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Editor

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 503

CLASSICAL AND ADVANCED THEORIES OF THIN STRUCTURES

MECHANICAL AND MATHEMATICAL ASPECTS

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PREFACE

Thin structures are frequently used in several areas of civil, mechanical and aeronautical engineering. Slender beams, plates and shells, just to mention some of the most common examples, are especially advantageous in structural design applications since they are able to ensure a high ratio between "strength" and weight.

Approximate analytical models for thin structures are hundreds of years old and go back to the pioneering works by Euler, D. Bernoulli, Navier and Kirchhoff, to mention a few. These classical theories are based on the introduction of some a-priori assumptions motivated by the smallness of certain dimensions with respect to others, on the stress field or on the deformation of the thin body. In the last few decades a considerable amount of work has been done in order to rigorously justify these a-priori assumptions. In particular, approaches based on rigorous asymptotic expansion (mainly due to the French school) or inspired by the Gamma-convergence of energy functionals (proposed by E. De Giorgi in 1979) have been successfully used in deriving one or two-dimensional classical mechanical models for thin structures, in linear and non-linear elasticity. Besides these asymptotic methods, another approach, called the method of internal constraints, was developed. This method is based on viewing the a-priori kinematical assumptions as internal constraints, which generate reactive stresses, and to obtain the lower dimensional theory by integration over the "small" dimensions. Asymptotic methods and the method of internal constraints could be seen as complementary: on one hand, asymptotic methods provide a rigorous justification of the starting point of the method of internal constraints, on the other hand, this latter method provides an intuitive and consistent mechanical deduction of lower dimensional theories without an extensive use of deep mathematical tools.

The aim of the CISM course entitled "Classical and Advanced Theories of Thin Structures: Mechanical and Mathematical Aspects", held in Udine on June 5-9 2006, was to present an up-to-date overview of the general aspects and applications of the theories for thin structures, through the interaction of several topics, ranging from nonlinear thin-films, shells, beams of different materials and in different contexts (elasticity, plasticity, etc.).

The course was addressed to PhD students and researchers in the fields of continuum mechanics, structural engineering and applied mathematics. The plan of the course and the lectures have taken into account the different background of the audience.

The first two chapters by Morassi and Paroni introduce the main notions of continuum mechanics: the concepts of stress, deformation and constitutive equations. Moreover, in discussing the existence of the solution for the elasticity problem, several mathematical tools such as Sobolev spaces, weak convergence, Lax-Milgram lemma and Korn's inequalities are recalled. As anticipated earlier, the notion of Gamma-convergence plays a fundamental role in the deduction of lower dimensional models. Percivale in the third chapter introduces the notion and describes the main properties of Gamma-convergence. Besides presenting some illustrative examples, he also discusses the properties of the limit problem for an elastic-plastic beam. In the fourth chapter Podio-Guidugli presents a unified approach to classic rod and plate theories by means of a procedure which is an outgrowth of the method of internal constraints. He also discusses how to approximate the stress field in a linearly elastic structure-like body. The next chapter by Ciarlet is devoted to a detailed description of the nonlinear and linear equations proposed by W.T. Koiter for modelling thin elastic shells. The existence, uniqueness and regularity of solutions to the linear Koiter equations is then established, thanks to a fundamental Korn inequality on surfaces. At the beginning of the chapter the basic notions about differential geometry for surfaces are introduced. The book ends with the lectures by Fonseca. In the first two she applies Gamma-convergence techniques to deduce the energy for brittle and non-brittle thin films. The last four lectures are dedicated to the study of various interesting aspects of nonlinear membranes, including the deduction for two nonlinear membrane models: one based on the so called Cosserat vector and the other written in terms of Young measures. This latter model is appropriate for the study of phase transitions.

Antonino Morassi and Roberto Paroni

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Strain, Stress and Linearized Elasticity

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Abstract In these notes I introduce the concepts of strain and stress, and I present the linearized elasticity problem.

1 Introduction

In these notes I present some basic elements of Continuum Mechanics. The first two chapters introduce the concepts of strain and stress, respectively. The third chapter is devoted to the derivation of the linearized elasticity and to the presentation of some related existence and uniqueness results.

The manuscript reproduces more or less the pattern followed in the notes of the three lectures delivered at CISM. The presentation of the arguments is deliberately simplified and there is no claim to general and exhaustive treatment. Indeed, the aim of this first series of lectures, and of those by Paroni (2008), was to introduce the audience of the course to the principles of Continuum Mechanics and to the mathematical tools adequate for the correct formalization of the problems.

The reader interested to further investigate the issues outlined in these notes is invited to consult the classical treatises of Continuum Mechanics and Theory of Elasticity, an essential list of which can be found in the references.

2 Strain

2.1 Points, vectors and tensors

Let us denote by $\mathbb{R}^3$ the cartesian space formed by the usual vectorial space of all ordered triplets, or *points*, $X = (X_1, X_2, X_3)$, $X_i \in \mathbb{R}$ for $i = 1, 2, 3$, endowed with an euclidean structure induced by the distance

$$d(X,Y) = (\sum_{i=1}^{3}(X_i - Y_i)^2)^{1/2} \qquad (1)$$

between any two points X and $Y = (Y_1, Y_2, Y_3)$ of $\mathbb{R}^3$. Here, X_1, X_2, X_3 are the *orthogonal cartesian coordinates* of X in $\mathbb{R}^3$.

We shall denote by V the vector space associated with $\mathbb{R}^3$. The elements of V are called *vectors*. Every vector $\mathbf{v}$ may be interpreted as difference between two points of $\mathbb{R}^3$, namely the head, Y, and the end, X, of the pointed arrow used to denote the vector $\mathbf{v}$, e.g. $\mathbf{v} = Y - X$. The euclidean distance of $\mathbb{R}^3$ endows V with its natural metric structure: for every $\mathbf{v} \in V$, the norm $|\mathbf{v}|$ of $\mathbf{v}$ is the length of the arrow which represents $\mathbf{v}$. In particular, $|\mathbf{v}| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$, where $\cdot$ is the usual inner product of V.

We think V equipped with a fixed cartesian frame $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, where O is a chosen point of $\mathbb{R}^3$, called *origin*, and $\{\mathbf{e}_i\}_{i=1}^3$ is an orthonormal basis of V, e.g. $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Therefore, the (cartesian) components of a vector $\mathbf{v}$ are given by

$$v_i = \mathbf{v} \cdot \mathbf{e}_i, \tag{2}$$

$i = 1, 2, 3$, so that

$$\mathbf{v} \cdot \mathbf{u} = \sum_{i=1}^{3} v_i u_i. \tag{3}$$

Similarly, if $\{\mathbf{e}_i\}_{i=1}^3$ are the versors of the axes of the cartesian space $\mathbb{R}^3$, recalling that $\mathbf{X} = X - O$, the ith coordinate of the point X is $X_i = \mathbf{X} \cdot \mathbf{e}_i$, $i = 1, 2, 3$.

The set of all the linear transformations of V into itself is denoted by Lin. Each element of Lin is called second-order tensor. Equipped with the usual operations of addition and scalar multiplication, Lin is a vector space. Given an orthonormal basis $\{\mathbf{e}_i\}_{i=1}^3$ of V, the components A_{ij} of $\mathbf{A} \in \mathrm{Lin}$ with respect to this basis are defined by

$$A_{ij} = \mathbf{e}_i \cdot \mathbf{A}\mathbf{e}_j, \quad i, j = 1, 2, 3. \tag{4}$$

2.2 Bodies and deformations

For us, a *body* B is a set of particles. These particles can be considered as primitive elements of Mechanics. The body B is assumed to be smooth, that is the particles $X \in B$ can be set into one-to-one correspondence with a bounded *domain*[1] in $\mathbb{R}^3$ with regular boundary, and the mapping is assumed to be differentiable as many times as desired (usually two or three times).

