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Damage Mechanics and Micromechanics of Localized Fracture Phenomena in Inelastic Solids

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

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DAMAGE MECHANICS AND MICROMECHANICS OF LOCALIZED FRACTURE PHENOMENA IN INELASTIC SOLIDS

EDITED BY

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PREFACE

This book resulted from a series of lecture notes presented in CISM, Udine. It has two major contributions: Damage Mechanics and Localization in Inelastic Deformation. The course is intended to provide researchers and graduate students with a clear and thorough presentation of the recent advances in continuum damage mechanics for both metals and metal matrix composites as well as the micromechanics of localization in inelastic solids.

Damage Mechanics and Applications: *The major goal of this part of the course is to present many of the different constitutive damage models that have recently appeared in the literature. Another goal is to clearly present the different approaches to this topic in a single complete course that is easily accessible to researchers and graduate students in civil engineering, mechanical engineering, engineering mechanics, aerospace engineering, and material science. The course material was delivered in well-organized lectures that started with the preliminaries and proceeded to advanced topics.*

Micromechanics of Localization in Inelastic Solids and Applications: *The main objective of this contribution is to discuss very efficient procedures of the numerical investigation of localized fracture in inelastic solids generated by impact-loaded adiabatic processes. Particular attention is focused on the proper description of a ductile mode of fracture propagating along the shear band for high impact velocities. This procedure of investigation is based on utilization the finite element or finite difference methods for regularized thermo-elasto-viscoplastic constitutive model of damage material using both rate dependency and non-local approaches.*

It is noteworthy to stress that all considered numerical examples are motivated by recent experimental observations. Qualitative comparison of numerical results with experimental observation data is presented. The numerical results obtained have proven the usefulness of the thermo-elasto-viscoplastic theory in the numerical investigation of dynamic shear band propagation and localized fracture using rate dependency and nonlocal formulations.

Invited Lecturers:
Tomasz Lodygowski (Poznan University of Technology, Poland), Piotr Perzyna (Institute of Fundamental Technological Research, Poland), Antonio Rinaldi (University of Rome "Tor Vergata", Italy), and George Z. Voyiadjis (Louisiana State University, USA).

Coordinator: *George Z. Voyiadjis*

*The authors present in **Part I** the micro-mechanical damage model that accounts for the nonlocal microscopic interactions in the simulation of metal/composite impact and severe contact stress problems. This is achieved by introducing the contributions of damage and its corresponding gradients in the virtual power relations as measures of micro motion of damage within the bulk. By using these internal state variables together with displacement and temperature, the constitutive model is formulated with state laws based on the free energies and the complimentary laws based on the dissipation potentials.*

In this work also the mechanics of small damage is presented using a consistent mathematical and mechanical framework based on the equations of damage mechanics. In this regard, the new scalar damage variable is investigated in detail. The investigation in this work has been carried out for seeking a physical basis is sought for the damage tensor (M) that is used to link the damage state of the material with effective undamaged configuration. The approach presented here provides for a strong physical basis for this missing link. In particular, the authors have made an important link between the damage tensor and fabric tensors.

Computational aspects of the presented theory are also discussed. Numerical integration algorithms, verification and validation process of the theory are discussed. The finite element simulations are also performed by implementing the presented model in the commercial finite element code ABAQUS (6.8.3) as a user defined subroutine (VUMAT).

*The outline of the material presented in **Part I** is as follows: in section 2, general mechanisms of the perforation and penetration mechanisms are discussed. A coupled rate-dependent (viscoplasticity) continuum damage theory is presented. In section 3, mechanics of small damage in fiber reinforced composite material is presented. In section 4, a comparative study has been made on the damage variables of the continuum. In section 5, computational aspects of the theory are discussed. The elastic predictor and coupled viscoplastic-viscodamage corrector algorithm that allows for total uncoupling of geometrical and material nonlinearities are presented. The nonlinear algebraic system of equations is solved by consistnt linearization and the Newton–Raphson iteration. A derivation of a new definition for the consistent tangent stiffness matrix that is essential to maintaining the asymptotic quadratic rate of convergence is also presented in section 5. In section 6, numerical simulations of material instability are introduced in order to validate and test the proposed finite strain approach along with the proposed algorithm and its implementation in the ABAQUS finite element code. In section 7, various numerical examples are presented in order to validate the reliability and capability of the theory in simulating various impact and contact stress problems. Experimental comparisons of the adiabatic shear localizations between the proposed model simulations and other independent results are presented. Effect of initial temperatures and loading rates on the development of shear localizations is also* *investigated in this section. Model capabilities are preliminarily illustrated for the dynamic localization of inelastic flow in adiabatic shear bands and compared with the experimental results of steel plates by deformable blunt projectiles at various impact speeds.*

A handful of methodologies can be pursued in alternative to (and in combination with) continuum mechanics to develop advanced damage models that are intrinsically suited to address complex issues, such as strain localization phenomena and sample-size dependence of structural failure, which are intimately related to microscale phenomena. Statistical Damage Mechanics (SDM) is one of them.

In ***Part II*** *a brief discourse about advances in SDM introduces this methodology and shows its potential. In general, SDM is a multidisciplinary field that seeks to devise non-conventional approaches to fracture and damage by means of discrete damage models accounting for the essential microscale properties of a material or structural system. The ultimate goal is the development of multiscale methodologies that can reliably predict the materials macroscale response in consideration of the microstructural evolution caused by the damage process. Starting from the lower length scale characteristic of the microstructure, the SDM multiscale approach delivers a surprising amount of new insight and makes for a precious companion to the micromechanics methods discussed previously in the book. The bottom-up approach proves to be very effective in understanding the physics of the damage localization process. Although SDM is not yet a mature discipline, by now it has a clear footprint, after nearly two decades of research. This part of the book describes the state of the art and exposes recent trends in SDM for consideration and discussion by the larger solid mechanics community. Some of the concepts discussed herein □yet at their infancy □possess the potential to develop into full blown, innovative, and successful engineering tools. Designing the materials from the microstructure is already a current industry practice to achieve superior damage tolerance, partly driven by widespread usage of composites. SDM may help to do it even better. Another brand new application field for SDM is represented by nanotechnology, especially as far as the design of small (micro- and nano-sized) mechanical systems, such as MEMS and NEMS, where sample-size effects and discreteness are pervasive aspects of the design. After the introductory section 1, Part II unfolds a logical overview of concepts and solutions that begins with simpler one-dimensional models (i.e. the parallel bar system or PBS) in section 2 and ends with multidimensional systems (i.e. spring networks) in section 3. Great emphasis is placed on the derivation of physically-based definitions of the damage parameter for quasi-brittle systems that stem from a pervasive statistical rationale. In section 2, though, a coupled damage-plasticity model is also covered for ductile systems, limited to the one dimensional case. Finally, the last portion of section 3 focuses on sample-size effects in quasi-brittle materials and* *shows how to obtain and harness scaling laws into fractal-based constitutive relations for engineering applications.*

The main objective of ***Part III*** *is to show the broad application of the thermodynamic theory of elasto-viscoplasicity for the description of important problems in modern manufacturing processes, and particularly for meso-, micro-, and nano-mechanical issues. This description is particularly needed for the investigation by using the numerical methods how to avoid unexpected plastic strain localization and localized fracture phenomena in new manufacturing technology.*

