Peter Wriggers
Carsten Carstensen
Editors

CISM

International Centre for Mechanical Sciences

Mixed Finite Element Technologies

CISM Courses and Lectures, vol. 509

SpringerWienNewYork

SpringerWienNewYork

CISM COURSES AND LECTURES

The series presents lecture notes, monographs, edited works and proceedings in the field of Mechanics, Engineering, Computer Science and Applied Mathematics.
Purpose of the series is to make known in the international scientific and technical community results obtained in some of the activities organized by CISM, the International Centre for Mechanical Sciences.

INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 509

MIXED FINITE ELEMENT TECHNOLOGIES

EDITED BY

CARSTEN CARSTENSEN
HU BERLIN, GERMANY

PETER WRIGGERS
UNIVERSITY OF HANNOVER

SpringerWienNewYork

This volume contains 55 illustrations

Printed in Italy
SPIN 12641321

All contributions have been typeset by the authors.

ISBN 978-3-211-99092-6 SpringerWienNewYork

PREFACE

The finite element method is the dominating tool in computational mechanics. To quote the late Simo: "Mechanics without finite elements is like rock'n roll without electricity". In straightforward finite element schemes, there is an elastic energy which is minimized over some conforming discrete subspace X_h*. This leads to a linear system of equations of the form*

$$a(u_h, v_h) = b(v_h) \qquad \text{for all } v_h \in Y_h$$

with unknown discrete solution u_h *in* X_h *for a given right-hand side* $b(v_h)$ *and the bilinear form* $a(u_h, v_h)$*. In the context of an elastic energy* $\frac{1}{2}a(v_h, v_h) - b(v_h)$*, the bilinear form is symmetric and one choice for trial and test functions reads* $X_h = Y_h$*.*

In saddle-point problems, which are related to different engineering applications in mechanics, a mixed finite element method may consider the general situation where $X_h \neq Y_h$*. This is the starting point of the more involved mathematical justification of the mixed finite element method. The choice of trial and test functions* X_h *and* Y_h *is subject to an inf-sup condition named after Babuska, Brezzi, Ladyzhenskaya in order to guarantee stability.*

Therefore, the use of arbitrary discrete spaces - often employed in engineering finite element analysis - fails in general. More often even in everydays practise, there are methods applied which "work on Mondays but not on Tuesdays" (saying after R. Stenberg). Hence it is mandatory to consult some mathematics in order to construct reliable and fast mixed finite element simulation tools. This CISM course brought together leading experts in the field of nonstandard finite element methods to highlight the state of the art on mathematical and engineering aspects of current mixed finite element technology. The basic lectures on the necessary mathematical and mechanical background for linear and nonlinear application areas of mixed finite elements methods were given by the two editors and followed by particular applications and methods discussed by the other lecturers.

Carsten Carstensen and Peter Wriggers

CONTENTS

Lectures on Adaptive Mixed Finite Element Methods*

C. Carstensen†

†Department of Mathematics, Humboldt-Universität Berlin, Germany

Abstract These lectures concern the three most simple model problems of elliptic second order partial differential equations which allow for some mixed formulations. The introduction to the Poisson problem, to the Stokes problem, and to linear elasticity is completely standard and hence kept rather short.

The first aim is a general discussion of the mixed formulations around various statements of the inf-sup condition often named after Ladyzhenskaya, Babuška, Brezzi. Some details on the implementation of Raviart-Thomas mixed finite elements in MATLAB complement this introduction.

The second aim is a particular outline of the author's own research on a posteriori error analysis and adaptive algorithms of mixed finite element methods displayed for the Poisson problem. The presentation is motivated by the author's research Braess et al. (2004); Carstensen (1997, 1999, 2005); Carstensen and Dolzmann (1998); Carstensen et al. (2000); Carstensen and Funken (2001a,b); Carstensen and Verfürth (1999) with essential help of many researchers including S. Bartels, D. Braess, G. Dolzmann, S.A. Funken and R. Hoppe.

These lecture notes are written with the help of Wolfgang Boiger, Andreas Byfut, David Günther, Athina Konstantinidou, Hella Rabus, and Jan Reininghaus. The support of all these scientists, especially that of the younger ones is thankfully acknowledged.

1 Three Mixed Formulations

This lecture introduces mixed formulations for three model examples and displays them in a unified abstract way, which the remaining lectures are based on.

*Supported by the DFG Research Center MATHEON "Mathematics for key technologies" in Berlin.

1.1 Mixed Formulations

If the main interest is on an accurate stress or flux approximation and some strict equilibration condition, it might be advantageous to consider an operator split: Instead of one partial differential equation of order $2m$ one considers two equations of order m.

To be more precise, given one equation in an abstract form $\mathcal{L}u = G$ with some differential operator $\mathcal{L} = \mathcal{A}\mathcal{B}$ composed of $\mathcal{A}$ and $\mathcal{B}$, define $p := \mathcal{B}u$ and solve the two equations

$$\mathcal{A}p = G \quad \text{and} \quad \mathcal{B}u = p. \tag{1}$$

Throughout these lecture notes, a and b are two bilinear forms associated with $\mathcal{A}$ and $\mathcal{B}$ on the Hilbert spaces L and H with dual spaces L^* and H^*, typically, L is some Lebesgue space and H is some Sobolev space. Given bounded bilinear forms $a : H \times H \to \mathbb{R}$, $b : H \times L \to \mathbb{R}$ and right-hand sides $g \in L^*$, $f \in H^*$, a weak form of (1) reads

$$\begin{aligned} a(p,q) + b(q,u) &= f(q) \quad \text{for all} \quad q \in H; \\ b(p,v) &= g(v) \quad \text{for all} \quad v \in L. \end{aligned} \tag{2}$$

Note that $p - \mathcal{B}u = f$ represents the strong form of $(2)_a$ and $\mathcal{A}p = G$ is the strong form of $(2)_b$.