Bodies are available to us only in their *configurations*, that is the regions they happen to occupy in $\mathbb{R}^3$. It is often convenient to select one particular configuration and refer everything concerning the body to that

[1] Let Ω be an open and connected set of $\mathbb{R}^3$. The closure of Ω, $\overline{\Omega} = \Omega \cup \partial\Omega$, is called here *domain*.

configuration. Let $\boldsymbol{\kappa}$ be such configuration

$$\begin{array}{rccl}\boldsymbol{\kappa}: & B & \to & \mathbb{R}^3 \\ & X & \mapsto & \boldsymbol{\kappa}(X) = \mathbf{X}\,, \end{array} \tag{5}$$

where $\mathbf{X}$ is the place occupied by the particle X in the configuration $\boldsymbol{\kappa}$. $\boldsymbol{\kappa}$ is assumed to be smooth, in particular $X = \boldsymbol{\kappa}^{-1}(\mathbf{X})$ is well-defined. $\boldsymbol{\kappa}(B)$ denotes the set of all the reference placements of the body; it will be called *reference configuration* and denoted by $\overline{\Omega}$.

The deformation of the body will be described as a mapping acting on the reference configuration $\boldsymbol{\kappa}$ onto the *actual configuration* $\boldsymbol{\chi}$. Therefore, a deformation of the body

$$\begin{array}{rccl}\boldsymbol{\chi}: & \overline{\Omega} & \to & \mathbb{R}^3 \\ & \mathbf{X} & \mapsto & \boldsymbol{\chi}(\mathbf{X}) = \mathbf{x} \end{array} \tag{6}$$

gives the actual placement $\mathbf{x}$ occupied by the particle $\mathbf{X}$ in the reference configuration. We indicate with $\boldsymbol{\chi}(\overline{\Omega})$ the image of $\overline{\Omega}$ under the map $\boldsymbol{\chi}$ or, briefly, the *deformation* of $\overline{\Omega}$ under $\boldsymbol{\chi}$.

We accept the following a priori assumptions on $\boldsymbol{\chi}$:

i) There is a bijection between $\overline{\Omega}$ and $\boldsymbol{\chi}(\overline{\Omega})$.

ii) $\boldsymbol{\chi} \in C^1(\overline{\Omega})$.

iii) The orientation preserving condition

$$\det \nabla\boldsymbol{\chi}(\mathbf{X}) > 0 \tag{7}$$

holds in $\overline{\Omega}$. Here, ∇ is the gradient operator[2] and det is the determinant operator.[3]

By above assumptions, the map $\boldsymbol{\chi}$ is globally invertible in $\overline{\Omega}$, e.g. there exists the inverse of $\boldsymbol{\chi}$, say $\boldsymbol{\chi}^{-1}$, such that

$$\boldsymbol{\chi}^{-1}(\mathbf{x}) = \mathbf{X} \quad \text{in } \boldsymbol{\chi}(\overline{\Omega}),$$

[2] Let $\boldsymbol{\chi} : \Omega \to \mathbb{R}^3$ be a function of class C^1 in Ω. The *gradient* of $\boldsymbol{\chi}$ at $\mathbf{X} \in \Omega$ is the unique element $\nabla\boldsymbol{\chi}$ such that

$$\lim_{|\mathbf{h}|\to 0} \frac{|\boldsymbol{\chi}(\mathbf{X}+\mathbf{h}) - \boldsymbol{\chi}(\mathbf{X}) - \nabla\boldsymbol{\chi}(\mathbf{X})\mathbf{h}|}{|\mathbf{h}|} = 0.$$

If $\boldsymbol{\chi}$ is of C^1-class up to the (smooth) boundary $\partial\Omega$ of Ω, then the gradient of $\boldsymbol{\chi}$ at $\mathbf{X} \in \partial\Omega$ is well-defined and it can be evaluated as limit of $\nabla\boldsymbol{\chi}(\mathbf{Y})$, $\mathbf{Y} \in \Omega$, as $\mathbf{Y} \to \mathbf{X}$.

[3] Let $\mathbf{A} \in \mathrm{Lin}$; $\det \mathbf{A} \in \mathbb{R}$ is defined as

$$(\det \mathbf{A})\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = (\mathbf{A}\mathbf{a} \times \mathbf{A}\mathbf{b}) \cdot \mathbf{A}\mathbf{c} \quad \text{for every } \mathbf{a}, \mathbf{b}, \mathbf{c} \in V \text{ such that } \mathbf{a} \times \mathbf{b} \cdot \mathbf{c} \neq 0.$$

Here $\times$ is the vectorial product in $V \times V$.

and it still satisfies conditions i)-iii).

The *displacement field* $\mathbf{u}(\mathbf{X})$ associated with the deformation $\boldsymbol{\chi}$ is defined as follows:

$$\begin{array}{rccl} \mathbf{u}: & \overline{\Omega} & \rightarrow & \mathbb{R}^3 \\ & \mathbf{X} & \mapsto & \mathbf{u}(\mathbf{X}) = \boldsymbol{\chi}(\mathbf{X}) - \mathbf{X} \end{array} \tag{8}$$

The vector $\mathbf{u}(\mathbf{X})$ is the displacement from the reference configuration to the deformed shape at the point $\mathbf{X}$. The *displacement gradient* is given by

$$\nabla \mathbf{u} = \nabla \boldsymbol{\chi} - \mathbf{1}\ , \tag{9}$$

where $\mathbf{1} \in \mathrm{Lin}$ is the identity tensor.

Following a standard notation, we shall denote by $\mathbf{F}$ and $\mathbf{H}$ the deformation gradient and the displacement gradient, respectively, i.e.

$$\begin{array}{l} \nabla \boldsymbol{\chi}(\mathbf{X}_0) = \mathbf{F}(\mathbf{X}_0)\ , \\ \nabla \mathbf{u}(\mathbf{X}_0) = \mathbf{H}(\mathbf{X}_0), \quad \mathbf{X}_0 \in \overline{\Omega}. \end{array} \tag{10}$$

The values $\mathbf{F}(\mathbf{X}_0)$, $\mathbf{H}(\mathbf{X}_0)$, $\mathbf{X}_0 \in \overline{\Omega}$, are elements of Lin. In particular, the values $\mathbf{F}(\mathbf{X}_0)$ are elements of the set of positive linear transformations Lin^+:

$$\mathrm{Lin}^+ = \{\mathbf{A} \in \mathrm{Lin} \mid \det \mathbf{A} > 0\} \tag{11}$$

2.3 Deformation: examples

i) *Rigid deformation*

We say that a deformation is rigid if it leaves the distance between any pair of points of $\overline{\Omega}$ unchanged, i.e.

$$|\boldsymbol{\chi}(\mathbf{X}) - \boldsymbol{\chi}(\mathbf{Y})| = |\mathbf{X} - \mathbf{Y}| \quad \text{for every } \mathbf{X},\ \mathbf{Y} \in \overline{\Omega}\ .$$

One can show that

$$\boldsymbol{\chi}(\mathbf{X}) = \boldsymbol{\chi}(\mathbf{X}_0) + \mathbf{Q}(\mathbf{X} - \mathbf{X}_0)\ , \tag{12}$$

where $\mathbf{X}_0$ is any point of $\overline{\Omega}$ and $\mathbf{Q}$ is a constant *orthogonal tensor*[4] with $\det \mathbf{Q} = 1$. Then, we have

$$\mathbf{F}(\mathbf{X}) = \mathbf{Q} \quad \text{in } \overline{\Omega}$$

[4] A second-order tensor $\mathbf{Q} \in \mathrm{Lin}$ is called *orthogonal* if $\mathbf{Q}\mathbf{Q}^T = \mathbf{1}$. We shall denote by Orth the collection of all second-order orthogonal tensors. We call *rotation* an orthogonal tensor $\mathbf{Q}$ with $\det \mathbf{Q} = 1$ and we shall indicate

$$\mathrm{Orth}^+ = \{\mathbf{Q} \in \mathrm{Orth} \mid \det \mathbf{Q} = 1\}.$$

and the *rigid displacement field* is given by

$$\mathbf{u}(\mathbf{X}) = \mathbf{u}(\mathbf{X}_0) + (\mathbf{Q} - \mathbf{1})(\mathbf{X} - \mathbf{X}_0) \quad \text{in } \overline{\Omega}\ .$$

A rigid deformation is a translation if $\mathbf{Q} = \mathbf{1}$, a rotation (about $\mathbf{X}_0$) if $\mathbf{u}(\mathbf{X}_0) = \mathbf{0}$, cf. Ex. 3.

ii) *Simple extension*

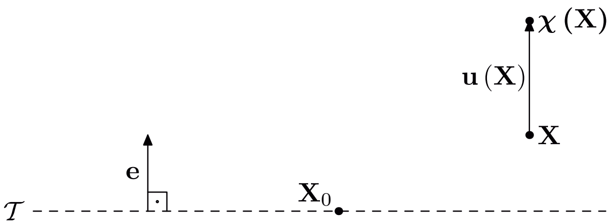

Figure 1. Simple extension in the direction **e**.