In the first part the development of thermo-elasto-viscoplastic constitutive model of a material which takes into consideration the induced anisotropy effects as well as observed contribution to strain rate effects generated by microshear banding is presented. Analysis of recent experimental observations concerning investigations of fracture phenomena under dynamic loading processes suggests that there are two kind of induced anisotropy: (i) the first caused by the residual type stresses produced by the heterogeneous nature of the finite plastic deformation in polycrystalline solids and (ii) the second called the fracture induced anisotropy generated by the evolution of the microdamage mechanisms. It is noteworthy to stress that both these induced anisotropy effects are coupled and this property has to be taken into account in the proposed constitutive description. On the other hand we very well know from recent experimental observations concerning investigation of dynamic loading processes that formation of microshear bands influences the evolution of microstructure of a material. The basic assumption is that microshear banding contributes to viscoplastic strain rate effects.The model is developed within the thermodynamic framework of a unique, covariance constitutive structure with a finite set of the internal state variables. A set of internal state variables consists of one scalar and two tensors, namely the equivalent inelastic deformation, the second order microdamage tensor with the physical interpretation that defines the volume fraction porosity and the residual stress tensor (the back stress). To describe suitably the influence of both induced anisotropy effects and the stress triaxiality observed experimentally the new kinetic equations for the microdamage tensor and for the back stress tensor are proposed. To describe the contribution to strain rate effects generated by microshear banding we propose to introduce certain scalar function which affects the relaxation time in the viscoplastic flow rule. The relaxation time is used as a regularization parameter. Fracture criterion based on the evolution of the anisotropic intrinsic microdamage is formulated. The fundamental features of the proposed constitutive theory have been carefully discussed.

The objective of the second part is to discuss very efficient procedure of the numerical investigation of localized fracture in inelastic solids generated by impact-loaded adiabatic processes. Particular attention is focused on the proper description of a ductile mode of fracture propagating along the shear band for *high impact velocities. This procedure of investigation is based on utilization the finite difference and finite element methods for regularized thermo-elasto-viscoplastic model of damaged material. The viscoplastic regularization procedure assures the stable integration algorithm by using the finite difference or finite element methods. Particular attention is focused on the well-posedness of the evolution problem (the initial boundary value problem) as well as on its numerical solutions.*

In ***Part I*** *the behavior of selected brittle materials and structures (concrete and masonry) subjected to explosive loadings is discussed. For concrete the accepted Cumulative Fracture*

Criterion (CFC) is exposed. It describes the degradation of the material under fast dynamic processes accompanied by the strong waves propagation phenomenon and large strain rates of deformation. To overcome the computational difficulty in the analyses of such complex problems, the sub-modeling technique as well as splitting of the calculations into two separate parts: analysis of acoustic wave in the air and the propagation of stresses in structures, are used. Some instructive numerical examples of concrete and masonry walls are in focus of the presentation. The numerical tools and computer simulations allow for proper estimation of the structures safety and for taking the design decisions on how to ensure their expected strength.

George . Voyiadjis , Baton Rouge, 2010

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The authors dedicate this volume to the Memory of Dusan Krajcinovic

a Friend, Mentor and Inspirator

CONTENTS

PART I

Consistent Non Local Coupled Damage Model and Its Application in Impact Response of Composite Materials

by

George Z. Voyiadjis , Babur Deliktas and Peter I. Kattan

Consistent Non Local Coupled Damage Model and Its Application in Impact Response of Composite Materials

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Abstract: In this work also the mechanics of small damage are also presented using a consistent mathematical and mechanical framework based on the equations of damage mechanics. In this regard, the new scalar damage variable is investigated in detail. The investigation in this work has been carried out for seeking a physical basis is sought for the damage tensor $[M]$ that is used to link the damage state of the material with effective undamaged configuration. The approach presented here provides for a strong physical basis for this missing link. In particular, the authors have made an important link between the damage tensor and fabric tensors.Computational aspects of the presented theory are also discussed. Numerical integration algorithms, verification and validation process of the theory are discussed. The finite element simulations are also performed by implementing the presented model in the commercial finite element code ABAQUS [6.8.3] as a user defined subroutine (VUMAT).

1. PROLOGUE

The recent advances in aerospace and war capabilities have made necessary the modification of the design of structures that are proned to hypervelocity impact in order to increase their resistancy againts the penetration and perforation by projectiles with much higher impact energies. In this respect, high performance materials need to be developed so that they can offer significant advantages over the currently used materials. Metals and composites are among those materials that are oftenly used in various part of the structural components of the engineering structure in aerospace, automible and defense industries. In general, these materials are subjeced to tremendeous microstructural changes due to perforation and penetration phenomena and have complex material response due to a strong rate and temperature dependence when deformed non-uniformly into the inelastic range. Therefore, the high velocity impacting mechanism needs to be understood properly in order to be able to design materials of high ballistic resistant response. However, the exact mechanism by which the impacting target materials undergo fracture and ablation is a relatively complex process (Zukas, (1990). Generally, strong shock wave-material interactions are generated and propagated along both the projectile and the target, which can lead to fracture at low global inelastic strains.

The key role in the numerical simulation of the impact damage related problems is the accurate modeling of the material behavior at high strain rates and temperatures. Many researchers, therefore, have investigated the material failure mechanism during high velocity impact conditions with the ultimate goal of developing a micromechanical constitutive model that can effectively simulate the impact damage problem (Armstrong and Zerilli, 1994, Bammann, et al., 1990, Borvik, et al., 2004, Borvik, et al., 2006, Borvik, et al., 2002, Camacho and Ortiz, 1997, Curran, et al., 1987, Eftis, et al., 2003, Johnson and Cook, 1985, Steinberg and Lund, 1989). It is noted that none of these constitutive models address the problem of describing high shock compression and subsequent material degradation and failure in which the latter is expressed as an evolving micro-flaw having a damage rate determined from micro-mechanical analysis. Moreover, these models cannot consider the actual sizes, shapes, and orientations of the individual micro-voids and micro-cracks, which may have a predominant influence on the mechanical response of the material. The mechanical behavior of these applications cannot be characterized by classical (rate-independent) continuum theories because they incorporate no ‘material length scales.’ It is therefore necessary to develop a coupled rate-dependent (viscoplasticity) continuum damage theory bridging the gap between the classical continuum theories and the microstructure simulations.

The authors of this work have recently recognized the need for a micro-mechanical damage model (Abu Al-Rub and Voyiadjis, 2006, Voyiadjis, et al., 2003, Voyiadjis, et al., 2004) that accounts for the nonlocal microscopic interactions between material points (i.e. to take into account the influence of an internal state variable at a point on its neighborhood) in the simulation of metal impact problems This nonlocal microdamage model is formulated based on the enhanced gradient-dependent theory which is successful in explaining the size effects encountered at the micron scale and in preserving the well-posed nature of the initial boundary value problem that governs the solution of material instability triggering strain localization. Moreover, the viscoplasticity theory (rate-dependency) allows the spatial difference operator in the governing equations to retain its ellipticity and consequently the initial boundary value problem is hence well-posed (Batra, 2006, Batra and Chen, 1999, Batra and Kim, 1988, Batra and Kim, 1990, Loret and Prevost, 1990, Molinari, 1997, Needleman, 1988, Sluys, 1992). However, the gradient dependent theories enhance a stronger regularization of the localization problem than the rate-dependent theory. Moreover, the rate-dependent theory cannot explain the size effect of the microdamage zone (i.e. the void/crack size and spacing) on the material failure while the gradient theory can address that.

Therefore, the objective of this work is to present for high speed impact damage problems a novel microdamage constitutive model that possesses several material length scales. This model can be used to produce physically meaningful and numerically converging results within strain localization computations by finite element codes. Moreover, the algorithmic aspects and numerical implementation of this model in finite element codes are presented in this chapter.