1.2 Lebesgue and Sobolev Spaces

The reader is expected to be familiar with the basic features of Lebesgue and Sobolev functions. For the understanding of these lecture notes, it suffices to know that $v \in L^2(\Omega)$ for some domain $\Omega \subset \mathbb{R}^n$ means that v is measurable and the (Lebesgue) integral of $|v(x)|^2$ over Ω is finite,

$$\|v\|^2_{L^2(\Omega)} := \int_\Omega |v(x)|^2\, dx < \infty.$$

Then, this defines the norm of v in $L^2(\Omega)$. The same notation applies to more components such as $L^2(\Omega;\mathbb{R}^m)$ and matrices $L^2(\Omega;\mathbb{R}^{m\times n})$, where each component belongs to $L^2(\Omega)$.

For a smooth function v, the Sobolev norm $\|\cdot\|_{H^1(\Omega)}$ is defined by

$$\|v\|^2_{H^1(\Omega)} := \int_\Omega |v(x)|^2\, dx + \int_\Omega |\nabla\, v(x)|^2\, dx$$

and $H^1(\Omega)$ denotes the completion of smooth functions with respect to this norm. To be more precise, let $C^\infty_c(\Omega)$ denote the set of arbitrarily smooth functions with a compact support inside Ω. Then, the Sobolev space $H_0^1(\Omega)$ is defined as the closure of $C_c^\infty(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{H^1(\Omega)}$, while $H^1(\Omega)$ is the closure of $C_c^\infty(\mathbb{R}^n)$ restricted to Ω under the same norm. One says that $v \in H^1(\Omega)$ satisfies $v|_{\partial\Omega} = 0$ in the sense of traces, written $\gamma(v) = 0$, if and only if $v \in H_0^1(\Omega)$.

It can be proven that Sobolev functions (i.e., the elements in such a Sobolev space) have a weak derivative in $L^2(\Omega;\mathbb{R}^n)$ and satisfy the integration by parts formula: For all $u, v \in H^1(\Omega)$ and $j = 1, \dots, n$ there holds

$$\int_\Omega u \frac{\partial v}{\partial x_j}\, dx + \int_\Omega \frac{\partial u}{\partial x_j}\, v\, dx = \int_{\partial\Omega} \gamma(u)\, \gamma(v)\, \nu_j\, ds. \tag{3}$$

Here ν is the outer unit normal vector along the boundary $\partial\Omega$ and the trace operator $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is well defined, linear, continuous, and satisfies $(\gamma u)(x) = u(x)$ for all $x \in \partial\Omega$ and all $u \in C(\overline{\Omega})$. An alternative notation reads $\gamma(u) = u|_{\partial\Omega}$ or one simply replaces $\gamma(u)$ by u on the right-hand side of (3).

Moreover, $H(\mathrm{div}, \Omega) \subset L^2(\Omega)^n$ is the set of functions whose divergence is integrable. The norm in $H(\mathrm{div}, \Omega)$ is given by

$$\|v\|^2_{H(\mathrm{div},\Omega)} = \int_\Omega |v(x)|^2\, dx + \int_\Omega |\,\mathrm{div}\, v(x)|^2\, dx.$$

Note that $H^1(\Omega)^n \subset H(\mathrm{div}, \Omega) \subset L^2(\Omega)^n$, but $H^1(\Omega)^n \neq H(\mathrm{div}, \Omega) \neq L^2(\Omega)^n$ for $n > 1$. Finally,

$$H^k_{\mathrm{loc}}(\Omega) = \{V : \Omega \to \mathbb{R} \text{ measurable} : \forall K \subset\subset \Omega,\, v|_K \in H^k(\Omega)\},$$

where $K \subset\subset \Omega$ denotes an open bounded subset with $\overline{K} \subset \Omega$.

1.3 Poisson Problem

The stationary heat equation and many mathematical models in applications of solid and fluid mechanics lead to the Laplace operator $\triangle = \frac{\partial^2 \cdot}{\partial x_1^2} + \dots + \frac{\partial^2 \cdot}{\partial x_1^n}$. Given $g \in L := L^2(\Omega)$, let $u \in H^1(\Omega)$ with $p := \nabla u \in H := H(\mathrm{div}, \Omega)$ denote the solution to the *Poisson problem*

$$\Delta u + g = 0 \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{4}$$

The weak form of (4) is given by (2) with $p, q \in H$, $u, v \in L$, and (where $\cdot$ denotes the scalar product in $\mathbb{R}^n$)

$$a(p,q) := \int_\Omega p \cdot q\, dx, \quad b(q,v) := \int_\Omega v\ \mathrm{div}\, q\, dx,$$
$$f(q) := 0, \qquad g(v) := -\int_\Omega g\, v\, dx.$$

It is known that there exists a unique solution and, at least locally, $u \in H^2_{\mathrm{loc}}(\Omega)$ and $p \in H^1_{\mathrm{loc}}(\Omega)$.

1.4 Stokes Problem

The stationary incompressible fluid flow can be modeled by the Stokes equations for given $f \in L^2(\Omega)$ and unknown velocity u and pressure p,

$$\Delta u + \nabla p = -f \text{ in } \Omega, \quad \operatorname{div} u = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{5}$$

The weak form of (5) is given by (2) with $p, q \in L := L^2_0(\Omega)$ and $u, v \in H := H^1_0(\Omega)^n$ and, (where $:$ denotes the scalar product in $\mathbb{R}^{n\times n}$)

$$\begin{aligned} a(u,v) &:= \int_\Omega \nabla u : \nabla v\,dx, & b(v,q) &:= \int_\Omega q \operatorname{div} v\,dx,\\ f(v) &:= \int_\Omega f\,v\,dx, & g(q) &:= 0. \end{aligned}$$

Here, in comparison to (2), the roles of u and p are interchanged, which typically leads to some confusion. Moreover, $L^2_0(\Omega) := \{q \in L^2(\Omega) : \int_\Omega q\,dx = 0\}$ fixes a global additive constant in the pressure (because of lacking Neumann boundary conditions). Then there exists a unique solution (u,p).