Let $\mathbf{X}_0$ be a point on $\overline{\Omega}$ which remains fixed during deformation. Let $\mathcal{T}$ the plane passing through $\mathbf{X}_0$ with normal $\mathbf{e}$, $|\mathbf{e}| = 1$. The simple extension, of amount α in the direction $\mathbf{e}$, is such that the displacement $\mathbf{u}(\mathbf{X})$ of point $\mathbf{X}$ is parallel to $\mathbf{e}$-direction and its modulus is proportional to the distance of $\mathbf{X}$ from $\mathcal{T}$. That is

$$\chi(\mathbf{X}) = \mathbf{X} + \alpha[(\mathbf{X} - \mathbf{X}_0) \cdot \mathbf{e}]\mathbf{e} = \mathbf{X} + \alpha(\mathbf{e} \otimes \mathbf{e})(\mathbf{X} - \mathbf{X}_0)\ , \tag{13}$$

where, for any two vectors $\mathbf{a}, \mathbf{b} \in V$, $\mathbf{a} \otimes \mathbf{b}$ denotes the element of Lin such that $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$ for every $\mathbf{v} \in V$. We have

$$\mathbf{u}(\mathbf{X}) = \alpha((\mathbf{X} - \mathbf{X}_0) \cdot \mathbf{e})\mathbf{e}\ ,$$

$$\mathbf{F}(\mathbf{X}) = \mathbf{1} + \alpha(\mathbf{e} \otimes \mathbf{e}), \quad \mathbf{H}(\mathbf{X}) = \alpha(\mathbf{e} \otimes \mathbf{e})\ ,$$

$$\det \mathbf{F} > 0 \Longleftrightarrow \alpha > -1\ .$$

iii) *Simple shear*

Let $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$, $|\mathbf{e}_1| = |\mathbf{e}_2| = 1$. With the notation of Figure 2, the deformation of simple shear, of amount γ in the plane $(\mathbf{e}_1, \mathbf{e}_2)$, is such that

$$\chi(\mathbf{X}) = \mathbf{X} + \gamma((\mathbf{X} - \mathbf{X}_0) \cdot \mathbf{e}_2)\mathbf{e}_1 = \mathbf{X} + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2)(\mathbf{X} - \mathbf{X}_0) \tag{14}$$

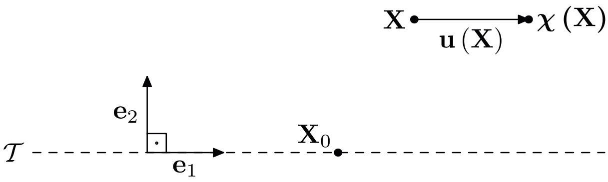

Figure 2. Simple shear in the plane $(\mathbf{e}_1, \mathbf{e}_2)$.

and then

$$\mathbf{u}(\mathbf{X}) = \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2)(\mathbf{X} - \mathbf{X}_0),$$

$$\mathbf{F}(\mathbf{X}) = \mathbf{1} + \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2), \quad \mathbf{H}(\mathbf{X}) = \gamma(\mathbf{e}_1 \otimes \mathbf{e}_2),$$

$$\det \mathbf{F} = 1.$$

These three cases are examples of *homogeneous deformations*, that is deformations with constant gradient in $\overline{\Omega}$:

$$\boldsymbol{\chi}(\mathbf{X}) = \boldsymbol{\chi}(\mathbf{X}_0) + \mathbf{F}(\mathbf{X} - \mathbf{X}_0), \quad \mathbf{X}_0 \in \overline{\Omega}, \quad \text{in } \overline{\Omega}, \tag{15}$$

with $\mathbf{F} = const$ in $\overline{\Omega}$.

2.4 Changes in length, area, volume, angle. Strain measures

The knowledge of the gradient $\mathbf{F}$ at $\mathbf{X}_0 \in \Omega$ allows to characterize locally the deformation. In fact, by the regularity of $\boldsymbol{\chi}$ we have

$$\boldsymbol{\chi}(\mathbf{X}_0 + d\mathbf{X}) = \boldsymbol{\chi}(\mathbf{X}_0) + \mathbf{F}(\mathbf{X}_0)d\mathbf{X} + o(d\mathbf{X};\ \mathbf{X}_0), \tag{16}$$

with $\lim_{|d\mathbf{X}| \to 0} \frac{|o(d\mathbf{X};\ \mathbf{X}_0)|}{|d\mathbf{X}|} = 0$. When $|d\mathbf{X}|$ is *sufficiently small*, then the deformation can be locally approximated by

$$\boldsymbol{\chi}(\mathbf{X}_0 + d\mathbf{X}) = \boldsymbol{\chi}(\mathbf{X}_0) + \mathbf{F}(\mathbf{X}_0)d\mathbf{X}. \tag{17}$$

Denoting by

$$d\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}_0 + d\mathbf{X}) - \boldsymbol{\chi}(\mathbf{X}_0), \tag{18}$$

we have

$$d\mathbf{x} = \mathbf{F}(\mathbf{X}_0)d\mathbf{X}. \tag{19}$$

This equation can be regarded as a local description of the deformation in a neighborhood of $\mathbf{X}_0$. In this description the vector $d\mathbf{X}$ is mapped linearly into $\mathbf{F}(\mathbf{X}_0)d\mathbf{X}$.

If $d\mathbf{X}$ in Eq. (19) is identified with an *infinitesimal* oriented line element (or *material fiber*) through $\mathbf{X}_0$, then Eq. (19) can be used for evaluating the local changes in length, area, etc.

Change in length

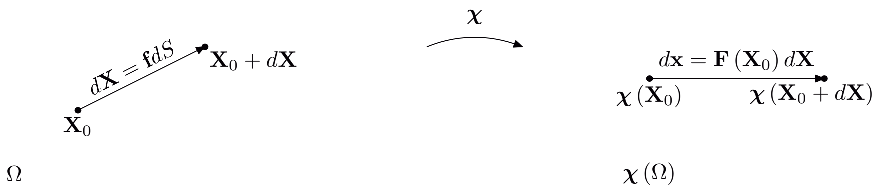

Figure 3. Local change in length.

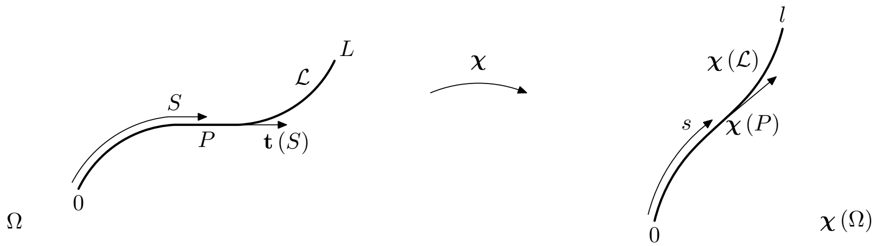

Figure 4. Change in length of finite curves.

With reference to Figure 3, let $d\mathbf{X} = dS\mathbf{f}$ with $|\mathbf{f}| = 1$. The length of the material fiber $d\mathbf{X}$ after the deformation is (here $\mathbf{F} = \mathbf{F}(\mathbf{X}_0)$)

$$ds = |d\mathbf{x}| = \sqrt{\mathbf{F}^T\mathbf{F}\mathbf{f} \cdot \mathbf{f}}\, dS$$

and, therefore, the local change in length is controlled by the *Cauchy-Green* *right tensor* $\mathbf{C} = \mathbf{F}^T\mathbf{F}$[5]:

$$\frac{(ds)^2 - (dS)^2}{(dS)^2} = (\mathbf{C} - \mathbf{1})\mathbf{f} \cdot \mathbf{f}. \tag{20}$$

$\mathbf{C}$ is a symmetric positive tensor.[6]

We can use the above result for estimating the change in length of finite (simple and regular) curves $\mathcal{L} \subset \overline{\Omega}$, see Figure 4. Let S and s be the arch length on $\mathcal{L}$ and the corresponding on $\boldsymbol{\chi}(\mathcal{L})$, respectively. The length of the deformed curve is given by $l = \int_{0(\boldsymbol{\chi}(\mathcal{L}))}^{l} ds$. By making the change of variables $s = s(S)$ we have $l = \int_{0(\mathcal{L})}^{L} (\frac{ds}{dS})dS$ and $\frac{ds}{dS} = \sqrt{\mathbf{C(S)}\boldsymbol{\tau} \cdot \boldsymbol{\tau}}$, where $\boldsymbol{\tau} = \boldsymbol{\tau}(S)$ is the unit tangent vector to $\mathcal{L}$ at the point of abscissa S.