The outline of the presented materials is as follows: in section 2, general mechanisms of the perforation and penetration mechanisms are discussed. A coupled rate-dependent (viscoplasticity) continuum damage theory is presented. In section 3, mechanics of small damage in fiber reinforced composite material is presented. In section 4, a comaparative study has been made on the damage variables of the continuum. In section 5, computational aspects of the theory are discussed. The elastic predictor and coupled viscoplastic-viscodamage corrector algorithm that allows for total uncoupling of geometrical and material nonlinearities are presented. The nonlinear algebraic system of equations is solved by consistnt linearization and the Newton–Raphson iteration. A derivation of a new definition for the consistent tangent stiffness matrix that is essential to maintaining the asymptotic quadratic rate of convergence is also presented in section 5. In section 6, numerical simulations of material instability are introduced in order to validate and test the proposed finite strain approach along with the proposed algorithm and its implementation in the ABAQUS finite element code. In section 7, various numerical example are presented in order to validate the reliability and capability of the the theory in simulating various impact and contact stress problems. Experimental comparisons of the adiabatic shear localizations between the proposed model simulations and other independent results are presented. Effect of initial temperatures and loading rates on the development of shear localizations is also investigated in this section. Model capabilities are preliminarily illustrated for the dynamic localization of inelastic flow in adiabatic shear bands and compared with the experimental results of Borvik et al. (2002) for the perforation of 12mm thick Weldox 460E steel plates by deformable blunt projectiles at various impact speeds.

2. Nonlocal Formulation of High Velocity Impact Induced Damage

2.1. Penetration and perforation mechanisms

The penetration and perforation mechanisms are interdisciplinary subjects and are based on the laws of conservation and compatibility. Penetration implies movement of a projectile into a target. Phenomenologically, as illustrated in

Figure 1, the penetration can be viewed as a process to generate a coneshaped macro crack in the material, in which, the kinetic energy of the penetrator is dissipated.

Perforation implies penetration all the way through a target. Projectile exit through a finite target is often accompanied by delamination and plugging. Delamination refers to a tensile failure parallel to the target rear surface and is often initiated by spall. However, it can also occur in quite thick targets, where it seems to be caused by *shear bands* around the projectile head that can propagate as cracks near the exit surface (Chelluru, 2007). In general, a penetration equation is a set of equations that are used to predict the outcome of an important event, such as the residual velocity or mass of the projectile after impact. Empirical penetration equations are essentially curve fits and take the general form $f(x_1, x_2, \ldots x_n)$ where $x_1, x_2, \ldots x_n$ are parameters such as projectile size and target thickness.

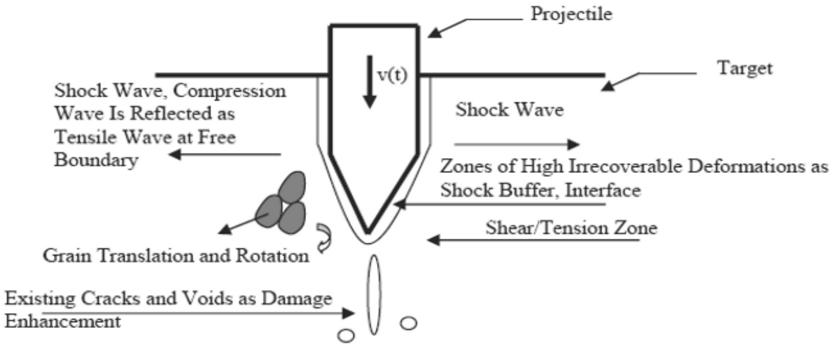

Figure 1. Schematics Illustration of the Penetration Process

It is important to note that results of these equations are only accurate if the case for which it is being used is close to that of the experimental data to which the equations are fit (Christiansen and Friesen, 1997). Prediction of the Ballistic limit is a very difficult task for which complete success may not be possible. Therefore, extensive experimental work is being done to date to understand which parameters affect the impact in composite structures. The analysis is still complex because the events that occur at the projectile/target interface are somewhat unknown. Although many studies have been performed, only highly controlled velocities, shape, sizes and trajectories have been examined. As a result, numerous approximations and assumptions must be made in order to apply to these analyses. Impact is a much localized phenomenon. Stress and strain effects are usually limited to within 3-6 projectile diameters of the impacted zone (Chelluru, 2007). Impacted target materials may fail by a combination of several modes including spalling, plugging, petaling, ductile or brittle fracture, and adiabatic shearing.

Spalling is the tensile failure of the target material due to reflection of the initial compressive waves from the far side of the target. Failure by spalling can occur on either the front or back of a target and is characterized by the formation of petals or ejects. In the event of impact there is an exchange of energy that takes place:

$$E_{tans} = E_{plate}^{internal} + E_{plate}^{kinetic} + E_{eroded}^{internal} + E_{eroded}^{kinetics} \quad (1)$$

Law of conservation is observed in any physical phenomena. Kinetic energy of the projectile is spent in raising the internal energy and kinetic energy of the plate and some part of the energy is lost in the form of eroded material. The amount of energy dissipated also differs with geometry. Blunt projectiles like cylinders are found to cause plugging because of pure shear failure, while the conical projectiles are found to cause petaling effects.

High velocity impact will localize compression of the composite and subsequently shearing the fiber and spalling of the resin during impact. Once the projectile has slowed, the composite deforms causing fiber stretching, pullout and delamination of the composite layers and thus lower the load carrying capacity.

2.2 A coupled rate-dependent (viscoplasticity) continuum damage theory

The theoretical model presented in this section is considered within a thermodynamic framework, where it is assured that the principles of thermodynamics are satisfied. Therfore, the virtual power relations are first defined and the principle of virtual power along with varitational formulations are used to develop the governing differential equations and their corresponding boundary conditions of the proposed theory. The principle of virtual power used by Voyiadjis and Deliktas (2009, 2009) is different than those used by Voyiadjis and Co-workers (Abu Al-Rub and Voyiadjis, 2004, Abu Al-Rub and Voyiadjis, 2006, Abu Al-Rub, et al., 2007, Dorgan and Voyiadjis, 2003, Dorgan and Voyiadjis, 2006, Voyiadjis and Abu Al-Rub, 2005, Voyiadjis and Abu Al-Rub, 2007, Voyiadjis, et al., 2003, Voyiadjis and Almasri, 2008, Voyiadjis and Deliktas, 2000, Voyiadjis and Deliktas, 2000), Gudmundson and co-workers (Fredriksson and Gudmundson, 2005, Fredriksson and Gudmundson, 2007, Gudmundson, 2004, Nygards and Gudmundson, 2004, Tjernlund, et al., 2006), Willis and coworkers (Aifantis and Willis, 2005, Aifantis and Willis, 2006, Fleck and Willis, 2009, Fleck and Willis, 2009) and, Gurtin and coworkers (Anand, et al., 2005, Bittencourt, et al., 2003, Cermelli, et al., 2004, Gudmundson, 2004, Gurtin, 2000, Gurtin, 2004, Gurtin and Anand, 2009, Gurtin and Needleman, 2005) where the principle of virtual power is modified by adding the contributions from damage and its corresponding gradients as a measure of micro motion of damage within the bulk. In addition two internal state variables are introduced on the contact interface, one measuring the tangential slip and another measuring the wear. By using these internal state variables together with displacements and temperature, the constitutive model is formulated with state laws based on the free energies and the complimentary laws based on the dissipation potentials. This model provides a potential feature for enabling one to relate the non-local continuum plasticity and damage of the bulk material to friction and wear at the contact interfaces.

