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RUSSIAN CONTRIBUTIONS TO GAME THEORY AND EQUILIBRIUM THEORY

Edited by Theo S. H. Driessen, Gerard van der Laan, Valeri A. Vasil'ev and Elena B. Yanovskaya

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Russian Contributions to Game Theory and Equilibrium Theory

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Preface and acknowledgments

The purpose of this collection of papers is to report about highly qualified research by Soviet game theorists during the two decades 1968-1988 to the international readership. The research by Soviet scientists within the field of game theory, starting from the early period around 1965 under the supervision of their former leader N.N. Vorob'ev, has resulted in many high-level contributions. The contributions in this collection have not been published before, except in Russian language journals. In the past the Russian literature in general, and the Russian game theoretical literature in particular, was not available for colleagues outside the Soviet Union, and even nowadays the papers in the Russian language are inaccessible for the international readership. Therefore this collection of English translations of specifically chosen former research articles in Russian will help to close the gap in the international knowledge about the Soviet advances in game theory.

In 1968, the Econometric Research Program at Princeton University published a bundle of English translations of Soviet research, entitled "Selected Russian Papers on Game Theory, 1959-1965". This bundle, yet recently downloadable from www.econ.princeton.edu/ERParchives, played an important role to inform the international game theory community about interesting game theoretic results in the former Soviet Union. In particular, because of this bundle the international researchers in game theory learned that the famous existence result for the core had been proven by the Russian PhD student Olga Bondareva already five years before the first paper by Shapley on this topic was published.

Since the 1990s also the Russian researchers are publishing in English and therefore their results are available to the international readership. However, the period 1965-1990 remains as a period in which within the former Soviet Union game theory was successfully developing, but separately from the world science. This collection of papers attempts to acquaint English language readers with some contributions in game theory and the related field of equilibrium theory, which never had been published in English before. Since some Soviet journals of high level were translated into English, the most important contributions of the Soviet researchers to these fields are already accessible in English. For this reason there are no papers of the Soviet game-theoretic leader N.N.Vorob'ev within this volume. Nevertheless, many papers containing very nice results are still unattainable for foreign readers. The twelve selected papers in this volume, all from the period 1965-1990, have been translated by the authors themselves. Some of the papers are slightly adapted to improve the readability. In addition, the volume starts with an introductory chapter on the history of Soviet game theory before the 1990s. This chapter also contains a short summary of the selected contributions.

The idea of editing a volume of English translations of former Soviet contributions emerged from the continuous scientific communication between game theorists from The Netherlands and The Russian Federation during the last decade 1996-2005. The mutual scientific communication was initiated in June 1996 by the participation of Dutch researchers into the international conference on game theory in memoriam of N.N. Vorob'ev, held at St. Petersburg. This conference initiated subsequent scientific visits by Russian game theorists to The Netherlands, in particular the University of Twente at Enschede. These visits, realized through funding from the Dutch mathematical organization SWON (Stichting Wiskunde Onderzoek Nederland), resulted in several joint publications. In 2001 the individual SWON funding was upgraded by the Dutch scientific organization NWO (Nederlandse Organisatie voor Wetenschappelijk Ondezoek) within the framework of a bilateral scientific agreement between The Netherlands and The Russian Federation. We are very grateful to NWO and the Russian Foundation for Basic Research (RFBR) for the financial support for our projects "Axiomatic Approach to the Elaboration of Cooperative Game Solutions: Theory and Applications" (2001-2005) and "Game Theoretic Models for Cooperative Decision Making and Their Applications to Mathematical Modelling in Economics and Social Sciences" (2005-2008). Within these projects researchers participate from University of Twente at Enschede, Vrije Universiteit at Amsterdam, Tilburg University, St. Petersburg Institute for Economics and Mathematics, Sobolev Institute of Mathematics at Novosibirsk and Central Institute of Economics and Mathematics at Moscow. As part of these projects three international three-days workshops on cooperative game the-

ory have been organized in Enschede (2002, 2005) and Amsterdam (2004). From the ongoing cooperation and mutual discussions the Dutch editors learned that the Russian literature on game theory and related fields contains many contributions which deserve to become accessible for the international readership. We owe many thanks to the authors for their efforts to translate their old papers in English. The current volume will contribute to a better scientific knowledge of the treasury hidden in the former Soviet journals.