1.5 Elasticity Problem

We adopt the notation of the previous two subsections and continue with a linear stress-strain relation with the linear Green strain $\varepsilon(u) := \operatorname{sym}(\nabla u) := \big(\nabla u + (\nabla u)^T\big)/2$ of the form

$$g + \operatorname{div} \mathbb{C}\,\varepsilon(u) = 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega. \tag{6}$$

The weak form of (6) is given by (2) with $\sigma, \tau \in H := H(\operatorname{div},\Omega)^n \cap L^2(\Omega; \mathbb{R}^{n\times n}_{\mathrm{sym}})$, $u, v \in L := L^2(\Omega)^n$, and

$$\begin{aligned} a(\sigma,\tau) &:= \int_\Omega (\mathbb{C}^{-1}\sigma) : \tau\,dx, & b(\tau,v) &:= \int_\Omega \operatorname{div}\tau \cdot v\,dx,\\ f(\tau) &:= 0, & g(v) &:= -\int_\Omega g\,v\,dx. \end{aligned} \tag{7}$$

Here, $\mathbb{C}$ denotes the linear fourth-order tensor, namely

$$\mathbb{C}A := \lambda\,\operatorname{tr}(A)\,\mathbf{1} + 2\mu\,A \quad \text{for } A \in \mathbb{R}^{n\times n},$$

with inverse relation

$$\mathbb{C}^{-1}A = 1/(2\mu)\,A - \lambda/(2\mu(n\lambda + 2\mu))\,\operatorname{tr}(A)\,\mathbf{1} \quad \text{for } A \in \mathbb{R}^{n\times n}.$$

The material parameters λ and μ are positive and hence (6) is an elliptic PDE with a unique solution $u \in H^1_0(\Omega) \cap H^2_{\text{loc}}(\Omega)$.

2 First Analysis for Mixed Formulations

Let X, Y be real Hilbert spaces with duals X^*, Y^* and let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form, i.e., for all $u, w \in X, v, z \in Y$ there holds

$$a(\lambda\, u, \mu\, v) = \lambda\, \mu\, a(u, v) \text{ for all } \lambda, \mu \in \mathbb{R}, \qquad \text{(homogenity)}$$
$$a(u + w, v + z) = a(u, v) + a(u, z) + a(w, v) + a(w, z), \qquad \text{(additivity)}$$
$$\|a\| := \sup_{\substack{x \in X \\ \|x\|=1}} \sup_{\substack{y \in Y \\ \|y\|=1}} a(x, y) < \infty. \qquad \text{(boundedness)}$$

Given such a bounded bilinear form $a : X \times Y \to \mathbb{R}$, one defines the two associated linear operators

$$\begin{aligned} &A_1 \in L(X; Y^*) \quad \text{and} \quad A_2 \in L(Y; X^*) \quad \text{by} \\ &A_1 x := a(x, \cdot) \quad \text{and} \quad A_2 y := a(\cdot, y) \quad \text{for } x \in X \text{ and } y \in Y. \end{aligned} \tag{8}$$

The operator norms of A_1 and A_2 read

$$\|A_1\| := \|A_1\|_{L(X,Y^*)} = \sup_{x \in X \setminus \{0\}} \sup_{y \in Y \setminus \{0\}} \frac{a(x, y)}{\|x\|_X \|y\|_Y} = \|A_2\|_{L(Y,X^*)} = \|a\|.$$

Remark 2.1. Conversely, any $A_1 \in L(X; Y^*)$ defines some bilinear form $a : X \times Y \to \mathbb{R}$ which is bounded with $\|a\| = \|A_1\|$. Analogous remarks apply to A_2.

An operator A_1 (resp. A_2) from (8) is called an isomorphism if it has a bounded inverse so that the linear equation $A_1 x = f$ (resp. $A_2 y = g$) has a unique solution, which is bounded with respect to the data f (resp. g), in other words, the problem is *well posed*. Sufficient and necessary conditions for an isomorphism A_j are well known Babuška and Azis, Brezzi and Fortin (1991). The proof is sketched for convenient reading to narrow the gap between lectures in functional analysis and on mixed FE formulations.

Theorem 2.2 (Invertibility of $A_j \Leftrightarrow$ Inf-Sup Condition)**.** *Let X, Y be Hilbert spaces and let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form with A_1 and A_2 as in* (8)*.*

a. The following conditions (i),(ii),(iii),(iv) are pairwise equivalent

i) $\forall\, f \in Y^*\ \exists!\, x \in X, \quad a(x, \cdot) = f;$

ii) $\forall\, g \in X^*\ \exists!\, y \in Y, \quad a(\cdot, y) = g;$

iii) $\alpha_1 := \inf_{\substack{x \in X \\ \|x\|_X=1}} \sup_{\substack{y \in Y \\ \|y\|_Y=1}} a(x, y) > 0$ *and*

$\forall\, y \in Y \setminus \{0\}\ \exists\, x \in X, \quad a(x, y) \neq 0;$

iv) $\alpha_2 := \inf_{\substack{y\in Y\\ \|y\|_Y=1}} \sup_{\substack{x\in X\\ \|x\|_X=1}} a(x,y) > 0$ *and*

$\forall\, x \in X \setminus \{0\}\ \exists\, y \in Y, \quad a(x,y) \neq 0.$

b. There holds $A_1^* = A_2$, $A_2^* = A_1$ *and each of the above conditions (i),(ii),(iii),(iv) implies that* A_1 *and* A_2 *are continuously invertible and*

$$\|A_1^{-1}\|_{L(Y^*;X)} = \frac{1}{\alpha_1} = \frac{1}{\alpha_2} = \|A_2^{-1}\|_{L(X^*;Y)}\ .$$

Proof of Theorem 2.2.
Claim 1*:* If $\alpha_1 > 0$, then Range(A_1) is a closed subspace of Y^*.