One now defines a region $V \subset \Re^d$ (d=2, 3) with a piecewise smooth boundary Ω that occupies a continuously deformable body (Figure 2). The boundary Ω is divided into three disjoint parts; Ω_t is the part of the boundary where tractions are prescribed whereas the displacements are prescribed at the boundary Ω_u, and the unilateral contact interface is defined by the boundary Ω_c. The contact interfaces surface energy is considered as material boundary following the concept of Voyiadjis and Deliktas (2009) and enhanced by the following pioneering work of Fremond (1996) and its corresponding modification by Iremena et al.(2003).

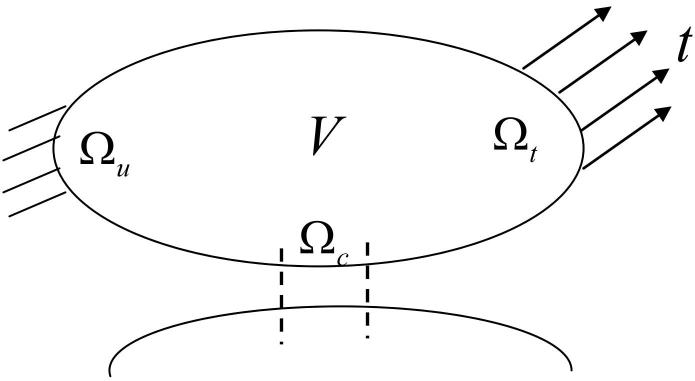

Figure 2. A Deformable Body with Unilateral Contact

The modified form of the internal virtual power is expressed here as follows:

$$P_{\text{int}} = \int_V \left(\sigma_{ij}\dot{\varepsilon}_{ij}^{e} + X_{ij}\dot{\varepsilon}_{ij}^{p} + S_{ijk}\dot{\varepsilon}_{ij,k}^{p} + Y_{ij}\dot{\phi}_{ij} + \Gamma_{ijk}\dot{\phi}_{ij,k} \right) dV - \int_{\Omega_c} p_i v_i \, d\Omega + \int_{\Omega_c} M_{ij}^{c}\dot{\varepsilon}_{ij}^{c} d\Omega_c + \int_{\Omega_c} \aleph_{ij}^{c}\dot{\phi}_{ij}^{c} d\Omega_c + \int_{\Omega_c} Q^{c}\varphi^{c} d\Omega_{cc} + \int_{\Omega_c} q_i v_i^{t} d\Omega_c \tag{2}$$

The superscripts are used to describe c for contact, n for normal and t for traction. The tensor X_{ij} is the driving stress work conjugate to the plastic strain, $\dot{\varepsilon}_{ij}^{p}$, and S_{ijk} is the micro-stress work conjugate to the plastic strain gradient, $\dot{\varepsilon}_{ij,k}^{p}$. The tensors Y_{ij} and Γ_{ijk} are the damage related internal forces work conjugate to ϕ_{ij} and $\phi_{ij,k}$, respectively. The external power is expressed as

$$P_{ext} = \int_{\Omega_t} t_i v_i \, d\Omega_t + \int_{\Omega_t} m_{ij}^{t}\dot{\varepsilon}_{ij}^{p} d\Omega_t + \int_{\Omega_c} \eta_{ij}^{t}\dot{\phi}_{ij} d\Omega_t \tag{3}$$

Furthermore, forces associated with friction and wear are also introduced. The non local differential equation of the flow rule and its relevant nonstandard boundary conditions are derived by using the principle of virtual power $P_{ext} - P_{\text{int}} = 0$ and imposing integration by parts, and the divergence theorem

$$\begin{aligned} &\int_{\Omega_t} \left(t_i - \sigma_{ij} n_j \right) v_i d\Omega_t + \int_{\Omega_t} \left(-p_i - \sigma_{ij} n_j \right) v_i d\Omega_t + \\ &\int_V \left(\sigma_{ij,j} \right) v_i dV + \int_V \left(\tau_{ij} - X_{ij} + S_{ijk,k} \right) \dot{\varepsilon}_{ij}^{p} dV + \int_V \left(\Gamma_{ijk,k} - Y_{ij} \right) \dot{\phi}_{ij} dV + \\ &\int_{\Omega_c} \left(-M_{ik} - S_{ijk} n_k \right) \dot{\varepsilon}_{ij}^{p} d\Omega_c + \int_{\Omega_c} \left(-\aleph_{ij} - \Gamma_{ijk} n_k \right) \dot{\phi}_{ij} d\Omega_c + \\ &\int_{\Omega_c} Q^{c}\dot{\varphi} d\Omega_c + \int_{\Omega_c} q_i v_i^{t} d\Omega_c + \int_{\Omega_t} \left(m_{ik} - S_{ijk} n_k \right) \dot{\varepsilon}_{ij}^{p} d\Omega_c + \int_{\Omega t} \left(\eta_{ij} - \Gamma_{ijk} n_k \right) \dot{\phi}_{ij} d\Omega_t = 0 \end{aligned} \tag{4}$$

From Eq. (3) one can obtain the balance laws as follows

$$\left. \begin{aligned} \sigma_{ij,j} &= 0 \\ \tau_{ij} &= X_{ij} + S_{ijk,k} \\ \Gamma_{ijk,k} - Y_{ij} &= 0 \end{aligned} \right\} \text{ in } V \tag{5}$$

Standard and non standard boundary conditions on the boundaries, Ω_t and Ω_c are obtained respectively

$$\left.\begin{aligned} t_i &= \sigma_{ij} n_j \\ m_{ij} - S_{ijk} n_k &= 0 \\ \eta_{ij} - \Gamma_{ijk} n_k &= 0 \end{aligned}\right\} \text{ on } \Omega_t \tag{6}$$

$$\left.\begin{aligned} M_{ik} + S_{ijk} n_k &= 0 \\ \aleph_{ij} + \Gamma_{ijk} n_k &= 0 \\ -p_i &= \sigma_{ij} n_j \\ Q^c &= 0 \\ q_i &= 0 \end{aligned}\right\} \text{ on } \Omega_c \tag{7}$$

Consequently forces Q^c and q_i are simply equal to zero. In addition to balance laws one needs constitutive assumptions that couple internal forces to the state variables. A general framework for the thermodynamical forces and state variables are obtained by using free energies and the dissipation potential in the style of the standard material. Such an approach is consistent with satisfying the second law of thermodynamics which states that the rate change of the total free energy must be less than the external power i.e. $\dot{\Upsilon} \le P_{ext}$, where the total free energy can be defined as

$$\Upsilon = \int_V \rho \psi_v dV + \int_{\Omega_c} \psi_c d\Omega_c \tag{8}$$

where ρ is the mass density, $\psi_v = \psi_v\left(\varepsilon^e, \varepsilon^p, \phi, \nabla\varepsilon^p, \nabla\phi\right)$ is the free energy in the volume, V and $\psi_c = \psi_c\left(u, \varpi, \varphi, \varepsilon^p, \phi\right)$ is the free energy on the contact surface Ω_c. Thermodynamical Clausius-Duhem inequalities for both the bulk and the contact interface are given as follows

$$\begin{gathered} \Phi_v = \sigma : \dot{\varepsilon}^e + X : \dot{\varepsilon}^p + Y : \dot{\phi} + S : \nabla\dot{\varepsilon}^p + \Gamma : \nabla\dot{\phi} - \rho\dot{\psi}_v \ge 0 \\ \Phi_c = p : \dot{u} + \beta\dot{\varepsilon}^p + \varsigma\dot{\phi} + q_t\dot{\varpi} + \chi\dot{\varphi} - \dot{\psi}_c \ge 0 \end{gathered} \tag{9}$$

These contact surfaces interpret the grain interactions with reference to separation and sliding as an additional source of form of the inelastic deformation. The vector $\boldsymbol{p}$ is the interface traction force vector. In addition χ and ς are two new thermodynamic forces work conjugate to the measure of grain boundary separation defined by the flux, φ, and the measure of the grain boundary sliding defined by the flux, ϖ, respectively. The term β represents the interfacial strength parameter associated with the plastic strain jump at the sub interface surface.