We conclude to express our thanks to Hans Peters from Maastricht University for his invaluable help as managing editor of this volume and his secretary Yolanda Paulissen for processing the papers into their final form.

Theo Driessen — Enschede
Gerard van der Laan — Amsterdam
Valeri Vasil'ev — Novosibirsk
Elena Yanovskaya — St. Petersburg

December 2005

Contents

Chapter 1

Introduction

V.A. Vasil'ev and E.B. Yanovskaya

1 Game theory in the USSR before 1990

In this introductory chapter first a short historical information about the development of game theory in the USSR before the 1990s is given. It should help to understand the choice of the papers in this volume. The development of game theory in the USSR began in the early sixties. N.N.Vorob'ev (1925-1995) was the author of the first papers about game theory in Russian; he was also the leader of a group of PhD students and young researchers at Leningrad who studied this new field in mathematics.

The early papers by Vorob'ev on enumerating equilibrium points in bimatrix games ([74]), coalitional games where a player may belong to different coalitions ([77] and [78]) and equivalence of different types of strategies in games in extensive form ([75] and [76]), were published in Soviet journals having English translations, and so they became known in the West. Later on, game theory in the USSR exhibited significant progress, which deserved world-wide attention. However, many game-theoretic papers were published in Soviet journals and edited volumes which have never been translated into English.

In the 1960s, the theory of zero-sum games was popular in the USSR. Some mathematicians even considered minmax theorems as the only truly mathematical results in game theory. New minmax theorems were obtained at that time, see e.g. Yanovskaya [82] and [83]. Also the structure of the set of optimal strategies was studied and for some particular classes of zero-sum games optimal strategies were found. Similar results for general non-cooperative games were obtained as well. The axiomatizations of the value of a matrix game and equilibrium payoffs in non-cooperative games were obtained by the Lithuanian researcher Vilkas [72], who was one of the first PhD students of Vorob'ev. A review of all these results can be found in the comments chapters to Vorob'ev's book "Foundations of Game Theory: Non-cooperative Games" [80].

One of the main problems of the theory of games in extensive form is to find the classes of the most simple mixed strategies which are sufficient for determining the optimal ones. This field of research began with the famous Kuhn theorem in which he proved for extensive form games with perfect recall the equivalence of each mixed strategy of a player to some behavioural strategy. This theorem has been generalized by several researchers in different directions. In 1957 Vorob'ev published his first paper [75] within a series of generalizations of Kuhn's equivalence theorem about behavioral strategies in extensive form games with perfect recall. He considered more complicated structures of players' recall. For the case of "ordered" recall he introduced "reduced" strategies which were extensions of behavioral strategies and he proved the corresponding equivalence theorem. In [76], Vorob'ev considered another structure of recall and proved an equivalence theorem generalizing that of Thompson for signalling strategies.

Romanovsky [61] found a method of reducing the problem of finding optimal behavioural strategies in finite two-person games in extensive form with perfect recall to the solution of matrix games with linear restrictions on the set of their mixed strategies. For games with partially ordered recall Yanovskaya [81] defined "quasi-strategies" described by the probabilities of vertices and alternatives of the information structure of each player. She proved the corresponding equivalence theorem for quasi-strategies and reduced optimal quasi-strategies of a two-person game in extensive form to the solution of a polyhedral game. Konurbaeva [40] extended this result to games described by acyclic graphs instead of trees. In [41], she also considered games in extensive form where (as Isbell defined) information sets may precede themselves. For such games she defined special "linear-like strategies" and proved the corresponding equivalence theorem. At last, recently Liapunov [49] introduced "H-strategies" for games in extensive form maximizing the entropy of mixed strategy in the class of all strategies equivalent to a given one. The corresponding equivalence theorem has been proved. For games with perfect recall the H-strategies coincide with the behavioural ones.