Change in area

Consider two material fibers through $\mathbf{X}_0$, say $d\mathbf{X}_1 = dS_1\mathbf{f}_1$ and $d\mathbf{X}_2 = dS_2\mathbf{f}_2$, with $|\mathbf{f}_1| = |\mathbf{f}_2| = 1$ and $\mathbf{f}_1 \times \mathbf{f}_2 \neq \mathbf{0}$. Denote by $d\mathbf{A} = d\mathbf{X}_1 \times d\mathbf{X}_2$ the *infinitesimal oriented surface* in the referential configuration $\overline{\Omega}$. Under the deformation $\boldsymbol{\chi}$ the images of $d\mathbf{X}_1$, $d\mathbf{X}_2$ are respectively $d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1$, $d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$, where $\mathbf{F} = \mathbf{F}(\mathbf{X}_0)$. Therefore, the image of $d\mathbf{A}$ is $d\mathbf{a} = \mathbf{F}d\mathbf{X}_1 \times \mathbf{F}d\mathbf{X}_2$, which corresponds to the oriented infinitesimal surface area in the actual configuration $\boldsymbol{\chi}(\overline{\Omega})$, see Figure 5.

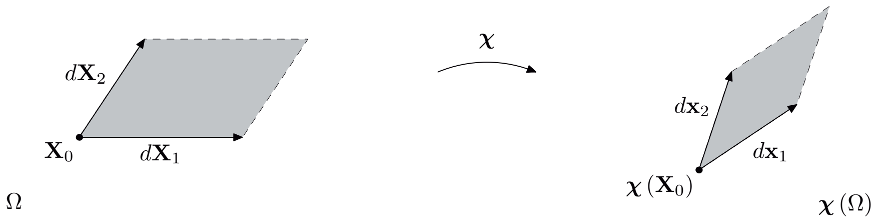

Figure 5. Local change in area.

We have

$$dA = |d\mathbf{A}| = |d\mathbf{X}_1 \times d\mathbf{X}_2|,$$

[5]For each $\mathbf{A} \in \mathrm{Lin}$, the transpose $\mathbf{A}^T$ of $\mathbf{A}$ is the unique tensor $\mathbf{A}^T\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{A}\mathbf{b}$ for every $\mathbf{a}, \mathbf{b} \in V$.

[6]A tensor $\mathbf{A} \in \mathrm{Lin}$ is *symmetric* if $\mathbf{A} = \mathbf{A}^T$. The set of second-order symmetric tensors is denoted by Sym. $\mathbf{A} \in \mathrm{Sym}$ is *positive definite*, and we write $\mathbf{A} \in \mathrm{Sym}^+$, if $\mathbf{A}\mathbf{a} \cdot \mathbf{a} \geq 0$ for every $\mathbf{a} \in V$ and $\mathbf{A}\mathbf{a} \cdot \mathbf{a} = 0$ implies $\mathbf{a} = \mathbf{0}$.

$$da = |d\mathbf{a}| = |d\mathbf{x}_1 \times d\mathbf{x}_2|$$

and, therefore, using the vectorial identities $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$ and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, we obtain

$$\frac{da}{dA} = \left\{ \frac{(\mathbf{Cf}_1 \cdot \mathbf{f}_1)(\mathbf{Cf}_2 \cdot \mathbf{f}_2) - (\mathbf{Cf}_1 \cdot \mathbf{f}_1)^2}{(\mathbf{f}_1 \cdot \mathbf{f}_1)(\mathbf{f}_2 \cdot \mathbf{f}_2) - (\mathbf{f}_1 \cdot \mathbf{f}_2)^2} \right\}^{1/2} . \tag{21}$$

The change in area of a finite regular surface $\mathcal{S}$ can be evaluated as follows. Let $\mathbf{X} = \mathbf{X}(\vartheta_1, \vartheta_2)$ be a representation of the surface $\mathcal{S}$, where the two parameters ϑ_α, $\alpha = 1, 2$, belong to some bounded domain D of $\mathbb{R}^2$, see Figure 6. The tangent space $\mathcal{T}(\mathcal{S}, \mathbf{X}_0)$ of $\mathcal{S}$ at the point $\mathbf{X}_0$ is spanned by the vectors $\mathbf{A}_\alpha = \frac{\partial \mathbf{X}}{\partial \vartheta_\alpha}(\mathbf{X}_0)$. Similarly, the vectors $\mathbf{a}_\alpha = \frac{\partial \boldsymbol{\chi}}{\partial \vartheta_\alpha} = \mathbf{F}(\mathbf{X}_0)\mathbf{A}_\alpha$, $\alpha = 1, 2$, constitute a basis of the tangent space $\mathcal{T}(\boldsymbol{\chi}(\mathcal{S}), \boldsymbol{\chi}(\mathbf{X}_0))$.

The initial area of $\mathcal{S}$ is $A = \int_{\mathcal{S}} dA$, whereas the area after the deformation is

$$a = \int_{\boldsymbol{\chi}(\mathcal{S})} da = \int_{\mathcal{S}} (\frac{da}{dA}) dA \ ,$$

and $\frac{da}{dA}$ can be evaluated as in Eq. (21) with $\mathbf{f}_\alpha$ replaced by $\mathbf{a}_\alpha$, $\alpha = 1, 2$.

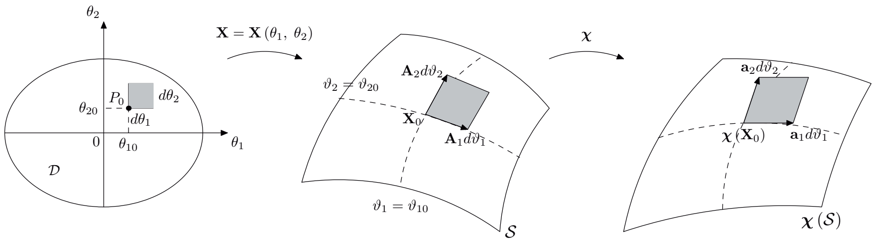

Figure 6. Change in area of a finite surface $\mathcal{S}$.

Change in volume

Let $d\mathbf{X}_i$, $i = 1, 2, 3$, be three non coplanar fibers along edges through $\mathbf{X}_0$ of a parallelepiped volume element in the reference configuration $\overline{\Omega}$. The local change in volume is equal to

$$\frac{dv - dV}{dV} = \frac{\mathbf{F}d\mathbf{X}_1 \times \mathbf{F}d\mathbf{X}_2 \cdot \mathbf{F}d\mathbf{X}_3 - d\mathbf{X}_1 \times d\mathbf{X}_2 \cdot d\mathbf{X}_3}{d\mathbf{X}_1 \times d\mathbf{X}_2 \cdot d\mathbf{X}_3}, \tag{22}$$

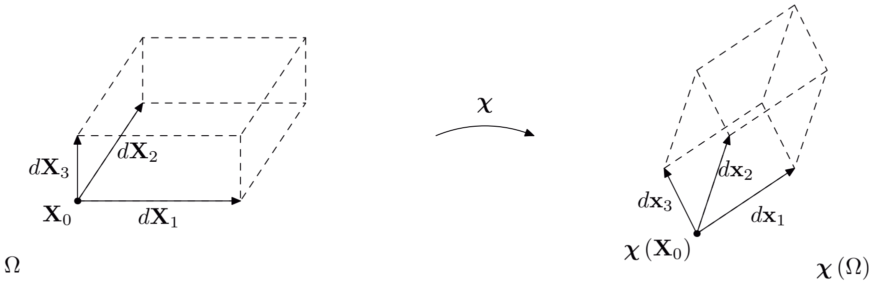

Figure 7. Local change in volume.