The number of unknown internal state variables of the model can be determined by solving the constraint minimization problem of the energy dissipation by using the Lagrange multiplier method and assuming the existence of the constraint surfaces for plasticity and damage respectively such that

$$L_p\left(X,S\right)=\Phi_v-\lambda^p f\left(X,S\right)$$
$$L_d\left(Y,\Gamma\right)=\Phi_v-\lambda^d g\left(Y,\Gamma\right) \qquad (10)$$
$$L_c\left(p,\beta,\varsigma,q_t,\chi\right)=\Phi_c-\lambda^c f\left(p,\beta,\varsigma,q_t,\chi\right)$$

The mathematical aspects of the thermodynamical consideration of the theory for such form of the free and dissipation potential, dissipation inequalities, minimization problem and existence of the uniqueness of the solutions are discussed in detail in recent studies by Voyiadjis and Deliktas (2009). Here, the governing equations for the coupled viscoplastic damage behavior are defined by the following constitutive relations :

$$\sigma_{ij}=E_{ijkl}\left(\phi\right)\varepsilon_{kl}^{e} \qquad (11)$$
$$\tau_{ij}=X_{ij}+S_{ijk,k}$$
$$f=\left(X_{ij}X_{ij}+\ell^{-2}S_{ijk}S_{ijk}\right)^{1/2}-\sigma_f\left(\dot{E}^p\right)\leq 0$$
$$\dot{\lambda}^{cp}\geq 0 \quad \dot{\lambda}^{cp}f^c\geq 0 \quad \dot{\lambda}^{cp}\dot{f}^c=0$$
$$g=\left(Y_{ij}Y_{ij}+\ell_d^{-2}\Gamma_{ijk}\Gamma_{ijk}\right)^{1/2}-\sigma\left(\dot{\kappa}\right)\leq 0$$
$$\Gamma_{ijk,k}-Y_{ij}=0$$
$$\dot{\lambda}^d\geq 0 \quad \dot{\lambda}^d g\geq 0 \quad \dot{\lambda}^d\dot{g}=0$$

and for the contact interface

$$M_{ik} + S_{ijk} n_k = 0 \tag{12}$$
$$f^c = \left(M_{ij} M_{ij}\right)^{1/2} - \sigma_f\left(\dot{\varepsilon}^p\right) \le 0$$
$$\dot{\lambda}^p \ge 0 \quad \dot{\lambda}^p f^c \ge 0 \quad \dot{\lambda}^p \dot{f}^c = 0$$
$$\aleph_{ij} + \Gamma_{ijk} n_k = 0$$
$$g^c = \left(\aleph_{ij} \aleph_{uj}\right)^{1/2} - \sigma\ \left(\dot{\phi}\right) \le 0$$
$$\dot{\lambda}^d \ge 0 \quad \dot{\lambda}^d g^c \ge 0 \quad \dot{\lambda}^d \dot{g}^c = 0$$

where $\dot{E}^p$ is the effective nonlocal flow rate such that $\dot{E}^p = \sqrt{\dot{\varepsilon}^p_{ij}\dot{\varepsilon}^p_{ij} + \ell_p^2 \dot{\varepsilon}^p_{ij,k}\dot{\varepsilon}^p_{ij,k}}$. The nonlocal effective damage flow rate is defined by $\dot{\kappa} = \sqrt{\dot{\phi}_{ij}\dot{\phi}_{ij} + \ell_d^2 \dot{\phi}_{ij,k}\dot{\phi}_{ij,k}}$. In this constitutive modeling two characteristic material length scales are introduced. They are the plastic length scale, ℓ_p and the damage length scale, ℓ_d respectively. The term α is the thermal expansion coefficient; $\dot{T}$ is the rate of absolute temperature; $\boldsymbol{I}$ is the second order identity tensor; and $\boldsymbol{A}$ is the fourth order tensor defined as

$$\boldsymbol{A} = \frac{\partial \boldsymbol{M}^{-1}}{\partial \emptyset} : \boldsymbol{M} : \boldsymbol{\sigma} + \boldsymbol{E} : \boldsymbol{M} : \frac{\partial \boldsymbol{M}^{-1}}{\partial \emptyset} : \boldsymbol{E}^{-1} : [\boldsymbol{\sigma} + \alpha(T - T_r)\boldsymbol{I}] \tag{13}$$

The fourth order damage tensor $\boldsymbol{M}$ is the function of the second order damage tensor, $\emptyset$. Its explicit form can be found in the works of Voyiadjis and co-workers (Abu Al-Rub and Voyiadjis, 2003, Abu Al-Rub and Voyiadjis, 2005, Abu Al-Rub and Voyiadjis, 2006, Abu Al-Rub, et al., 2007, Dorgan and Voyiadjis, 2003, Dorgan and Voyiadjis, 2006, Dorgan and Voyiadjis, 2007, Dorgan and Voyiadjis, 2007, Kattan and Voyiadjis, 1993, Kattan and Voyiadjis, 1993, Kattan and Voyiadjis, 1996, Park and Voyiadjis, 1997, Park and Voyiadjis, 1998, Voyiadjis and Kattan, 1992, Voyiadjis and Park, 1997). The functional form of the flow rule of plasticity is given as follows :

$$\sigma(\dot{E}^p) = \sigma_y + R(\dot{E}^p)\left[1 + \left(c_p(T - T_r)\dot{E}^p\right)^{1/m}\right] \tag{14}$$

where σ_y is the initial yield strength; and c_p is the specific heat. The evolution equation for isotropic and hardening laws are given as follows

$$\dot{R} = \mu \dot{E}^p q e^{-\mu E^p} \tag{15}$$

and

$$\dot{\boldsymbol{X}} = \frac{2}{3} C \dot{\boldsymbol{\varepsilon}}^p - \gamma \boldsymbol{X} \dot{E}^p + \beta \dot{\boldsymbol{\sigma}} \tag{16}$$