Another direction in game theory and decision making under uncertainty that has been developed in the 1970s at Leningrad concerns mixed extensions of arbitrary binary relations. Some papers on this topic have been published and the results are summed up in Kiruta *et al.* [39]. These results are about the existence of maximal elements of mixed extensions of binary relations and their applications to the existence of mixed equilibrium points in non-cooperative games with ordered outcomes as well as some mixed concepts in cooperative TU-games. The bibliography of [39] shows that the well-known skew-symmetric utility theory due to Fishburn [11] has also been developed at the same time in the former Soviet Union (see also the papers of Yanovskaya in this volume).

The theory of cooperative games was developing in Leningrad beginning from the famous paper due to Bondareva [25] (see also [26]) who obtained necessary and sufficient conditions for non-emptiness of the core of a TU-game. Later on sufficient conditions providing stability of the core were obtained by Diubin (see the paper in this volume), Bondareva [27], Kulakovskaya [44] and Vilkov [73] for NTU-games, and necessary and sufficient conditions by Kulakovskaya [43]. For cooperative games with a countable player set, Naumova [53], [54] and [55] respectively, obtained necessary and sufficient conditions for non-emptiness of the countably-additive core, found the NM-solutions for some classes of simple games and she obtained sufficient conditions for the existence of M_i^1 bargaining sets.

All the necessary and sufficient stability conditions, mentioned above, indicate that the core of a TU-game may be quite often unstable with respect to the classic domination relation. The same is true even for the standard transitive closure of this relation. Nevertheless, as it was established by Vasil'ev (see his paper in this volume), the core is externally stable with respect to the so-called limit transitive closure, being a fairly natural "sequential" transitive extension of the classic domination relation. An example of an NTU-game having a closed subset of imputations, which contains all the monotonic trajectories originating at its points and does not intersect with the core, demonstrates that the side-payment property is of crucial importance for this result (for more details, see the same paper).

The most important achievements of Soviet game theorists regarding to the classic VonNeumann-Morgenstern solution, is the well-known existence theorem established for all different cases in four-person TU-games in Bondareva [28], Kulakovskaya [44], [45], [46] and Bondareva *et al.* [29]. Also Vasil'ev's later obtained results in this field on the existence and NM-rank of the so-called generalized NM-solutions, clarifying the structure of the classic domination relation, are of interest (see [69], [70], and Vasil'ev's paper in this volume).

In the mid of seventies a new approach for studying the Shapley value and its nonsymmetric analogues, both in finite and infinite case, has been elaborated by Vasil'ev at Novosibirsk. This approach strongly relied upon the vector lattice theory in order to provide a systematic treatment of the values by applying the advanced methods of Kantorovich-Banach-space theory [37], [38] (for the basic notions see, e.g., [1]).[1] The important concept of a totally positive game has been introduced, and a new axiomatization of the Shapley value, treated as a positive (w.r.t. the cone of totally positive games) symmetric linear operator, has been given in [66]. By applying this axiomatization, so-called Harsanyi payoff vectors and the Harsanyi set (also known as the Selectope, independently introduced in Hammer *et al.* [14]), were proposed by Vasil'ev for to give a unified description of the core imputations in terms of some decentralization mechanisms [67]. Surprisingly, the Harsanyi set turned out to be of a core-type structure itself, a fundamental result first established by Vasil'ev in 1980 [68] (see also the paper in this volume), and has been independently rediscovered by Derks *et al.* [8].

To conclude this survey on the classic cooperative game theory results, obtained in Leningrad and Novosibirsk during 1968-1990, we mention that in 1975 Sobolev [65] published a very important result characterizing the prenucleolus axiomatically, and among the axioms the consistency axiom (or the reduced game property) was used for the first time. The very complicated proof can be found in Peleg and Sudhölter's introductory book [21] on cooperative game theory. In [65] also an alternative consistency property is defined and used to obtain a new axiomatization of the Shapley value.