where $\mathbf{F}$ is the gradient at $\mathbf{X}_0$. Recalling the definition of determinant of a tensor, see footnote 3, we have

$$\frac{dv}{dV} = \det \mathbf{F},$$

which is a positive number in view of condition (7). Therefore, the volume after the deformation can be calculated as

$$v = \int_{\boldsymbol{\chi}(\Omega)} dv = \int_{\Omega} (\frac{dv}{dV}) dV = \int_{\Omega} \det \mathbf{F} dV \ .$$

Change in angle

To fix the ideas, let $d\mathbf{X}_1$ and $d\mathbf{X}_2$ be two orthogonal fibers through $\mathbf{X}_0$ in the reference configuration $\overline{\Omega}$. Let $d\mathbf{X}_\alpha = dS_\alpha \mathbf{f}_\alpha$, $\alpha = 1, 2$. By definition of scalar product we have

$$\cos\varphi_{(\mathbf{f}_1,\, \mathbf{f}_2)} = \frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{|d\mathbf{x}_1||d\mathbf{x}_2|} = \frac{\mathbf{C}\mathbf{f}_1 \cdot \mathbf{f}_2}{(\mathbf{C}\mathbf{f}_1 \cdot \mathbf{f}_1)^{1/2}(\mathbf{C}\mathbf{f}_2 \cdot \mathbf{f}_2)^{1/2}} \tag{23}$$

and, since cosine function is uniquely invertible on $[0, \pi]$, the change in angle between the directions of $\mathbf{f}_1$ and $\mathbf{f}_2$ is equal to

$$\gamma_{(\mathbf{f}_1,\, \mathbf{f}_2)} = \frac{\pi}{2} - \arccos\left(\frac{\mathbf{C}\mathbf{f}_1 \cdot \mathbf{f}_2}{(\mathbf{C}\mathbf{f}_1 \cdot \mathbf{f}_1)^{1/2}(\mathbf{C}\mathbf{f}_2 \cdot \mathbf{f}_2)^{1/2}} \right) \ . \tag{24}$$

The above analysis (local and global) shows that the changes in length, area, volume and angle, or the so-called *essential* changes in geometry, are controlled by the tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. This suggests how to measure strain.

A reasonable requirement in the definition of a strain measure is that its value should be constant over the set of all rigid deformations. In fact, it is easy to verify that $\mathbf{C} = \mathbf{1}$ for any rigid deformation. Therefore, $\mathbf{C}$ can be chosen as a local strain measure. There are other strain measures, for example that provided by the tensor

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{C} - \mathbf{1}\right), \tag{25}$$

which is identically equal to $\mathbf{0}_{\mathrm{Lin}}$ at any rigid deformation.

2.5 Polar decomposition theorem

This theorem states that there exists a unique symmetric, positive definite tensor $\mathbf{U}$ and a unique rotation $\mathbf{R}$ such that the gradient deformation $\mathbf{F}$ can be expressed as

$$\mathbf{F} = \mathbf{R}\mathbf{U}\,. \tag{26}$$

The tensor $\mathbf{U}$ is called the (right) *stretch tensor* and $\mathbf{R}$ is the *rotation tensor* in the polar decomposition (26). This factorization is a well-known linear algebra result which holds for every non singular tensor (e.g., with non zero determinant) and tell us that the deformation correponding locally to $\mathbf{F}$ may be obtained by applying first the stretch $\mathbf{U}$, followed by the rotation $\mathbf{R}$.

To make this statement more expressive, let $\mathbf{C} = \sum_{i=1}^{3} u_i^2 \boldsymbol{\eta}_i \otimes \boldsymbol{\eta}_i$, where $u_i^2 > 0$ are the eigenvalues of $\mathbf{C}$ and $\{\boldsymbol{\eta}_i\}_{i=1}^{3}$, $|\boldsymbol{\eta}_i| = 1$ and $\boldsymbol{\eta}_i \cdot \boldsymbol{\eta}_j = \delta_{ij}$, are the corresponding eigenvectors. Therefore, since $\mathbf{U}^2 = \mathbf{C}$, $\mathbf{U}$ can be represented as $\mathbf{U} = \sum_{i=1}^{3} u_i \boldsymbol{\eta}_i \otimes \boldsymbol{\eta}_i$, $u_i > 0$, $i = 1, 2, 3$. The quantities u_i are the *principal stretches* and $\boldsymbol{\eta}_i$ are the *principal directions of strain* in the reference configuration. Therefore, $\mathbf{F}$ may be obtained by affecting pure stretches of amounts, say, u_i, along the principal directions of strain $\boldsymbol{\eta}_i$, $i = 1, 2, 3$, followed by a rotation of those directions of amount $\mathbf{R}$.

2.6 Exercises

1. Prove that the following conditions are equivalent:

i) $\boldsymbol{\chi}$ is a rigid deformation.

ii) There exists $\mathbf{Q} \in \mathrm{Orth}^{+}$, $\mathbf{Q}$ constant, and there exists $\mathbf{a} \in V$, $\mathbf{a}$ constant, such that $\boldsymbol{\chi}(\mathbf{X}) = \mathbf{Q}\mathbf{X} + \mathbf{a}$.

iii) $\mathbf{C}(\mathbf{X}) = \mathbf{1}$ in $\overline{\Omega}$.

2. Let $\mathbf{A} \in \mathrm{Lin}$. We define $\mathrm{cof}\mathbf{A} \in \mathrm{Lin}$ as follows:

$$\mathrm{cof}\mathbf{A}(\mathbf{a} \times \mathbf{b}) := \mathbf{A}\mathbf{a} \times \mathbf{A}\mathbf{b}, \quad \text{for every } \mathbf{a},\ \mathbf{b} \in V\,.$$

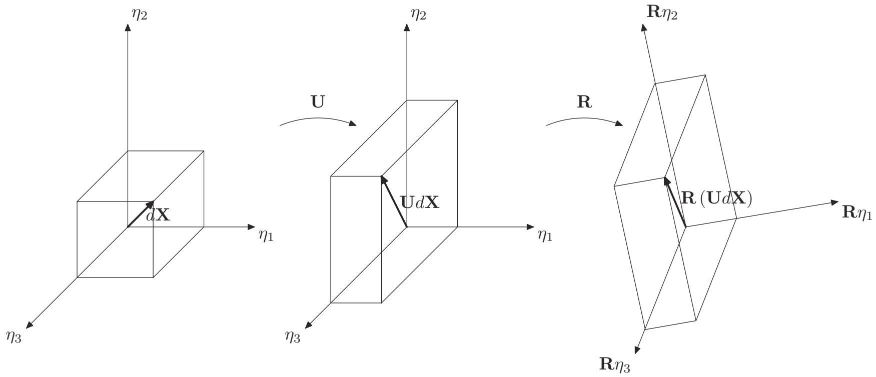

Figure 8. Polar decomposition of **F**.

Prove that $(\det \mathbf{A})\mathbf{1} = \mathbf{A}^T \mathrm{cof}\mathbf{A}$. Use the above identity to prove that the normal $\mathbf{n}_{\boldsymbol{\kappa}} = \frac{\mathbf{e}_1 \times \mathbf{e}_2}{|\mathbf{e}_1 \times \mathbf{e}_2|}$ of the oriented surface $d\mathbf{A} = \mathbf{e}_1 \times \mathbf{e}_2 dS_1 dS_2$ has (unitary) image under the deformation $\boldsymbol{\chi}$ equal to $\mathbf{n} = \frac{\mathrm{cof}\mathbf{F}\mathbf{n}}{|\mathrm{cof}\mathbf{F}\mathbf{n}|}$.

3. Prove that a rotation around an axis defined by the unit vector $\mathbf{w} \in V$ of an angle θ can be represented as

$$\mathbf{Q}(\mathbf{w},\ \theta) = \mathbf{1} + \sin\theta \mathbf{W} + (1 - \cos\theta)\,\mathbf{W}^2 \ ,$$

where $\mathbf{W}\mathbf{a} = \mathbf{w} \times \mathbf{a}$ for every $\mathbf{a} \in V$.

4. Prove the polar decomposition theorem.

5. Let $\boldsymbol{\chi} : \overline{\Omega} \to \mathbb{R}^3$ be a deformation. Denote by $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ a Cartesian frame. Prove that

$$\nabla \boldsymbol{\chi} = \sum_{i=1}^{3} \frac{\partial \boldsymbol{\chi}}{\partial X_i} \otimes \mathbf{e}_i,$$

where X_i, $i = 1, 2, 3$, are the cartesian coordinates of a point X. Compute the components of $\nabla \boldsymbol{\chi}$ in cylindrical and spherical coordinates.