Proof. Suppose (f_j) is a Cauchy sequence in Range(A_1). There exists a sequence (x_j) in X with $f_j = A_1\, x_j$ for all $j \in \mathbb{N}$. By definition of $\alpha_1 > 0$ there holds, for all $j,k \in \mathbb{N}$,

$$\alpha_1 ||x_j - x_k||_X \le ||A_1\, x_j - A_1\, x_k||_{Y^*} = ||f_j - f_k||_{Y^*}.$$

Hence (x_j) is also a Cauchy sequence in the complete space X, and so is convergent towards some x. Since A_1 is bounded,

$$(f_j) = (A_1\, x_j) \longrightarrow f := A_1\, x \quad \text{in } Y^*.$$

Hence $(f_j) \to f$ has a limit in Range(A_1). Therefore, Range(A_1) is complete, whence closed.

Claim 2*:* If Range(A_1) closed and $\ker(A_2) = \{0\}$, then Range$(A_1) = Y^*$.

Proof. Let $R_2 : Y^* \to Y$ be the Riesz representation in the Hilbert space Y, i.e., for each $f \in Y^*$, $z := R_2\, f \in Y$ satisfies

$$\langle z, y\rangle_Y = f(y) \quad \text{for all } y \in Y.$$

Suppose Range$(A_1) \subsetneq Y^*$. Then $(R_2 \text{ Range}(A_1)) \subsetneq R_2\, Y^* = Y$ and there exists some $y \in Y \setminus \{0\}$ such that $y \perp (R_2 \text{ Range}(A_1))$. Hence, for all $x \in X$,

$$0 = \langle R_2\, A_1\, x, y\rangle_Y = (A_1\, x)(y) = a(x,y) = (A_2\, y)(x).$$

This reads $y \in \ker(A_2) = \{0\}$ and contradicts $y \neq 0$.

Proof of $(iii) \Rightarrow (i)$*:* This follows from Claim 1 and 2.

Proof of $(i) \Rightarrow (iii)$*:* The inverse mapping theorem applies to $A_1 : X \to Y^*$ and shows

$$0 < ||A_1^{-1}||_{L(Y^*;X)}^{-1} = \left(\sup_{\substack{f \in Y^* \\ ||f||_{Y^*}=1}} ||A_1^{-1} f||_X \right)^{-1} = \inf_{\substack{f \in Y^* \\ ||f||_{Y^*}=1}} ||A_1^{-1} f||_X^{-1}$$
$$= \inf_{f \in Y^* \setminus \{0\}} \frac{||f||_{Y^*}}{||A_1^{-1} f||_X} = \inf_{x \in X \setminus \{0\}} \frac{||A_1\, x||_{Y^*}}{||x||_X} = \alpha_1.$$

This proves the first assertion in (iii) and $1/\alpha_1 = ||A_1^{-1}||_{L(Y^*;X)}$ in (b). The second assertion in (iii) is verified as follows: Given any $y \in Y \setminus \{0\}$, a corollary to the Hahn-Banach theorem guarantees the existence of some $f \in Y^*$ with $f(y) \neq 0$. From (i), there exists $x \in X \setminus \{0\}$ with $f = A_1\, x$. Hence, $a(x, y) = (A_1\, x)(y) = f(y) \neq 0$. This completes the proof of (iii).

Since $A_1^* \in L(Y; X^*)$ satisfies, for all $x \in X$ and $y \in Y$, that

$$(A_1^*\, y)(x) = A_1(x)(y) = a(x, y) = (A_2\, y)(x),$$

there holds $A_1^* = A_2$. If A_1 is invertible, so is its dual A_2 and vice versa. This proves $(i) \Leftrightarrow (ii)$. Moreover, the norms of A_2^{-1} and A_1^{-1} coincide. Hence $\alpha_1 = \alpha_2$. This indicates the proof of (b). The remaining details and implications in the proof of (a) are omitted. □

Definition 2.3. Some bilinear form $a : X \times X \to \mathbb{R}$ is called $X-$elliptic if there exists $\alpha > 0$ such that, for all $x \in X$, there holds

$$\alpha\, \|x\|_X^2 \leq a(x, x).$$

Example 2.4. Suppose $a : X \times X \to \mathbb{R}$ is bounded and X-elliptic. Clearly, $\alpha \leq \alpha_1$ in (iii) and, by Theorem 2.2, A_1 is an isomorphism. This statement is known as the generalised Lax-Milgram lemma and plays a dominant role in the existence theory of elliptic PDEs.

