B_2} &= W^H Q B_2, \\ \widehat{C_2} &= C_2 V, & \widehat{D_2} &= D_2, \\ \widehat{E} &= W^H Q E V. \end{aligned}$$

For the choice of the matrices V and W, various possibilities exist for the linear systems:

1. In the case of Krylov methods (Gugercin and Antoulas, 2006), it corresponds to the projection on the Krylov subspace of the controllability gramian coupled with an identification of the moments of the transfer function.
2. For balanced realizations, this choice corresponds to the projection on dominant modes of the controllability and observability gramians as already discussed in section 2.2.2.4.

[15] When $V \neq W$, it corresponds to an oblique projection, and when $V \equiv W$ it is called Galerkin projection or orthogonal projection.

3. For instabilities, the projection is made on the global and adjoint global modes (Schmid and Henningson, 2001; Barbagallo et al., 2009).
4. Finally, in the case of the Proper Orthogonal Decomposition (Lumley, 1967; Sirovich, 1987), it corresponds to the projection on the subspace determined optimally with snapshots of the system (see the contribution by B. Noack et al. in this book).

For the non-linear systems, the situation is different because, until now, there exists only the Proper Orthogonal Decomposition what explains its intensive use in the past years.

3 Optimal control theory

3.1 Constrained optimization problems

3.1.1 Abstract description

All the constrained optimization problems appearing in fluid mechanics and heat transfers (shape optimization, active flow control, optimal growth, control of thermal systems, ...) can be described mathematically by the following quantities[16] (Gunzburger, 1997a, 2003):

state variables ϕ which describe the flow. Depending on the problem, these variables might be mechanical or thermodynamic, for instance velocity vectors, pressure, temperature, ...

control parameters c**.** In practice, these variables occur as boundary conditions of the state equations[17], when the control is applied at the boundaries of the domain, or directly as a source term in the state equations if the control is distributed inside the domain (volume forcing). In data assimilation (meteorology, oceanography) and for optimal growth (see section 5) these control parameters intervene as initial conditions. According to the application, these parameters might be velocities prescribed at the boundaries (suction/blowing), heat flux or temperature at a wall, or for a shape optimization problem (Mohammadi and Pironneau, 2001), it might be variables allowing to describe

[16] To simplify the presentation, all the variables are here considered as scalars. However, the method extends naturally to the case of vectorial variables. For instance, an optimal control problem is solved for the Linear Quadratic Regulator approach in section 4, and for the three-dimensional Navier-Stokes equations in Bewley et al. (2001) or El Shrif (2008).

[17] Here, we use the traditional terminology in optimal control and call state equations, the equations which govern the dynamics of the system. Other terminologies are primal or direct equations.

geometrically the shape of the boundary. In this last case, the control parameters are rather called design variables.

a cost or objective functional $\mathcal{J}$ which describes a measure of the objectives we wish to achieve. It might be drag minimization, maximization of lift or heat flux, stabilization of a temperature, flow targets, ... This functional $\mathcal{J}$ depends on the state variables ϕ and on the control parameters c, *i.e.* $\mathcal{J}(\phi, c)$.

physical constraints F which represent the evolution of the state variables ϕ in terms[18] of the control parameters c with respect to the physical laws. Mathematically, these constraints are noted:

$$F(\phi, c) = 0.$$

In fluid mechanics, these constraints correspond generally to the Navier-Stokes equations and their associate initial and boundary conditions. If a problem of optimal disturbance is concerned then the initial condition is imposed as a constraint (see section 5). If the control is exerted at the boundaries of the flow domain, the boundary condition can also be included as constraint (see section 6 for an example). Moreover, we will see in section 3.1.2 that an additional constraint must in general be added so that the problem is well posed mathematically.

Finally, the constrained optimization problem can be stated in the following way:

> *determine the state variables ϕ and the control parameters c, such that the objective functional $\mathcal{J}$ is optimal (minimum or maximum according to the case) under the constraints F.*

3.1.2 Ill-posed optimization problem and choice of the cost functional

The choice of the cost functional $\mathcal{J}$ is central in an optimization problem. From a mathematical point of view, the physical quantity to be optimized is represented by

$$\mathcal{J} = \mathcal{M}$$

where $\mathcal{M}$ is an appropriate measure of any physical quantity of interest: drag, lift, disturbance energy, ... The choice of this cost functional is essential in practice so that the optimization problem is well posed. This choice

[18] Rigorously, it would be necessary to note the variables $\phi(c)$ because ϕ depend on the control variables c via the constraints. However, to reduce the notations, we will note the state variables simply as ϕ.

is sometimes difficult to achieve. For instance, it is not obvious to know in advance if it is better to choose as cost functional a measure of the drag to minimize this quantity. In some cases (Bewley et al., 2001; El Shrif, 2008), it seems that it is preferable to minimize the averaged kinetic energy of the flow in order to minimize the drag. In addition, beyond the mathematical difficulty that is raised, we can imagine that the implementation of the control will be eased if the cost functional is based on a relevant quantity for the physics of the problem.

In general, there is no explicit relation between the objective to be reached and the control variable. This can involve that the optimization problem is ill-posed and that its solution is then divergent. To solve this difficulty, the cost of the control should be limited[19]. Let $\mathcal{M}_c$ be a measure of the cost of the control, this limitation can be done:

1. By adding an additional constraint to the physical constraints (F)

   This constraint corresponds to a maximum value which should not be exceeded by the control cost. Let $(\mathcal{M}_c)_{max}$ be an arbitrary positive constant, the problem is then equivalent to impose that $\mathcal{M}_c \leq (\mathcal{M}_c)_{max}$. In optimization, the inequality constraints make intervene optimality conditions known as Karush-Kuhn-Tucker (Bonnans et al., 2003) which are often delicate to take into account. For this reason, it is generally preferred to retain equality type constraints which can be imposed more easily using Lagrange multipliers (see section 3.2). It will thus be sufficient to set an additional constraint of the type $\mathcal{M}_c = \mathcal{M}_c^u$ where $\mathcal{M}_c^u > 0$ is a cost imposed by the user, to do not have to change the nature of the optimization problem to be solved.

2. By modifying the cost functional $\mathcal{J}$

   A possible modification of the cost functional is to consider

$$\mathcal{J} = \mathcal{M} + \ell \mathcal{M}_c$$

   where ℓ is a positive real constant whose value is fixed by the user according to the importance given to the cost of the control. If the value of the parameter ℓ is low then it means that the cost of the control is not a priority in the practical implementation (low costs of control). On the contrary, if the value of ℓ is high, then the cost of the control is a priority (expensive control). A more thorough discussion is given in section 4 for the LQR control.

[19] Apart from a mathematical justification, a limitation of the control cost is necessary since from an economic point of view the ratio saving/cost is a determining factor.