6. Let Ω be a regular domain in $\mathbb{R}^3$ and let $\mathbf{v} = \mathbf{v}(\mathbf{X})$ be a regular vector field on $\overline{\Omega}$. Prove the following identities:

$$\int_{\Omega} \nabla \mathbf{v} = \int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n} \ ,$$

$$\int_{\Omega} \mathrm{Div}\, \mathbf{v} = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \ ,$$

where $\mathbf{n}$ is the unit outer normal to $\partial\Omega$ and $\mathrm{Div}\,\mathbf{v} = \mathrm{tr}(\nabla\mathbf{v})$.[7]

Let $\mathbf{A} = \mathbf{A}(\mathbf{X})$ a regular second order tensor field over Ω. Prove that

$$\int_\Omega \mathrm{Div}\mathbf{A} = \int_{\partial\Omega} \mathbf{A}\mathbf{n}\,,$$

where the divergence of a tensor field is defined as $\mathrm{Div}\left(\mathbf{A}^T\mathbf{a}\right) = \mathrm{Div}\mathbf{A}\cdot\mathbf{a}$ for every $\mathbf{a} \in V$ constant.

3 Stress

3.1 Forces and Moments

Forces are primitive elements of Mechanics and they express one of the most simple models describing the interactions between a body and the surrounding environment or between different parts of the same body.

We will introduce this notion with reference to a body B undergoing a deformation $\boldsymbol{\chi}$. Let p be a part of the body B and let $\boldsymbol{\chi}(p)$ be the deformation of p under $\boldsymbol{\chi}$. We shall postulate the presence of two kinds of forces acting on p in $\boldsymbol{\chi}(p)$:

- a body force $\mathbf{f}_b(p)$, or a volume force, caused, for example, by the gravity field or by electrostatic effects;

- a contact force $\mathbf{f}_c(p)$, due, for example, to the interaction of the body and the environment by means of the *external* boundary of the body or, more interestingly, caused by the interaction between different parts of the same body exerted through their common boundary.

We shall assume that $\mathbf{f}_b(p)$ is an absolutely continuous function of the volume of $\boldsymbol{\chi}(p)$, that is

$$\mathbf{f}_b(p) = \int_{\boldsymbol{\chi}(p)} \mathbf{b}\,dv, \tag{27}$$

where the function $\mathbf{b} : \boldsymbol{\chi}(\Omega) \to V$ is the body-force density (per unit volume). Moreover, also $\mathbf{f}_c(p)$ is assumed to be absolutely continuous function of the surface area of the boundary $\partial(\boldsymbol{\chi}(p))$ of $\boldsymbol{\chi}(p)$, that is

$$\mathbf{f}_c(p) = \int_{\partial\boldsymbol{\chi}(p)} \mathbf{t}\,da, \tag{28}$$

[7] The *trace* is the linear operator that assigns to each tensor $\mathbf{A}$ a scalar $\mathrm{tr}(\mathbf{A})$ and satisfies

$$\mathrm{tr}(\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\cdot\mathbf{b},$$

for all vectors $\mathbf{a}$ and $\mathbf{b}$.

where the function $\mathbf{t}$ defined in $\boldsymbol{\chi}(\overline{\Omega})$ is the contact-force density (per unit area). $\mathbf{t}$ will be called *traction* in the sequel.

The *resultant force* $\mathbf{r}(p)$ acting on p in $\boldsymbol{\chi}$ is given by

$$\mathbf{r}(p) = \mathbf{f}_b(p) + \mathbf{f}_c(p) = \int_{\boldsymbol{\chi}(p)} \mathbf{b}\, dv + \int_{\partial\boldsymbol{\chi}(p)} \mathbf{t}\, da. \tag{29}$$

The *resultant moment* $\mathbf{m}(p;\mathbf{x}_0)$ acting on p with respect to $\mathbf{x}_0$, $\mathbf{x}_0 \in \mathbb{R}^3$, is defined as

$$\mathbf{m}(p;\mathbf{x}_0) = \int_{\boldsymbol{\chi}(p)} (\mathbf{x}-\mathbf{x}_0) \times \mathbf{b}\, dv + \int_{\partial\boldsymbol{\chi}(p)} (\mathbf{x}-\mathbf{x}_0) \times \mathbf{t}\, da, \tag{30}$$

where we recall that $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ denotes the actual placement of the particle $\mathbf{X} \in \boldsymbol{\kappa}(B)$. We have, for any $\mathbf{x}_0$, $\mathbf{x}_0' \in \mathbb{R}^3$,

$$\mathbf{m}(p;\mathbf{x}_0') = \mathbf{m}(p;\mathbf{x}_0) + (\mathbf{x}_0 - \mathbf{x}_0') \times \mathbf{r}(p) \tag{31}$$

and $\mathbf{m}(p;\mathbf{x}_0)$ does not depend on $\mathbf{x}_0$ if and only if $\mathbf{r}(p) = \mathbf{0}$.

Remark 3.1. The expression given for $\mathbf{m}(p;\mathbf{x}_0)$ in Eq. (30) implicitly assumes that the *contact-couple density* is identically equal to zero in $\boldsymbol{\chi}(\Omega)$. This assumption can be weakened, with deep consequences on the remaining of the theory.

3.2 Euler's Laws

We can now formulate the *Euler's laws of equilibrium.* We shall say that the body B undergoing a deformation $\boldsymbol{\chi}$ is in *equilibrium*, under a system of contact and body forces, if the resultant force and the resultant moment on each part of B vanish:

$$\mathbf{r}(p) = \mathbf{0} \quad \text{for every } \boldsymbol{\chi}(p) \subset \boldsymbol{\chi}(\overline{\Omega}) \quad (\textit{force balance}), \tag{32}$$

$$\mathbf{m}(p;\mathbf{x}_0) = \mathbf{0} \quad \text{for every } \boldsymbol{\chi}(p) \subset \boldsymbol{\chi}(\overline{\Omega}) \quad (\textit{couple balance}), \tag{33}$$

where $\mathbf{x}_0$ is a given point of $\mathbb{R}^3$. Under the above conditions, we say that $\boldsymbol{\chi}$ is an *equilibrium deformation* and that $\boldsymbol{\chi}\left(\overline{\Omega}\right)$ is an *equilibrium shape* for $\overline{\Omega}$.

3.3 The Euler-Cauchy Stress Principle

The body-force density $\mathbf{b}$ and the contact-force density $\mathbf{t}$ may have very general expression and they may depend on $\mathbf{x}$, p, B, etc. In the sequel we shall restrict our attention to the so-called *external* body forces

$$\mathbf{b} = \mathbf{b}(\mathbf{x}). \tag{34}$$

Concerning the contact-force density, we shall make the following assumptions. Let $\mathbf{x}$ be a point of $\boldsymbol{\chi}(\Omega)$ and consider an oriented, regular surface Σ through $\mathbf{x}$, see Figure 9. Let $\mathbf{n}$ be the unit normal to Σ in $\mathbf{x}$. We indicate with $\mathbf{t}$ the contact-force density exerted on $\mathbf{x} \in \Sigma$ through Σ by the surrounding particles of the body which are on the portion of the body, adjacent to Σ, toward which the normal vector $\mathbf{n}$ points.

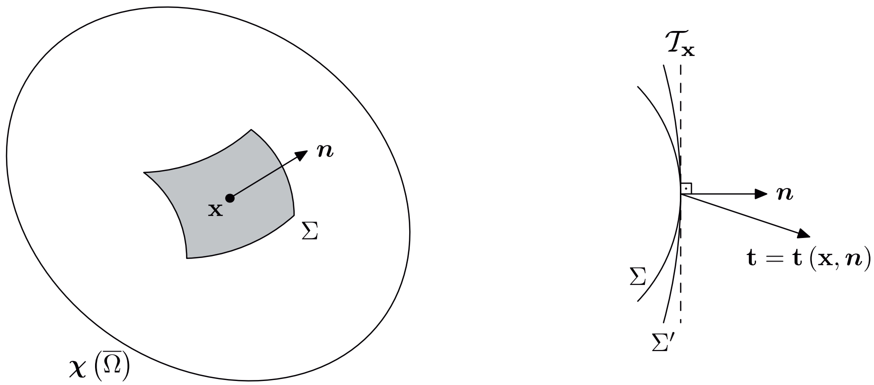

Figure 9. Traction on an oriented surface Σ through $\mathbf{x} \in \boldsymbol{\chi}(\Omega)$.