Theorem 2.5 (Main Theorem for Mixed Formulations)**.** *Let X,Y be Hilbert spaces and let $a : X \times X \to \mathbb{R}$ and $b : X \times Y \to \mathbb{R}$ be bounded bilinear forms. Set*

$$V := \ker(B_1) = \{x \in X \;:\; b(x, \cdot) = 0 \;\textit{in}\; Y^*\}$$

and suppose that

$$0 \leq \alpha := \inf_{\substack{x \in V \\ \|x\|_X=1}} \sup_{\substack{u \in V \\ \|u\|_X=1}} a(x, u) \leq \|a\| < \infty,$$
$$0 \leq \beta := \inf_{\substack{y \in Y \\ \|y\|_Y=1}} \sup_{\substack{x \in X \\ \|x\|_X=1}} b(x, y) \leq \|b\| < \infty.$$

Then the linear continuous map

$$L : \begin{cases} X \times Y \to (X \times Y)^*, \\ (x, y) \mapsto a(x, \cdot) + b(\cdot, y) + b(x, \cdot) \end{cases}$$

is an isomorphism if and only if

$$\alpha, \beta > 0 \text{ and } \forall\, y \in V \setminus \{0\}\ \exists\, x \in V, \quad a(x, y) \neq 0\,.$$

Remark 2.6. The summands in $a(x, \cdot) + b(\cdot, y) + b(x, \cdot)$ act on different spaces according to their domain, such that, in more details,

$$L(x, y)(u, w) := a(x, u) + b(u, y) + b(x, w).$$

Remark 2.7. Since X, Y are Hilbert spaces, infima und suprema are attained in the definition of α, β, etc. For example, $\operatorname{dist}(x, V) = \|x - v\|$ for some $v \in V$. In fact, for a minimal sequence (x_j) in V we have that $\|x_j\| \leq \|x - x_j\| + \|x\|$ is bounded and there is a subsequence $(x_k) \rightharpoonup v$ in V. Because $\|x - \cdot\|$ is convex and continuous, it is weakly lower semi continuous and so $\|x - v\| \leq \lim\limits_{k \to \infty} \|x - x_j\| = \operatorname{dist}(x, V)$.

Remark 2.8. The condition $\beta > 0$ for some bilinear form $b(\cdot, \cdot)$ is called LBB-condition (written 'b satisfies (LBB)') after Ladyzhenskaya, Babuška, and Brezzi. However, the notion of an LBB condition is more often applied to the discrete situation, cf. Definition 3.1.

Proof of Theorem 2.5.

"$\Longrightarrow$" Consider $L : X \times Y \to (X \times Y)^*$ with $(x, y) \mapsto a(x, \cdot) + b(\cdot, y) + b(x, \cdot)$ which satisfies the inf-sup condition with a constant $\gamma > 0$. Then, for all $y \in Y$ there exists $(u, w) \in X \times Y$ such that

$$\gamma\, \|y\|_Y\, \|u\|_X \leq \gamma\, \|(0, y)\|_{X \times Y}\, \|(u, w)\|_{X \times Y} \leq b(u, y).$$

This implies $\beta \geq \gamma > 0$. Hence, the restriction of B_1 to $V^\perp := \{x \in X :\ x \perp V\}$, namely

$$B :\ V^\perp \to Y^*, \quad x \mapsto b(x, \cdot),$$

is an isomorphism by Theorem 2.2, since (iv) and hence (a)-(b) hold with

$$\inf_{\substack{x \in V^\perp \\ \|x\|_X = 1}}\ \sup_{\substack{y \in Y \\ \|y\|_Y = 1}}\ b(x, y) = \beta > 0.$$

It remains to show that

$$A:\ V \to V^*, \quad v \mapsto a(v,\cdot)|_V$$

is an isomorphism, as well. Once this is established, the remaining part of the claim follows from Theorem 2.2 applied to $a|_{V\times V}$.

Let $f \in V^*$. By the Hahn-Banach theorem, there exists a continuation $F \in X^*$ of f. Since L is isomorphic, there exists $(x,y) \in X \times Y$ such that

$$L(x,y)(u,w) = a(x,u) + b(u,y) + b(x,w) = F(u)$$

for all $(u,w) \in X \times Y$. This expression does not depend on w and so $b(x,\cdot) = 0$, whence $x \in V$. For $u \in V$, this leads to

$$a(x,u) = F(u) = f(u),$$

and so to $Ax = a(x,\cdot)|_V = f$. Consequently, A is surjective.

Let $x \in \ker(A) \subset V$, i.e., $a(x,v) = 0$ for all $v \in V$. Since B is an isomorphism the adjoint operator

$$B^*:\ Y \to (V^\perp)^*, \quad y \mapsto b(\cdot,y)|_{V^\perp}$$

is an isomorphism, as well. Hence there exists a unique $y \in Y$ with

$$b(\cdot,y)|_{V^\perp} = -a(x,\cdot)|_{V^\perp} \in (V^\perp)^*.$$

Since $b(\cdot,y)|_V = 0 = -a(x,\cdot)|_V$, there holds

$$b(\cdot,y) = -a(x,\cdot) \in X^*.$$

This and $x \in V$, i.e., $b(x,\cdot) = 0$, implies

$$L(x,y) = a(x,\cdot) + b(\cdot,y) + b(x,\cdot) = 0.$$

By assumption, L is isomorphic and so $(x,y) = (0,0)$. Hence, $x \in \ker(A)$ implies $x = 0$ and therefore A is injective.

“$\Longleftarrow$” Suppose $\beta > 0$. Then the restriction of B_1 to $V^\perp := \{x \in X :\ x \perp V\}$, namely

$$B:\ V^\perp \to Y^*, \quad x \mapsto b(x,\cdot),$$

is an isomorphism by Theorem 2.2, since (iv) and hence (a)-(b) hold with

$$\inf_{\substack{x\in V^\perp\\ \|x\|_X=1}} \ \sup_{\substack{y\in Y\\ \|y\|_Y=1}} b(x,y) = \beta > 0.$$

For given $(x, y) \in X \times Y$ with $\|x\|_X^2 + \|y\|_Y^2 = 1$, there exist $x_1 \in V$, $x_2 \in V^\perp$ with $x = x_1 + x_2$. Define $t := \|x_2\|_X$, $s := \|x_1\|_X$, $r := \|x\|_X$, and so $r^2 = s^2 + t^2 \leq 1$ and $\|y\|_Y^2 = 1 - r^2$. By definition of α, β and the above inf-sup condition, there exist $u_1 \in V$, $u_2 \in V^\perp$ and $w \in Y$ with

$$\begin{aligned} a(x_1, u_1) &\geq \alpha \|x_1\|_X \|u_1\|_X, \\ b(x_2, w) &\geq \beta \|x_2\|_X \|w\|_Y, \\ b(u_2, y) &\geq \beta \|u_2\|_X \|y\|_Y. \end{aligned}$$