Moscow game theory was initiated in the late 1960s and early 1970s by Germeier (1918-1975), head of the Operations Research Departments of Moscow State University and the Computing Center of the USSR (now Russian) Academy of Sciences. The most popular topic was "hierarchic games", close in spirit to the principal-agent problems: when the principal agent lacks information, he is assumed to follow the maxmin approach rather than form subjective beliefs. Typical settings and solutions are presented in [33]. Unfortunately, even the original book was published posthumously, and its editing was not very good. An interesting model was developed by Germeier and Vatel' [34], who considered voluntary contributions to collective goods. Later on Gurvich [35] and [36], former PhD student of Germeier, characterized normal forms of games with perfect information, respectively

[1] Interestingly to note that both vector lattice theory, adapted to the description of the Shapley value, and linear programming, applied to the core existence problem, were created, among others, by the Soviet academician and Nobel prize winner L.V.Kantorovich, former vice-director of the Institute of Mathematics, Siberian Branch of the USSR Academy of Sciences.

two person game forms ensuring the existence of a Nash equilibrium.

Both in Moscow and in Leningrad bargaining solutions were investigated almost parallel to analogous researches in the west. In 1976 Butrim [31], pupil of Germeier, independently of Kalai and Smorodinsky [16], defined a system of axioms for two-person bargaining problems leading to the Kalai-Smorodinsky solution. Further, Butrim [32] extended the result to n-person bargaining problems. Vorob'ev [79] characterized the n-person Nash bargaining solution without the assumption about the existence of a feasible payoff vector strongly greater then the status quo. Independently of Myerson [19], Perles and Maschler [22] and Thomson [24], Pechersky studied the linear and superlinear bargaining solution (see his two papers in this volume).

In the 1960s, systematic research in the theory of differential games was started by the Soviet mathematicians to supply the scientific basement and rigorous techniques for the solution of important practical problems (such as pursuit-evasion problem, control under uncertainty, search problems and others). Initially main attention was paid to the zero-sum differential games. During the 1960-1970s the Soviet academicians Krasovsky [42] and Pontryagin [59], [60] (and the scientific schools of these mathematicians, see also Petrosjan [57] created a quite general and rigorous theory of so-called positional differential games.

During the 1970-1980s the general theory of n-person (non-cooperative and cooperative) differential games began to develop. In particular, Petrosjan [56] proposed the concept of time-consistency of differential n-person game solutions as a fundamental property of stability. This concept is actively developed at St. Petersburg University. The references can be found in [58].

In the middle of the seventies, working on a general conception of game theory as a mathematical discipline, Vorob'ev, impressed by a paper by Bubelis [30] on the algebraic reduction of n-person games to 3-person games and some results by Yanovskaya concerning equilibria in the mixed extension of general non-cooperative games (in this volume), formulated an idea of categorical approach to game theory. According to his original intuition, non-cooperative games form a category, and the set of equilibrium points (and, perhaps, some other solution concepts) is a functor from this category. Although the realization of this program turned out to be more complicated than was perceived by Vorob'ev, the problem was successfully resolved by Lapitsky (see his paper in this book and his later contributions [47] and [48] in Russian), who constructed a general theory of categories of games and corresponding solution concepts as properly chosen functors.

It is common knowledge that game theory methods and models are the corner-stones of modern general equilibrium theory in economics (see e.g. [20], [23] and [15]). We just mention the famous core equivalence results by Aumann [2], [3] and Aumann-Shapley [5], linking together the Walrasian equilibria, the core and the Shapley value in nonatomic pure exchange economies. Vice versa, it is a well-known fact that the core and Nash equilibrium, the most popular and highly elaborated game theory concepts, can be traced back to the Edgeworth contract curve [10] and the Cournot equilibrium [7], respectively (not speaking much about many other notions designed by the economists about one-two hundreds years ago before these notions were properly incorporated into the game theory thesaurus). Therefore we would pay some attention in this book to equilibria in economic models. Unlike the pure game theory papers, most of the main results obtained by Soviet researchers during the 1960-1980s concerning applications of game theory to the economical equilibrium analysis, were either published directly in the international journals (e.g., Econometrica, Journal of Mathematical Economics, Mathematical Social Sciences and others), or are translated in English somewhere, mostly in Matekon (M.E.SharpInc., Armonk, NY 10504), a well-known journal that helpfully served to integrate many Soviet mathematical economists into the international academic community (see e.g. [52]). That is why we restrict ourselves by presenting merely a small number of Russian papers in mathematical economics that seem to be of rather strong interest till now, but has not been translated in English yet.