We shall assume that the traction $\mathbf{t}$ at $\mathbf{x}$ has a common value for all those surfaces having a common tangent plane $\mathcal{T}_{\mathbf{x}}$ at $\mathbf{x}$, that is

$$\mathbf{t} = \mathbf{t}(\mathbf{x}, \mathbf{n}), \qquad \mathbf{t} : \boldsymbol{\chi}(\overline{\Omega}) \times S_1 \to \mathbb{R}^3, \tag{35}$$

$S_1 = \{\mathbf{v} \in V \mid |\mathbf{v}| = 1\}$, see Figure 9. Such tractions are called *simple*. The notion of simple traction can be extended also to points $\mathbf{x}$ belonging to the *external* boundary of $\boldsymbol{\chi}(\Omega)$. The assumption embodied in the expression of the contact-force $\mathbf{f}_c(p)$ and in Eq. (35) is the so-called *Euler-Cauchy Stress Principle.*

3.4 Consequences of the Euler-Cauchy Stress Principle

Cauchy's fundamental lemma

Assume that $\mathbf{b} \in C^0(\boldsymbol{\chi}(\Omega))$ and $\mathbf{t}(\cdot, \mathbf{n}) \in C^0(\boldsymbol{\chi}(\overline{\Omega}))$ (continuity with respect to the spatial variable). Then, the force balance equation (32) implies that there exists a continuous second-order tensor field $\mathbf{T}(\mathbf{x}) \in \mathrm{Lin}$, called *stress tensor* or *Cauchy tensor*, such that

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = \mathbf{T}(\mathbf{x})\mathbf{n} \qquad \text{in } \overline{\Omega}. \tag{36}$$

Equation (36) says that the traction $\mathbf{t}$ is in fact a linear homogeneous function of the unit normal $\mathbf{n}$ of the surface on which $\mathbf{t}$ is acting. More precisely, it can be shown that $\mathbf{T}(\mathbf{x})$ has the following representation

$$\mathbf{T}(\mathbf{x}) = \sum_{i=1}^{3} \mathbf{t}(\mathbf{x}, \mathbf{e}_i) \otimes \mathbf{e}_i, \tag{37}$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the usual orthonormal basis of $\mathbb{R}^3$ and $\mathbf{t}(\mathbf{x}, \mathbf{e}_i)$ is the traction acting on a plane surface through $\mathbf{x}$, with normal $\mathbf{e}_i$, $i = 1, 2, 3$.

Local form of equilibrium equations

Let the assumptions of the Cauchy's fundamental lemma be satisfied and let $\mathbf{T} \in C^1(\boldsymbol{\chi}(\Omega)) \cap C^0(\partial\boldsymbol{\chi}(\Omega))$. Then, the force balance equation (32) and the couple balance equation (33) imply that

$$\mathrm{div}\mathbf{T} + \mathbf{b} = \mathbf{0} \quad \text{in } \boldsymbol{\chi}(\Omega)^8 \tag{38}$$

and

$$\mathbf{T} = \mathbf{T}^T \quad \text{in } \boldsymbol{\chi}(\overline{\Omega}). \tag{39}$$

In fact, by the force balance equation and the Cauchy's representation of simple tractions, we have

$$\int_{\boldsymbol{\chi}(p)} \mathbf{b}\, dv + \int_{\partial\boldsymbol{\chi}(p)} \mathbf{T}\mathbf{n}\, da = \mathbf{0} \quad \text{for every } \boldsymbol{\chi}(p) \subset \boldsymbol{\chi}(\Omega).$$

By the Divergence Theorem (see Ex. 6 of Section 2.6) we can write

$$\int_{\boldsymbol{\chi}(p)} (\mathbf{b} + \mathrm{div}\mathbf{T})\, dv = \mathbf{0} \quad \text{for every } \boldsymbol{\chi}(p) \subset \boldsymbol{\chi}(\Omega).$$

By the continuity of the integrand function and by the arbitrariness of $\boldsymbol{\chi}(p)$, we obtain equation (38).

By the moment balance equation (33) and the Cauchy's Lemma, we have (let $\mathbf{x}_0 = \mathbf{0}$)

$$\int_{\boldsymbol{\chi}(p)} \mathbf{W}\mathbf{b} dv + \int_{\partial\boldsymbol{\chi}(p)} \mathbf{W}\mathbf{T}\mathbf{n} da = \mathbf{0} \quad \text{for every } \boldsymbol{\chi}(p) \subset \boldsymbol{\chi}(\Omega),$$

where $\mathbf{W} \in \mathrm{Skw} = \{\mathbf{A} \in \mathrm{Lin} \mid \mathbf{A}^T = -\mathbf{A}\}$ is the skew-symmetric tensor associated with the position vector $\mathbf{x}$, i.e. $\mathbf{W}\mathbf{v} = \mathbf{x} \times \mathbf{v}$ for every $\mathbf{v} \in V$.

[8] Note that $\mathrm{div}\mathbf{T}$ is calculated on $\boldsymbol{\chi}(\Omega)$.

By the Divergence Theorem and by continuity of the integrand function we find

$$\mathbf{Wb} + \mathrm{div}(\mathbf{WT}) = \mathbf{0} \qquad \text{in } \boldsymbol{\chi}(\Omega). \tag{40}$$

We now compute div($\mathbf{WT}$). Let $\mathbf{a}$ any constant vector. By the definition of divergence of a tensor field, we have

$$\mathrm{div}(\mathbf{WT}) \cdot \mathbf{a} = \mathrm{div}(\mathbf{T}^T\mathbf{W}^T\mathbf{a})^9 = \mathbf{W}\mathrm{div}\mathbf{T} \cdot \mathbf{a} + \mathbf{T} \cdot \nabla(\mathbf{W}^T\mathbf{a}).$$

Since $\mathbf{W} \in$ Skw, $\mathbf{W}^T\mathbf{a} = \mathbf{a} \times \mathbf{x} = \mathbf{Ax}$, where $\mathbf{A}$ is the constant skew-symmetric tensor associated to $\mathbf{a}$. Therefore:

$$\nabla(\mathbf{W}^T\mathbf{a}) = \nabla(\mathbf{Ax}) = \mathbf{A}. \tag{41}$$

Finally, by multiplying the differential equation (40) by $\mathbf{a}$ and recalling (41), we have

$$\mathbf{W}(\mathbf{b} + \mathrm{div}\mathbf{T}) \cdot \mathbf{a} + \mathbf{T} \cdot \mathbf{A} = 0 \quad \text{in } \boldsymbol{\chi}(\Omega),$$

for every constant vector $\mathbf{a}$. By the force balance equation and by the arbitrariness of $\mathbf{A}$, $\mathbf{A} \in$ Skw, the thesis follows.

3.5 Boundary conditions of traction and equilibrium problem

An appropriate boundary condition of traction is difficult to formulate. We recall that our main goal is to find the deformation caused by applying given forces to a given body in a reference configuration $\boldsymbol{\kappa}$. We wish to specify the tractions acting on $\partial\boldsymbol{\chi}(\Omega)$ but this configuration in unknown. Even if we try to assign the tractions on $\partial\boldsymbol{\kappa}(\Omega)$, the deformation produced will move the point of application and deform the surface, and still we find difficulties to impose given forces.

Therefore, it is customarily assumed that the boundary condition of traction in the actual state takes the form

$$\mathbf{t}(\mathbf{x}, \mathbf{n}) = \text{ given function of } \mathbf{x} \quad \text{on } \partial\boldsymbol{\chi}(\Omega),$$

that is

$$\mathbf{T}(\mathbf{x})\mathbf{n}(\mathbf{x}) = \boldsymbol{\varphi}(\mathbf{x}) \quad \text{on } \partial\boldsymbol{\chi}(\Omega), \tag{42}$$

where $\boldsymbol{\varphi} : \partial\boldsymbol{\chi}(\Omega) \to V$ is a (regular) given surface traction field.