The lengths of u_1, u_2, w are arbitrary, and with $(\cdot)_+ := \max\{0, \cdot\}$ one possible choice reads

$$\begin{aligned} \|w\|_Y &:= \beta\, t \geq 0, \\ \|u_1\|_X &:= \big(\alpha\, s - t\, \|a\|\big)_+, \\ \|u_2\|_X &:= \big(\beta(1 - r^2)^{1/2} - (t + s)\, \|a\|\big)_+. \end{aligned}$$

With $u = u_1 + u_2$ this implies

$$\begin{aligned} L(x, y)(u, w) &= a(x_1 + x_2,\, u_1 + u_2) + b(u_1 + u_2,\, y) \\ &\qquad + b(x_1 + x_2, w) \\ &\geq \alpha \|x_1\|_X \|u_1\|_X - \|a\| \|x_2\|_X \|u_1 + u_2\|_X \\ &\qquad - \|a\| \|x_1\|_X \|u_2\|_X + \beta \|u_2\|_X \|y\|_Y \\ &\qquad + \beta \|x_2\|_X \|w\|_Y \\ &\geq \|u_1\|_X \left(\alpha\, s - \|a\|\, t\right) + \|w\|_Y\, \beta\, t \\ &\qquad + \|u_2\|_X \left(\beta(1 - r^2)^{1/2} - \|a\|\, t - \|a\|\, s\right) \\ &= \|u\|_X^2 + \|w\|_Y^2 \\ &= \big(\alpha\, s - \|a\|\, t\big)_+^2 \\ &\qquad + \big(\beta(1 - r^2)^{1/2} - (t + s)\, \|a\|\big)_+^2 + \beta^2 t^2 \\ &=: f(r, s, t). \end{aligned}$$

Since $f(r, s, t) = 0$ implies $s = 0 = t$ and $r = 1$, there holds

$$0 < \bar{\beta}^2 := \min\{f(r, s, t) \,:\, 0 \leq r, s, t \leq 1 \text{ with } r \leq s + t\}.$$

The minimum $\bar{\beta}^2$ depends only on α, β, and $\|a\|$. Since $\|(u, w)\|_{X \times Y}^2 := \|u\|_X^2 + \|w\|_Y^2$, there holds

$$L(x, y)(u, w) \geq f(r, s, t) \geq \bar{\beta}\, \|(u, w)\|_{X \times Y}.$$

Therefore, L satisfies the inf-sup condition.

It remains to show that

$$\forall (u,w) \in (X \times Y) \setminus \{0\} \quad \exists (x,y) \in X \times Y,\, L(x,y)(u,w) \neq 0.$$

In fact, if $L(x,y)(u,w) = 0$ holds for all (x,y), then $b(u,\cdot) = 0$ and $a(\cdot,u) + b(\cdot,w) = 0$, hence $u \in V \subseteq X$.
Since a is non-degenerated and $u \in V$, the assumption $u \neq 0$ leads to some $z \in V$ with $a(z,u) \neq 0$ and $b(z,w) = 0$. This contradicts $a(z,u) + b(z,w) = 0$ and proves $u = 0$.

The assumption $w \neq 0$ and the Hahn-Banach theorem leads to some $f \in Y^*$ with $f(w) = 1$. Since the above operator B is an isomorphism, $x := B^{-1}(f) \in V^\perp$. This implies $b(x,w) = f(w) = 1$. But this contradicts $b(x,w) = 0$ and proves $w = 0$.

In conclusion, $(u,w) = 0$ and so L is non-degenerated. □

3 Discrete Mixed Formulations

The understanding of MFEM and its stability requires some mathematics, which will be introduced in this lecture. The instable methods sometimes even seem to work somehow in practice: *There are methods which work on Mondays but not on Tuesdays* is a saying due to Rolf Stenberg. However, a reliable computation is based on stable discretisations and therefore, has to satisfy the LBB conditions.

3.1 Abstract Framework

Throughout this lecture, let $a : X \times Y \to \mathbb{R}$ be a bounded bilinear form, and let $X_h \subseteq X$ and $Y_h \subseteq Y$ be closed (e.g., finite-dimensional) subspaces of the Hilbert spaces X and Y.

Definition 3.1 (LBB)**.** The discrete bilinear form a is said to satisfy the (LBB) condition if

$$\alpha_h := \inf_{\substack{x_h \in X_h \\ \|x_h\|_X = 1}} \sup_{\substack{y_h \in Y_h \\ \|y_h\|_Y = 1}} a(x_h, y_h) \geq 0 \tag{9}$$

satisfies $\alpha_h > 0$ uniformly with respect to a family $(X_h, Y_h)_h$ of discrete spaces. LBB abbreviates Ladyzhenskaya, Babuška, Brezzi and is also called (discrete) inf-sup condition.