Voyiadjis and Abu Al-Rub (2003) defined, q as

$$q = q_m + (q_m + q_o) e^{-2\mu q} \tag{17}$$

and $m, q_m, q_o,$, C, γ and β are the material constant to be calibrated from available experimental data. Similary one can define the flow rules of damage as follows

$$\sigma(\dot{\kappa}) = Y_o + K\big((\dot{\kappa})\big)\left[1 + \left(c_p (T - T_r)\dot{\kappa}\right)^{1/m}\right] \tag{18}$$

where Y_o is the initial damage threshold; $\left(\dot{\kappa} = \sqrt{\dot{\tilde{\boldsymbol{\phi}}} : \dot{\tilde{\boldsymbol{\phi}}}}\right)$; and the nonlocal damage tensor is defned as $\tilde{\boldsymbol{\phi}} = \boldsymbol{\phi} + {}^1/_2\, \boldsymbol{\ell}^2 \nabla^2 \boldsymbol{\phi}$ **.** The evolution of the damage anisotropic hardening equation is given by Voyiadjis and Deliktas (2000) as follows

$$K(\kappa) = \lambda \zeta \left(\frac{\kappa}{\lambda}\right)^{\xi} \delta_{ij} \tilde{\phi}_{\boldsymbol{ij}} + \delta_{ij} \lambda Y_o^2 \tag{19}$$

where λ is the Lame constant and is defined by Voyiadjis and Park (1997, 1995, Voyiadjis and Park, 1995, Voyiadjis and Park, 1997). The exponent, ζ represents the damage hardening parameter, and ξ denotes the damage growth rate. The influences of these parameters on the response of the material was studied by Voyiadjis and Deliktas (Voyiadjis and Deliktas, 1997)

3. Mechanics of Small Damage in Fiber-Reinforced Composite Materials

In this section the new concept of small damage is examined within the framework of continuum damage mechanics. In particular, special emphasis is given to a new damage variable that is defined in terms of the elastic stiffness of the material. Only the scalar case is studied here. The investigation of the new scalar damage variable and the new concept of small damage is carried out on fiber-reinforced composite materials. Furthermore, the two approaches to damage analysis in composite materials are re-examined in this work using the new damage variable. These are the well known overall and local approaches established in the literature of damage mechanics. It is noted that the examination of these two approaches that is presented here applies to both small and large damage mechanics. Finally, the two approaches are compared mathematically and are shown to be equivalent.

The damage variable (or tensor), based on the effective stress concept, represents average material degradation which reflects the various types of damage at the micro-scale level like nucleation and growth of voids, cracks, cavities, micro-cracks, and other microscopic defects (Budiansky and O'Connell, 1976, Krajcinovic, 1996, Lubarda and Krajcinovic, 1993). For the case of isotropic damage mechanics, the damage variable is scalar and the evolution equations are easy to handle. However, it has been shown by Ju (1990) and Cauvin and Testa (1999) that two independent damage variables must be used in order to describe accurately and consistently the special case of isotropic damage. It has been argued (Lemaitre, 1984) that the assumption of isotropic damage is sufficient to give good predictions of the load carrying capacity, the number of cycles or the time to local failure in structural components. However, the development of anisotropic damage has been confirmed experimentally (Chow and Wang, 1987, Lee, et al., 1985) even if the virgin material is isotropic. This has prompted several researchers to investigate the general case of anisotropic damage (Kattan and Voyiadjis, 1996, Kattan and Voyiadjis, 2001, Murakami, 1983, Voyiadjis and Kattan, 2006) In continuum damage mechanics, usually a phenomenological approach is adopted. In this approach, the most important concept is that of the Representative Volume Element (RVE). The discontinuous and discrete elements of damage are not considered within the RVE; rather their combined effects are lumped together through the use of a macroscopic internal variable. In this way, the formulation may be derived consistently using sound mechanical and thermodynamic principles (Doghri, 2000, Hansen and Schreyer, 1994, Luccioni and Oller, 2003).

Kachanov(1958) and Rabotnov (1969) introduced the concept of effective stress for the case of uniaxial tension. This concept was later generalized to three-dimensional states of stress by Lemaitre (1970) and Chaboche (1981). Let $\boldsymbol{\sigma}$ be the second-rank Cauchy stress tensor and $\overline{\boldsymbol{\sigma}}$ be the corresponding effective stress tensor. The effective stress $\overline{\boldsymbol{\sigma}}$ is the stress applied to a fictitious state of the material which is totally undamaged, i.e. all damage in this state has been removed. This fictitious state is assumed to be mechanically equivalent to the actual damaged state of the material. In this regard, one of two hypotheses (elastic strain equivalence or elastic energy equivalence) is usually used (Lemaitre, 1984, Sidoroff, 1981).

Ju (1990) pointed out that even for isotropic damage one should employ a damage tensor (not a scalar damage variable) to characterize the state of damage in materials. However, the damage generally is anisotropic due to the external agency condition or the material nature itself. Although the fourth-rank damage tensor (Murakami, 1983) can be used directly as a linear transformation tensor to define the effective stress tensor, it is not easy to characterize physically the fourth-rank damage tensor compared to the second-rank damage tensor.

In this work the new concept of small damage is examined within the framework of continuum damage mechanics. In particular, special emphasis is given to a new damage variable that is defined in terms of the elastic stiffness of the material. Only the scalar case is studied in this work. The scalar definition of the new damage variable was used recently by many researchers. The investigation of the new scalar damage variable and the new concept of small damage is carried out on fiber-reinforced composite materials.

Furthermore, the two approaches to damage analysis in composite materials are re-examined in this work using the new damage variable. These are the well known overall and local approaches established in the literature of damage mechanics. It is noted that the examination of these two approaches that is presented here applies to both small and large damage mechanics. Finally, the two approaches are compared mathematically and are shown to be equivalent.

3.1. New Scalar Damage Variable

This section addresses a new scalar damage variable that was used by researchers recently. For other damage variables, see the work of Lemaitre (1971, 1984) and Lemaitre and Chaboche (1990). This scalar damage variable ℓ may be defined in terms of the reduction in the elastic modulus as follows:

$$\ell = \frac{\overline{E} - E}{E} \tag{20}$$

where E is the elastic modulus in the damaged state while $\overline{E}$ is the effective elastic modulus (in the fictitious state) with $\overline{E} > E$ (see Figure 3). This damage variable was used recently by Celentano et. al (2004) and Nichols and Abell (2003) and Nichols and Totoev (1999). It was also used by Voyiadjis and Kattan (2009) in a comparative study of damage variables within the context of continuum damage mechanics. It should also be mentioned that Voyiadjis (1988) used a similar relation but in the context of elasto-plastic deformation.

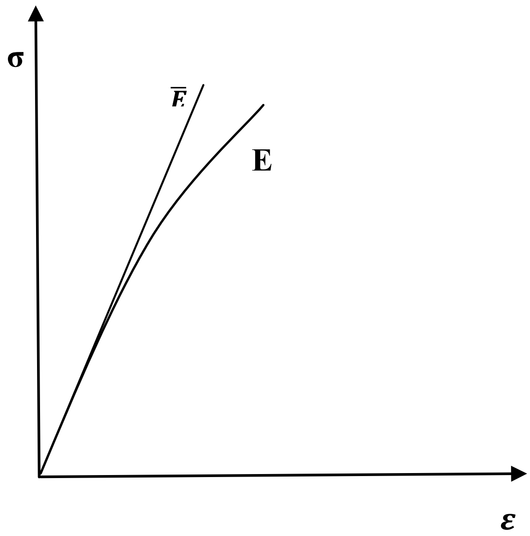

Figure 3. Damaged and Effective Moduli of Elasticity

The definition of the new damage variable of equation (20) may be re-written in the following more appropriate form:

$$\overline{E} = E(1+\ell) \tag{21}$$

It is clear from the definition in equation (21) that $\ell = 0$ when the body is undamaged, i.e. when $\overline{E} = E$. For a composite material consisting of a matrix and fibers, two local scalar damage variables may be similarly defined as follows:

$$\ell^m = \frac{\overline{E}^m - E^m}{E^m} \tag{22}$$

$$\ell^f = \frac{\overline{E}^f - E^f}{E^f} \tag{23}$$

where E^m and E^f are the elastic moduli for the matrix and fibers, respectively, while $\overline{E}^m$ and $\overline{E}^f$ are their effective counterparts. The two scalar local damage variables are denoted above by ℓ^m and ℓ^f for the matrix and fibers, respectively.