Since a brief survey of the papers on mathematical economics included in the book is given in the section below, we conclude with some core equivalence results, obtained in the former USSR for the Lindahlian type economies by Vasil'ev [71] and Makarov and Vasil'ev [51]. In their papers (in Russian and hardly available to the Western readers), seminal concepts of blocking in replicas of the pure public goods economies and economies with externalities were proposed and the validity of the Edgeworth conjecture on the shrinkability of the cores to the Lindahlian equilibria and so-called informational equilibria, respectively, has been proven (for more details, see also [52]).

2 Summary of the selected contributions

The twelve papers in this book can be subdivided into four groups: noncooperative game theory and social choice, cooperative game theory, bargaining theory and equilibrium theory. The two papers of Yanovskaya fit within the first group. In both papers mixed extensions of arbitrary binary relations are used as the key tool. Till the 1970s the probabilistic approach to decision-making was based mainly on expected utility which is applicable only for transitive preferences. Mixed extensions of non-transitive binary relations were begun to be investigated with the help of skew-symmetric bilinear theory of expected utility, introduced independently by Kiruta *et al.* [39] and Fishburn [11]. This theory was applied in both papers. In the first paper one probabilistic social choice rule on a finite set of alternatives is defined and an axiomatic characterization of the rule is given. It is shown that the set of maximal elements of this rule coincides with the set of all optimal mixed strategies of every player in the symmetric matrix game whose entries are the differences between the number of individuals prefering the row alternative above the column alternative and the number of individuals having the inverse opinions. In the second paper more general results on non-cooperative games with ordered and arbitrary outcomes are presented. In particular, theorems establishing nonemptiness of the sets of mixed equilibrium points for non cooperative games with ordered outcomes and also with non-transitive outcomes are proved. In the paper of Lapitsky a general theory of categories of games is developed and corresponding "equilibrium-type" solution concepts as properly chosen functors are given, see also the remarks on this topic in the previous section.

The second group of papers (about cooperative game) theory consists of three papers, that is two short papers of Diubin and one long paper of Vasil'ev. In the first paper of Diubin the results by Gillies [13] and Bondareva [27] on the stability of the core are strengthened. It is shown that the upper estimates of the characteristic function values in the class of symmetric estimates are the best possible. In the second paper it is shown that for cooperative games with a countable set of players the Shapley value can be determined as the unique measure on the set of players on which the minimum of some quadratic functional is attained. This is a generalization of a result of Keane [17] for finite TU-games.

Vasil'ev's paper is mostly devoted to the comparative analysis of some classes of nonsymmetric values, core allocations, generalized von Neumann-Morgenstern solutions and totally-contractual sets. It contains an extended survey of results obtained by the author on the problems in question and consists of three parts. In the first part a unified functional approach to the investigation of nonsymmetric analogues of the Shapley value, already mentioned in the previous section, is given. In the second part the so-called generalized VonNeumann-Morgenstern solutions (gNM-solutions, shortly) are introduced, based on the principle of sequential improvements of dominated alternatives. The notion of NM-rank, characterizing the number of improvements required to arrive at gNM-solution starting at the “most distant point", is proposed, and some gNM-existence theorems with evaluation of NM-rank are established. It is shown, that in contrast to the classic NM-solution, generalized NM-solutions always exist. Within the framework of a topological approach, inspired by the well-known Maschler-Peleg theory of dynamic systems [18], it is proven that for any balanced cooperative TU-game there exists a dynamic system, whose final set is globally stable and equals the core, while outside the core all the outcomes of the transition function dominate the current imputation. As a consequence one of the main results of the paper follows: for any cooperative TU-game there exists an NM-solution w.r.t. the limit transitive closure of the classic domination relation. This solution is unique and coincides with the core, if the latter is not empty, and can be chosen to contain a finite number of imputations otherwise. The third, concluding part of the paper contains a game-theoretical analysis of the totally-contractual sets, similar to that introduced by Makarov [50] in order to describe stable outcomes of some rather natural recontracting procedure in pure exchange economies. The structure of domination relations, induced by several rules of entering and breaking contract systems, is studied, and quite natural and mild conditions, providing the coincidence of the totally-contractual core and Walrasian equilibria, are established.