Remark 3.2. Any choice of some orthonormal basis $(\xi_1, \dots, \xi_m)$ in X_h and some orthonormal basis $(\eta_1, \dots, \eta_n)$ in Y_h leads to a coefficient matrix $A \in \mathbb{R}^{m\times n}$ via

$$A_{jk} := a(\xi_j, \eta_k) \quad \text{for } i = 1, \dots, m \text{ and } k = 1, \dots, n.$$

Then, α_h is the smallest singular value of A. This is seen from a singular value decomposition

$$A = Q\Sigma R$$

of A with a diagonal matrix $\Sigma \in \mathbb{R}^{m\times n}$ and orthogonal matrices $Q \in \mathbb{R}^{m\times m}$ and $R \in \mathbb{R}^{n\times n}$. In fact,

$$x_h = \sum_{j=1}^{m} \lambda_j \xi_j \neq 0 \quad \text{and} \quad y_h = \sum_{k=1}^{n} \mu_k \eta_k \neq 0$$

for real coefficients $\lambda_1, \dots, \lambda_m$, $\mu_1, \dots, \mu_n$ and (recall orthonormality of the basis)

$$\|x_h\|_X^2 = \sum_{j=1}^{m} \lambda_j^2 \quad \text{and} \quad \|y_h\|_Y^2 = \sum_{k=1}^{m} \mu_k^2.$$

Since the Euclidian norm $|\cdot|$ is invariant under orthonormal transformations, the vectors $\lambda = (\lambda_1, \dots, \lambda_m)^T$ and $\mu = (\mu_1, \dots, \mu_n)^T$ satisfy

$$\frac{a(x_h, y_h)}{\|x_h\|_X \, \|y_h\|_Y} = \frac{\lambda\, Q\Sigma R\, \mu}{|\lambda|\, |\mu|} = \frac{(Q^T\lambda)\, \Sigma\, (R\mu)}{|Q^T\lambda|\, |R\mu|}$$

and recall that Σ is diagonal with singular values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$ for $r = \min\{m, n\}$ in the diagonal, thus

$$\begin{aligned}
\alpha_n &= \inf_{\substack{\lambda \in \mathbb{R}^m \\ |\lambda| = 1}} \sup_{\substack{\mu \in \mathbb{R}^n \\ |\mu| = 1}} \sum_{j=1}^{m} \sum_{k=1}^{n} \lambda_j\, \Sigma_{jk}\, \mu_k \\
&= \inf_{\substack{\lambda \in \mathbb{R}^m \\ |\lambda| = 1}} \sup_{\substack{\mu \in \mathbb{R}^n \\ |\mu| = 1}} \sum_{\ell=1}^{r} \lambda_\ell\, \Sigma_{\ell\ell}\, \mu_\ell \\
&= \inf_{\substack{\lambda \in \mathbb{R}^m \\ |\lambda| = 1}} \sup_{\substack{\mu \in \mathbb{R}^n \\ |\mu| = 1}} \sum_{\ell=1}^{r} (\sigma_\ell \lambda_\ell)\, \mu_\ell.
\end{aligned}$$

Since the scalar product of $(\sigma_\ell \lambda_\ell)_{\ell=1,\dots,r}$ and $(\mu_1, \dots, \mu_r)$ is maximal if the two vectors are parallel,

$$\sup_{\substack{\mu \in \mathbb{R}^n \\ |\mu| = 1}} \sum_{\ell=1}^{r} (\sigma_\ell \lambda_\ell)\, \mu_\ell = \Big(\sum_{\ell=1}^{r} \sigma_\ell^2 \lambda_\ell^2\Big)^{1/2}.$$

Hence, there holds

$$\alpha_h = \inf_{\substack{\lambda \in \mathbb{R}^m \\ |\lambda| = 1}} \Big(\sum_{\ell=1}^{r} \sigma_\ell^2 \lambda_\ell^2 \Big)^{1/2}.$$

This is zero if $n > r$, and equals the singular value σ_r if $n \leq r$.

It is emphasised here, that $m = n$ if A is regular and then α_h is the smallest (positive) singular value of A.

Definition 3.3 (Exact Problem)**.** Given $f \in Y^*$, find $x \in X$ with

$$a(x, y) = f(y) \quad \text{for all } y \in Y. \tag{P}$$

Definition 3.4 (Discrete Problem)**.** Given $f \in Y^*$, find $x_h \in X_h$ with

$$a(x_h, y_h) = f(y_h) \quad \text{for all } y_h \in Y_h. \tag{P_h}$$

Theorem 3.5 (Existence and Uniqueness)**.**

(a) The discrete problem (P_h) *has a unique solution* $x_h \in X_h$ *for all* $f \in Y^*$ *if* $\alpha_h > 0$ *and a is non-degenerated in the sense of*

$$\forall\, y_h \in Y_h \; \exists\, x_h \in X_h,\; a(x_h, y_h) \neq 0. \tag{ND}$$

(Analogous results hold for the exact problem.)

(b) Assume (P) *to be uniquely solvable and let* $x \in X$ *be its unique solution. Furthermore, suppose* $\alpha_h > 0$ *in* (9), (ND), *and that* x_h *solves* (P_h)*. Then*

$$\|x - x_h\|_X \leq (1 + \|a\|/\alpha_h) \inf_{v_h \in X_h} \|x - v_h\|_X \,. \tag{10}$$

Remark 3.6. The estimate (10) is called quasi-optimal, if $\alpha_h \geq \alpha > 0$ is independent of $\dim X_h$ for a family of discrete spaces $(X_h)_h$. It links the Galerkin error $\|x - x_h\|_X$ with the best-approximation error $\operatorname{dist}(x, X_h)$ by

$$\|x - x_h\| \approx \operatorname{dist}(x, X_h).$$

Proof of Theorem 3.5.

(a) This follows from Theorem 2.2 by replacing $X \times Y$ with $X_h \times Y_h$.