Using the hypothesis of elastic energy equivalence (see Voyiadjis and Kattan, 2009), one now can write the following transformation relation between the stress σ and the effective stress $\overline{\sigma}$:

$$\overline{\sigma} = \sigma\sqrt{1+\ell} \tag{24}$$

In the same way, another transformation relation can be obtained between the strain ε and the effective strain $\overline{\varepsilon}$ as follows:

$$\overline{\varepsilon} = \frac{\varepsilon}{\sqrt{1+\ell}} \tag{25}$$

For a composite material consisting of a matrix and fibers, equivalent local relations to equations (24) and (25) may be written as follows:

$$\overline{\sigma}^m = \sigma^m\sqrt{1+\ell^m} \tag{26}$$

$$\overline{\sigma}^f = \sigma^f\sqrt{1+\ell^f} \tag{27}$$

$$\overline{\varepsilon}^m = \frac{\varepsilon^m}{\sqrt{1+\ell^m}} \tag{28}$$

$$\overline{\varepsilon}^f = \frac{\varepsilon^f}{\sqrt{1+\ell^f}} \tag{29}$$

For small values of the scalar damage variable ℓ, the square root appearing in equations (24) – (29) can be approximated as follows using the Taylor series expansion:

$$\sqrt{1+\ell} \approx 1+\frac{1}{2}\ell \tag{30}$$

Next, one uses the rule of mixtures for the total effective stress as follows:

$$\overline{\sigma} = \overline{c}^m \overline{\sigma}^m + \overline{c}^f \overline{\sigma}^f \tag{31}$$

where $\overline{c}^m$ and $\overline{c}^f$ are the effective volume fractions for the matrix and fibers, respectively. Substituting equations (24) and (27) into equation (31) and simplifying, one obtains the relation between the total stress and the local stresses as follows:

$$\sigma = \overline{c}^m \sigma^m \sqrt{\frac{1+\ell^m}{1+\ell}} + \overline{c}^f \sigma^f \sqrt{\frac{1+\ell^f}{1+\ell}} \tag{32}$$

For the case of small damage and using the approximation in equation (30), the relation in equation (32) becomes:

$$\sigma + \frac{1}{2}\sigma\ell = \overline{c}^m \sigma^m + \overline{c}^f \sigma^f + \frac{1}{2}\overline{c}^m \sigma^m \ell^m + \frac{1}{2}\overline{c}^f \sigma^f \ell^f \tag{33}$$

It is clear that the above special relation for the case of small damage is a linear relation in the respective variables. It should be noted that the above relation cannot be simplified further without making certain limiting assumptions. For example, for the special case where the values of the three damage variables appearing in the equation are equal, i.e. when $\ell^m = \ell^f = \ell$, one may obtain the following simplified relations only for this very special case:

$$\sigma = \overline{c}^m \sigma^m + \overline{c}^f \sigma^f \tag{34}$$

$$\sigma\ell = \overline{c}^m \sigma^m \ell^m + \overline{c}^f \sigma^f \ell^f \tag{35}$$

Going back now to the general case of damage, one uses the following two relations between the local stresses σ^m, σ^f and the total stress σ :

$$\sigma^m = B^m \sigma \tag{36}$$

$$\sigma^f = B^f \sigma \tag{37}$$

where B^m and B^f are the matrix and fiber stress concentration factors, respectively. Substituting equations (36) and (37) into equation (32), one obtains the following relation :

$$\bar{c}^m B^m \sqrt{1+\ell^m} + \bar{c}^f B^f \sqrt{1+\ell^f} = \sqrt{1+\ell} \tag{38}$$

The above relation is an exact relation between the local damage variables and the overall damage variable. For the special case of small damage and equal values of the three damage variables given in equations (34) and (35), the relation in equation (38) becomes:

$$\bar{c}^m B^m + \bar{c}^f B^f = 1 \tag{39}$$

$$\bar{c}^m B^m \ell^m + \bar{c}^f B^f \ell^f = \ell \tag{40}$$

It should be noted that the above simplified relations apply only to this special case.

3.2. Damage Evolution

In this section one derives the general relation for the evolution of the new damage variable. The evolution equation will be written in terms of the increments of stress or strain based on sound thermodynamic principles for a uniaxial state of stress. In general, the elastic strain energy for the effective configuration is given as follows:

$$U = \frac{1}{2}\bar{\sigma}\,\bar{\varepsilon} \tag{41}$$

where $\bar{\sigma}$ and $\bar{\varepsilon}$ are the effective stress and effective strain, respectively. In addition, one can also write the elastic constitutive relation in the effective configuration as follows;

$$\bar{\sigma} = \bar{E}\,\bar{\varepsilon} \tag{42}$$

where $\overline{E}$ is the effective elastic modulus. Substituting equation (42) into equation (41), one obtains:

$$U = \frac{1}{2}\overline{E}\,\overline{\varepsilon}^2 \tag{43}$$

Next, one substitutes equation (25) into equation (43) to obtain:

$$U = \frac{\overline{E}\,\varepsilon^2}{2(1+\ell)} \tag{44}$$

The above relation provides an expression for the elastic strain energy in terms of the new damage variable. Next, one takes the derivative of U in the above relation to obtain:

$$dU = \frac{\overline{E}\,\varepsilon\, d\varepsilon}{1+\ell} - \frac{\overline{E}\,\varepsilon^2\, d\ell}{2(1+\ell)^2} \tag{45}$$

Next, one defines the variable y to be the general thermodynamic force associated with the scalar damage variable ℓ as follows:

$$y = \frac{\partial U}{\partial \ell} \tag{46}$$

Substituting equation (44) into equation (46), one obtains:

$$y = -\frac{\overline{E}\,\varepsilon^2}{2(1+\ell)^2} \tag{47}$$

Next, one introduces $g(y,B)$ as the damage function or criterion given as follows:

$$g(y,B) \equiv \frac{1}{2}y^2 - B(\beta) = 0 \tag{48}$$

where $B(\beta)$ is the overall threshold of damage and β is an overall damage parameter. The power of dissipation Π can now be written as follows:

$$\Pi = -y\,d\ell - B\,d\beta \tag{49}$$

Next, one constructs an objective function Ψ as follows:

$$\Psi = \Pi - d\lambda \cdot g \tag{50}$$

where $d\lambda$ is a Lagrange multiplier. In order to extremize the function Ψ, the following two conditions must be satisfied:

$$\frac{\partial \Psi}{\partial y} = 0 \tag{51}$$

$$\frac{\partial \Psi}{\partial B} = 0 \tag{52}$$

Substituting equation (50) into equation (51) and using equation (49), one obtains;

$$d\ell = -d\lambda \frac{\partial g}{\partial y} \tag{53}$$

Substituting equation (50) into equation (52) and using equation (49), one obtains:

$$d\beta = d\ell \tag{54}$$

Next, one substitutes equation (54) into equation (53) to obtain:

$$d\ell = -d\beta \frac{\partial g}{\partial y} \tag{55}$$

Finally, one invokes the consistency condition $dg = 0$ and applies it to equation (48) to obtain:

$$d\ell = -\frac{y\,dy\,\dfrac{\partial g}{\partial y}}{\left(\dfrac{\partial B}{\partial \beta}\right)} \tag{56}$$