The third group of papers about bargaining theory consists of two papers by Pechersky and one paper by Kukushkin. In the first paper of Pechersky the linear bargaining solutions are characterized by efficiency and linearity. By that time such solutions had already been characterized by Myerson [19] and Thomson [24]. However, unlike these papers, Pechersky's paper does not impose any restriction (such as strict convexity or smoothness) on the bargaining sets. The solution is defined as the Steiner point of a contact set of feasible outcomes and the supporting hyperplane with a unit normal vector. In the second paper of Pechersky the superlinearity axiom is used for the first time and the existence of the superlinear solution for n-person bargaining games satisfying continuity, symmetry, weak Pareto optimality, superlinearity, and translation covariance, was proved. Later Perles and Maschler [22] defined and characterized the super-additive solution for two person bargaining games satisfying Pareto optimality, scale transformation covariance, super-additivity, symmetry and continuity. However, their solution cannot be extended to n-person bargaining games for $n > 2$. The paper of Kukushkin considers the following problem: two agents derive transferable utility from alternatives in a feasible set, but the choice is made by a third agent, who is only concerned in payments from the interested agents. The concept of equilibrium is given and it is shown that there exists an equilibrium that Pareto dominates (in the viewpoint of the two interested agents) any other equilibrium. In the terminology of Bernheim and Whinston [6] it is a menu auction.

The concept of equilibrium plays a key role in game theory. The fourth and last group of papers is on related equilibrium concepts in economic theory and consists of three papers by Zak, Shmyrev, and Danilov and Sotskov. The paper of Zak may seem to be rather technical, but is very interesting from both a theoretical and applied point of view. A differential approach is proposed to develop an original systematic consideration of stable and unstable economies. In the sense of Aumann and Peleg [4], a pure exchange economy is unstable when one of the participants can improve its position in equilibrium by throwing out, or hiding, a part of its initial endowment (see also [12] for a coalitional type of this phenomenon, which is less significant in competitive environments). The paper contains one of the main results in the field, establishing the stability of Walrasian economies with normal demand and equilibrium gross substitutability. It is also worth to note that an advanced differential topology technique, applied and partially developed by the author to study individual strategy-proofness aspects of Walrasian equilibrium, may be of particular importance to extend some well-known (at least in Russia) results of Polterovich and Spivak [63] on coalitional stability, as well. Shmyrev's paper deals with a new approach for searching Walrasian equilibrium in a linear pure exchange economy with fixed incomes. The approach is based on the consideration of a special linear parametric transportation problem, with prices to be taken as parameters. As it was shown in a later article [64], in comparison with the first well-known finite algorithm by Eaves [9], Shmyrev's finite methods for searching equilibrium, being elaborated by applying both complementarity theory and rather efficient linear programming-type procedures, seem to be more practical, even for the most general linear exchange models. Finally, the paper of Danilov and Sotskov concerns some analogies of the so-called equilibrated states as introduced by Polterovich [62]. A generalization of the famous Bergstrom-Ky-Fan theorem on maximal elements is given, and a unified approach to the existence theorems for both equilibrated states and the cores of cooperative games, based on this generalization, is proposed. Besides the cooperative game theory, the main results may also be of interest in some problems of optimal allocation of resources under rigid prices.

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