(b) For the exact solution x and a discrete solution x_h there holds the Galerkin orthogonality

$$a(x - x_h, y_h) = 0 \text{ for all } y_h \in Y_h.$$

Given $v_h \in X_h$ and $\varepsilon > 0$, (9) and $\alpha_h > 0$ imply that there exists $y_h \in Y_h$ with $\|y_h\|_Y = 1$ and

$$\begin{aligned}(\alpha_h - \varepsilon)\, \|x_h - v_h\|_X &\le a(x_h - v_h, y_h) \\ &= a(x - v_h, y_h) \\ &\le \|a\|\, \|x - v_h\|_X \,.\end{aligned}$$

Here, the second step is via the aforementioned Galerkin orthogonality and the last step is the boundedness of a. Since $\varepsilon > 0$ is arbitrarily small, this implies

$$\|x_h - v_h\|_X \le \frac{\|a\|}{\alpha_h}\, \|x - v_h\|_X \,.$$

This and a triangle inequality show the assertion:

$$\begin{aligned}\|x - x_h\|_X &\le \|x - v_h\|_X + \|v_h - x_h\|_X \\ &\le \left(1 + \frac{\|a\|}{\alpha_h}\right) \|x - v_h\|_X \,. \qquad \square\end{aligned}$$

3.2 Stable and Instable MFEM

This subsection considers one instable and one stable MFEM, i.e., violating and satisfying the (uniform) LBB condition, respectively.

Consider the following discrete problem (MFEM) for $\Omega \subset \mathbb{R}^n$ and finite dimensional subspaces $X_h \subset H(\mathrm{div}, \Omega)$ and $Y_h \subset L^2(\Omega)$ from Subsection 1.3. Compute $(p_h, u_h) \in X_h \times Y_h$ that satisfy

$$\begin{aligned}(p_h, q_h)_{L^2(\Omega)} + (u_h, \mathrm{div}\, q_h)_{L^2(\Omega)} &= 0 \quad \text{for all } q_h \in X_h, \\ (\mathrm{div}\, p_h, v_h)_{L^2(\Omega)} &= -(f, v_h)_{L^2(\Omega)} \quad \text{for all } v_h \in Y_h.\end{aligned} \tag{11}$$

Definition 3.7 (Triangulation)**.** A regular triangulation $\mathcal{T}$ into triangles (for $n = 2$) is a set of closed triangles T of positive area $|T|$ such that any two distinct triangles T_1 and T_2 are either disjoint, $T_1 \cap T_2 = \emptyset$, or share exactly one vertex z, $T_1 \cap T_2 = \{z\}$, or have one edge $E = T_1 \cap T_2$ in common. The set of all edges is denoted by $\mathcal{E}$, the set of nodes is denoted by $\mathcal{N}$. Each edge is associated to a length $h_E := \mathrm{diam}(E)$ and a unit normal and unit tangential vector ν_E and τ_E. The words mesh and triangulation are used as synonyms of each other.

Definition 3.8 (FE Spaces)**.** For a triangulation $\mathcal{T}$ of Ω and $k = 0, 1, 2, \ldots$ define

$$\begin{aligned} P_k(T) &:= \{\text{polynomials on } T \text{ with degree } \le k\}, \\ P_k(\mathcal{T}) &:= \{f \in L^2(\Omega) \,:\, \forall T \in \mathcal{T} \;\; f|_T \in P_k(T)\}, \\ \mathcal{S}^k(\mathcal{T}) &:= \{f \in P_k(\mathcal{T}) \,:\, f \text{ globally continuous in } \overline{\Omega}\,\}, \\ \mathcal{S}^k_0(\mathcal{T}) &:= \{f \in \mathcal{S}^k(\mathcal{T}) \,:\, f = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

Moreover, the nodal basis function $\phi_z \in \mathcal{S}^1(\mathcal{T})$ is defined by $\phi_z(z) = 1$ and $\phi_z(y) = 0$ for $z \in \mathcal{N}$ and all other nodes $y \in \mathcal{N} \setminus \{z\}$. One can prove that $(\varphi_z \,:\, z \in \mathcal{N})$ is indeed a basis of $\mathcal{S}^1(\mathcal{T})$ and $(\varphi_z \,:\, z \in \mathcal{N} \cap \Omega)$ is a basis of $\mathcal{S}^1_0(\mathcal{T})$.

Example 3.9 (Unstable P_1-P_0 MFEM)**.** The 2D $P_1 - P_0$ MFEM is defined by the discrete spaces

$$X_h := (\mathcal{S}^1(\mathcal{T}))^2 \quad \text{and} \quad Y_h := P_0(\mathcal{T}).$$

As an example for instability of the linear system of equations underlying (P_h), consider the Poisson problem (4) on an L-shaped domain for $g \equiv 1$. A simple implementation that results after some uniform refinement in a singular matrix A of the linear system associated with (P_h) is depicted in Figure 1. The outcome is that the program stops with the error message "Warning: Matrix is singular to working precision." To illustrate this singularity, Figure 3.2 shows the smallest singular value λ of the energy matrix A with respect to the degrees of freedom. For $\max_{E\in\mathcal{E}} h_E \to 0$ the singular value λ goes to 0. This yields instability as it can be observed in Figure 3.

Remark 3.10. Based on the Gauß divergence theorem, one can prove that any piecewise polynomial function $q_h \in P_k(\mathcal{T})^n \subset L^2(\Omega)^n$ belongs to $H(\operatorname{div}, \Omega)$ if and only if the jump of the normal component of q_h across E vanishes, i.e.,

$$[q_h]_E \cdot \nu_E = 0 \quad \text{on each interior edge } E.$$

This condition clearly indicates that the $P_1 - P_0$ MFEM is too restrictive by the request of global continuity.

Definition 3.11 (Raviart-Thomas FEM)**.** For a regular triangulation $\mathcal{T}$ of $\Omega \subseteq \mathbb{R}^2$ into triangles set

$$RT_k(T) := \left\{ (x,y) \mapsto \begin{pmatrix} p_1(x,y) \\ p_2(x,y) \end{pmatrix} + p_3(x,y)\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \,:\, p_1, p_2, p_3 \in P_k(T) \right\}$$