[9]The space Lin has a natural inner product $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}^T\mathbf{B})$, which in cartesian components reads as $\mathbf{A} \cdot \mathbf{B} = \sum_{i,j=1}^{3} A_{ij}B_{ij}$. Let $\mathbf{v}(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ a regular vector field and a regular tensor field defined on $\boldsymbol{\chi}(\Omega)$, respectively. We have: $\mathrm{div}(\mathbf{A}^T\mathbf{v}) = \mathrm{div}\mathbf{A} \cdot \mathbf{v} + \mathbf{A} \cdot \nabla\mathbf{v}$.

Summarizing, the equilibrium of the body B in configuration $\boldsymbol{\chi}$ is described by the following boundary value problem on the stress tensor $\mathbf{T} \in C^1(\boldsymbol{\chi}(\Omega)) \cap C^0(\partial\boldsymbol{\chi}(\Omega))$:

$$\begin{cases} \operatorname{div}\mathbf{T} + \mathbf{b} = \mathbf{0} & \text{in } \boldsymbol{\chi}(\Omega), \\ \mathbf{T} = \mathbf{T}^T & \text{in } \boldsymbol{\chi}(\overline{\Omega}), \\ \mathbf{T}\mathbf{n} = \boldsymbol{\varphi} & \text{on } \partial\boldsymbol{\chi}(\Omega), \end{cases} \tag{43}$$

where $\mathbf{b} \in C^0(\boldsymbol{\chi}(\Omega))$ and $\boldsymbol{\varphi} \in C^0(\partial\boldsymbol{\chi}(\Omega))$ are the given body-force and surface-traction densities and $\mathbf{n}$ is the unit outer normal to $\partial\boldsymbol{\chi}(\Omega)$. Note that $\boldsymbol{\varphi}$ and $\mathbf{b}$ must satisfy the necessary global equilibrium conditions $\mathbf{r}(\Omega) = \mathbf{0}$ and $\mathbf{m}(\Omega; \mathbf{x}_0) = \mathbf{0}$ for some $\mathbf{x}_0 \in \mathbb{R}^3$.

Problem (43) is largely undetermined and, accordingly, can have many solutions.

3.6 Virtual work theorem

There are situations in which it is useful to rewrite the equilibrium problem (43) in an integral form. Let $\mathbf{v} \in C^1(\boldsymbol{\chi}(\overline{\Omega}))$ be a vector field on $\boldsymbol{\chi}(\overline{\Omega})$. Scalar multiplication of the differential equilibrium equation by $\mathbf{v}$ and integration over $\boldsymbol{\chi}(\overline{\Omega})$ yield

$$\int_{\boldsymbol{\chi}(\Omega)} (\operatorname{div}\mathbf{T} \cdot \mathbf{v} + \mathbf{b} \cdot \mathbf{v}) dv = 0.$$

Recalling that $\operatorname{div}(\mathbf{T}^T\mathbf{v}) = \operatorname{div}\mathbf{T} \cdot \mathbf{v} + \mathbf{T} \cdot \nabla\mathbf{v}$, by Divergence Theorem, by the boundary condition on $\partial\boldsymbol{\chi}(\Omega)$ and by $\mathbf{T} = \mathbf{T}^T$, we have

$$\int_{\boldsymbol{\chi}(\Omega)} \mathbf{T} \cdot \nabla\mathbf{v} dv = \int_{\boldsymbol{\chi}(\Omega)} \mathbf{b} \cdot \mathbf{v} dv + \int_{\partial\boldsymbol{\chi}(\Omega)} \boldsymbol{\varphi} \cdot \mathbf{v} da \quad \text{for every } \mathbf{v} \in C^1(\boldsymbol{\chi}(\Omega)), \tag{44}$$

which is the usual form of the so-called *Virtual Work Theorem* on the equilibrium shape $\boldsymbol{\chi}(\Omega)$ ($(\text{VWT})_{\boldsymbol{\chi}}$). The field $\mathbf{v}$ is usually called *virtual displacement field.*

By using the arbitrariness of $\mathbf{v}$ and assuming enough regularity of the fields involved in (44), one can prove that (44) implies the equilibrium equations (43).

3.7 Reference description

A disadvantage of the equilibrium equations (43) over the deformed configuration (or, equivalently, of the $(\text{VWT})_{\boldsymbol{\chi}}$ in equation (44)) is that the equilibrium problem is formulated in terms of the unknown actual placement $\boldsymbol{\chi}(\Omega)$ of the body. To circumvent this difficulty, one can rewrite these equations in lagrangian form, that is in terms of the reference configuration $\overline{\Omega}$ of the body. This transformation involves the introduction of the so-called *Piola-Kirchhoff stress tensor*.

Let us assume that the body B, represented by the domain $\overline{\Omega}$ in its reference configuration $\boldsymbol{\kappa}$, is in equilibrium under the body forces $\mathbf{b}$ and the surface forces $\boldsymbol{\varphi}$ in the deformed configuration $\boldsymbol{\chi}(\Omega)$. We introduce the (regular) change of variables $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ within the $(\mathrm{VWT})_{\boldsymbol{\chi}}$ and we define

$$\mathbf{w}(\mathbf{X}) = (\mathbf{v} \circ \boldsymbol{\chi})(\mathbf{X}).$$

Then, the $(\mathrm{VWT})_{\boldsymbol{\chi}}$ expression becomes

$$\begin{aligned}\int_\Omega \nabla\mathbf{w} \cdot \mathbf{T} \quad & ((\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}})^{-1})^T \det\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}}\, dV = \\ & = \int_\Omega \mathbf{w}\cdot\mathbf{b}(\boldsymbol{\chi}(\mathbf{X}))\det\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}}\, dV + \int_{\partial\Omega}\mathbf{w}\cdot\boldsymbol{\varphi}(\boldsymbol{\chi}(\mathbf{X}))(\frac{da}{dA})dA \ ,\end{aligned}$$

or, using the standard notation $\frac{\partial\boldsymbol{\chi}}{\partial\mathbf{X}} = \mathbf{F}$,

$$\begin{aligned}\int_\Omega \nabla\mathbf{w} \cdot \mathbf{T} \quad & (\mathbf{F}^{-1})^T \det\mathbf{F}\, dV = \\ & = \int_\Omega \mathbf{w}\cdot\mathbf{b}(\boldsymbol{\chi}(\mathbf{X}))\det\mathbf{F}\, dV + \int_{\partial\Omega}\mathbf{w}\cdot\boldsymbol{\varphi}(\boldsymbol{\chi}(\mathbf{X}))(\frac{da}{dA})dA \ .\end{aligned} \tag{45}$$

Define

$$\mathbf{S}(\mathbf{X}) = \mathbf{T}(\boldsymbol{\chi}(\mathbf{X}))(\mathbf{F}^{-1})^T \det\mathbf{F}, \tag{46}$$

$$\mathbf{b}_{\boldsymbol{\kappa}}(\mathbf{X}) = \mathbf{b}(\boldsymbol{\chi}(\mathbf{X}))\det\mathbf{F}, \tag{47}$$

$$\boldsymbol{\varphi}_{\boldsymbol{\kappa}}(\mathbf{X}) = \boldsymbol{\varphi}(\boldsymbol{\chi}(\mathbf{X}))\frac{da}{dA} \ . \tag{48}$$

Then, Eq. (45) can be written as

$$\int_\Omega \nabla\mathbf{w}\cdot\mathbf{S}\, dV = \int_\Omega \mathbf{w}\cdot\mathbf{b}_{\mathbf{k}}\, dV + \int_{\partial\Omega}\mathbf{w}\cdot\boldsymbol{\varphi}_{\mathbf{k}}\, dA, \tag{49}$$

which expresses the Virtual Work Theorem written on the reference configuration $\boldsymbol{\kappa}$ from which the deformation $\boldsymbol{\chi}$ is applied ($(\mathrm{VWT})_{\boldsymbol{\kappa}}$).

The body-force density $\mathbf{b}_{\boldsymbol{\kappa}}$, $\mathbf{b}_{\boldsymbol{\kappa}} : \Omega \to \mathbb{R}^3$, measures the applied body force per unit volume in the reference configuration, and it depends on the deformation $\boldsymbol{\chi}$. The vector field $\boldsymbol{\varphi}_{\boldsymbol{\kappa}}$ is the density of the applied surface force per unit area on the deformation $\boldsymbol{\chi}$. The tensor $\mathbf{S}$ appearing in Eq. (46) is called the *Piola-Kirchhoff stress tensor*.