Substituting the equation for the function g from equation (48) into equation (56) and simplifying, one obtains the evolution equation for the new damage variable;

$$d\ell = -\frac{y^2 dy}{\left(\dfrac{\partial B}{\partial \beta}\right)} \tag{57}$$

It is clear from the above equation that the evolution of the damage variable is an ordinary differential equation between the variables ℓ and y. It should be noted that the term $\frac{\partial B}{\partial \beta}$ appearing in the denominator can be taken to be a constant (Voyiadjis and Kattan, 2006). In order to solve the above governing differential equation, one substitutes equation (47) into equation (57) to obtain:

$$\left(\frac{\partial B}{\partial \beta}\right) d\ell = \frac{\overline{E}^3 \varepsilon^5 d\varepsilon}{4(1+\ell)^6} - \frac{\overline{E}^3 \varepsilon^6 d\ell}{4(1+\ell)^7} \tag{58}$$

The above relation is an explicit differential equation between the new damage variable and the strain. In order to solve the above differential equation, one uses the substitution $x = \left(\frac{\varepsilon}{1+\ell}\right)^6$ and one obtains the final solution as follows:

$$\left(\frac{\partial B}{\partial \beta}\right) \ell = \frac{1}{24} \overline{E}^3 \left(\frac{\varepsilon}{1+\ell}\right)^6 \tag{59}$$

Note that the above equation is the sought explicit relation between the new damage variable ℓ and the strain ε. It is clear that this relation is nonlinear.

Using the elastic constitutive relation of equation (42) but written in the damaged configuration as $\sigma = E\,\varepsilon$, and substituting it into equation (59), one obtains the following relation between the stress and the new damage variable:

$$\sigma = \left[24 \left(\frac{\partial B}{\partial \beta} \right) \overline{E}^{\,3} \ell \right]^{1/6} \tag{60}$$

The above relation is an explicit relation between the stress and the new damage variable. It is clearly a nonlinear relation.

Next, one plots the two relations given in equations (59) and (60). Figure 4 shows the relation between the stress and the damage variable based on equation (60) while Figure 5 shows the relation between the strain and the damage variable based on equation (59). In plotting these graphs, one uses a SiC-Ti-Al composite lamina with the following material properties: $\overline{E} = 199$ GPa and $\left(\frac{\partial B}{\partial \beta} \right) = 1000$ (kPa)^3.

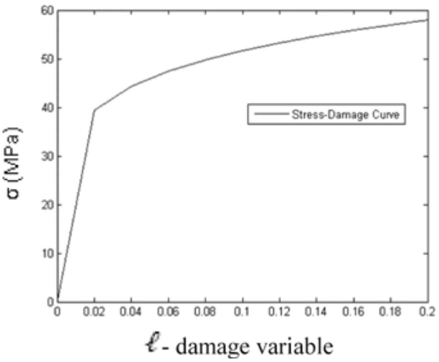

Figure 4. Stress-Damage Relation

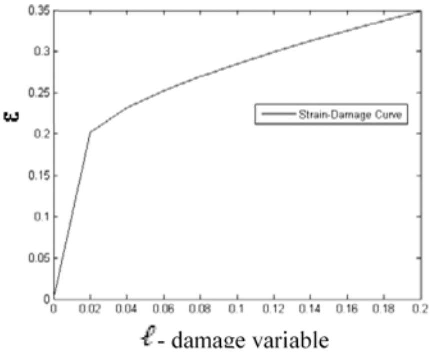

Figure 5. Strain-Damage Relation

It should be noted from Figures 4 and 5 that the stress-damage and strain-damage relations are linear for very small values of the damage variable, i.e for $\ell < 0.02$. For this small range of damage values, a linear and simplistic theory of damage mechanics may be formulated.

3.3. Mechanics of Small Damage

In this section, one provides certain equations for the mechanics of small damage, i.e. when the value of the damage variable is small. One will start by re-writing equation (60) as follows:

$$\sigma = (\alpha \, \ell)^{1/6} \tag{61}$$

where the coefficient α is given by:

$$\alpha = 24 \left(\frac{\partial B}{\partial \beta} \right) \bar{E}^{3} \tag{62}$$

Similarly, one can write the following two relations for the matrix and fiber stresses:

$$\begin{aligned} \sigma^{m} &= \left(\alpha^{m} \, \ell^{m} \right)^{1/6} \\ \sigma^{f} &= \left(\alpha^{f} \, \ell^{f} \right)^{1/6} \end{aligned} \tag{63}$$

where the coefficients α^{m} and α^{f} are given by (see equation (62)):

$$\begin{aligned} \alpha^{m} &= 24 \left(\frac{\partial B^{m}}{\partial \beta^{m}} \right) \bar{E}^{m3} \\ \alpha^{f} &= 24 \left(\frac{\partial B^{f}}{\partial \beta^{f}} \right) \bar{E}^{f3} \end{aligned} \tag{64}$$

Next, one uses the law of mixtures for the damaged configuration as follows:

$$\sigma = c^{m} \sigma^{m} + c^{f} \sigma^{f} \tag{65}$$

where c^{m} and c^{f} are the matrix and fiber volume fractions in the damaged configuration, respectively. Substituting equations (61) and (63) into equation (65) and simplifying, one obtains;

$$\ell^{1/6} = c^m \left(\frac{\alpha^m \ell^m}{\alpha} \right)^{1/6} + c^f \left(\frac{\alpha^f \ell^f}{\alpha} \right)^{1/6} \tag{66}$$

For small values of the damage variables ℓ, ℓ^m, and ℓ^f, and combining equations (66), (33), and (38), one obtains the following relation after some tedious mathematical operations involving some approximations for small damage:

$$\frac{\ell^f}{\ell^m} = \frac{\left[\left(\bar{c}^m B^m \right)^{1/6} - c^m \left(\frac{\alpha^m}{\alpha} \right)^{1/6} \right]^6}{c^f \left(\frac{\alpha^f}{\alpha} \right)^{1/6} - \left(\bar{c}^m B^m \right)^{-5/6} \bar{c}^f B^f \left[\left(\bar{c}^m B^m \right)^{1/6} - c^m \left(\frac{\alpha^m}{\alpha} \right)^{1/6} \right]^5} \tag{67}$$

The above relation is valid for small values of the parameter $\dfrac{\bar{c}^m B^m \ell^m}{\bar{c}^f B^f \ell^f}$.

Similarly, another relation can be obtained as follows:

$$\frac{\ell^m}{\ell^f} = \frac{\left[\left(\bar{c}^f B^f \right)^{1/6} - c^f \left(\frac{\alpha^f}{\alpha} \right)^{1/6} \right]^6}{c^m \left(\frac{\alpha^m}{\alpha} \right)^{1/6} - \left(\bar{c}^f B^f \right)^{-5/6} \bar{c}^m B^m \left[\left(\bar{c}^f B^f \right)^{1/6} - c^f \left(\frac{\alpha^f}{\alpha} \right)^{1/6} \right]^5} \tag{68}$$

The above relation is valid for small values of the parameter $\dfrac{\bar{c}^f B^f \ell^f}{\bar{c}^m B^m \ell^m}$.

In deriving equations (67) and (68), the following approximations have been used based on the Taylor series expansion of each term. For any variables x and y, and constants A, B, C, D, the following equation

$$Ax^{1/6} \cdot Bx^{-5/6} y = Cx^{1/6} + Dy^{1/6} \tag{69}$$

can be approximated by the following simpler relation:

$$(A - C)^{1/6} x \approx \left[D - 6B(A - C)^5 \right] y \tag{70}$$