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Alberto Cambini
Laura Martein

Generalized Convexity and Optimization

Theory and Applications

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Theory and Applications

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To Giovanna and Paolo

Foreword

In the latter part of the twentieth century, the topic of generalizations of convex functions has attracted a sizable number of researchers, both in mathematics and in professional disciplines such as economics/management and engineering. In 1994 during the 15th International Symposium on Mathematical Programming in Ann Arbor, Michigan, I called together some colleagues to start an affiliation of researchers working in generalized convexity. The international Working Group of Generalized Convexity (WGGC) was born. Its website at www.genconv.org has been maintained by Riccardo Cambini, University of Pisa.

Riccardo's father, Alberto Cambini, and Alberto's long-term colleague Laura Martein in the Faculty of Economics, University of Pisa, are the coauthors of this volume. My own contact with generalized convexity in Italy dates back to my first visit to their department in 1980, at a time when the first international conference on generalized convexity was in preparation. Thirty years later it is now referred to as GC1, an NATO Summer School in Vancouver, Canada. Currently WGGC is preparing GC9 which is to take place in Kaohsiung, Taiwan. As founding chair and also current chair of WGGC, I am delighted to see the continued interest in generalized convexity of functions, augmented by the topic of generalized monotonicity of maps.

Eight international conferences have taken place in this research area, in North America (2), Europe (5) and Asia (1). We thought it was now time to return to Asia since our membership has shifted towards Asia.

As an applied mathematician I have taught mostly in management schools. However, I am currently in the process of joining an applied mathematics department. One of the first texts I will try out with my mathematics students is this volume of my long-term friends from Pisa. I recommend this volume to anyone who is trying to teach generalized convexity/generalized monotonicity in an applied mathematics department or in a professional school. The volume is suitable as a text for both. It contains proofs and exercises. It also provides sufficient references for those who want to dig deeper as graduate students and as researchers. With dedication and much love the authors have written a book that is useful for anyone with a limited background in basic mathematics. At the same time, it also leads to more advanced mathematics.

The classical concepts of generalized convexity are introduced in Chaps. 2 and 3 with separate sections on non-differentiable and differentiable functions. This has not been done in earlier presentations. Chapter 4 deals with the relationship of optimality conditions and generalized convexity. One of the reasons for a study of generalized convexity is that convexity usually is just a convenient sufficient condition. In fact most of the time it is not necessary. And it is a rather rigid assumption, often not satisfied in real-world applications. That is the reason why economists have replaced it by weaker assumptions in more contemporary studies. In fact, some of the progress in this research area is due to the work of economists. I am glad that the new book emphasizes economic applications.

In Chap. 5 the transition from generalized convexity to generalized monotonicity occurs. Historically, this happened only around 1990 when I was working with the late Stepan Karamardian after joining the University of California at Riverside. He was a former PhD student of George Dantzig at the University of California at Berkeley. We collaborated on the last two papers he published, both on generalized monotonicity. (a new research area) We had opened up together.

In 2005 Nicolas Hadjisavvas, Sandor Komlosi and I completed the first *Handbook of Generalized Convexity and Generalized Monotonicity* with contributions from many leading experts in the field, including Alberto Cambini and Laura Martein, a proven team of co-authors who in their unique colorful way have left an imprint in the field. The new book is further evidence of their style.

Chapters 6 and 7 are devoted to specialized results for quadratic functions and fractional functions. With this the authors follow the outline of the first monograph in this research area, *Generalized Concavity* by Mordecai Avriel, Walter E. Diewert, Siegfried Schaible and Israel Zang in 1988. Chapter 8 contains algorithmic material on solving generalized convex fractional programs. It defeats the objection sometimes raised that the area of generalized convexity lacks algorithmic contributions. It is true that there could be more results in this important direction on a topic which by nature is theoretical. Perhaps the presentation in Chap. 8 will motivate others to take up the challenge to derive more results with a computational emphasis.

Today *Generalized Concavity* (1988) is available to us as the first volume on the topic, together with the comprehensive *Handbook of Generalized Convexity and Generalized Monotonicity* (2005), an edited volume of 672 pages, written by 16 different researchers including Alberto Cambini and Laura Martein. In addition, the published proceedings of GC1–GC8 are available from reputable publishing houses. The proceedings of GC9 will appear partially in the prestigious Taiwanese Journal of Mathematics.

As somebody who has participated in all the conferences, GC1–GC8, and who is co-organizing GC9 together with Jen-Chih Yao, Kaohsiung and who has been involved in most publications mentioned before, I congratulate the authors for having produced such a fine volume in this growing area of research. Like me they stumbled into it when no monographs on the topic were available. I can see the usefulness of the book for teaching and research for generations to come. Its technical level makes it suitable for undergraduate and graduate students. The level is pitched wisely. The book is more accessible than the Handbook as it assumes less background knowledge about the topic. This is not surprising as the purpose of the Handbook is different. The new book can serve as an up-to-date link to the Handbook. It also saves the reader from going through the earlier proceedings with more dated results.

As someone who, like the authors, has not departed from the area of generalized convexity in his career, I can highly recommend this excellent new volume in our community of researchers. WGGC has been the background for most recent publications in our field of study. It is the excitement of working in teams which has been promoted by WGGC. A sense of community very common in Italy is the background of this new volume. It made me happy when I reviewed the manuscript first. I hope that many readers will come to the same conclusion. My thanks and congratulations go to the authors for a job well done.

I want to thank the authors for having taken the time to write *Generalized Convexity and Optimization with Economic Applications* and for their diligent effort to produce an up-to-date text and wish the book much success among our growing community of researchers.

Riverside, California,
June 2008

Siegfried Schaible
Chair of WGGC

Contents

1 Convex Functions 1
- 1.1 Introduction 1
- 1.2 Convex Sets 1
  - 1.2.1 Topological Properties of Convex Sets 3
  - 1.2.2 Relative Interior of Convex Sets 4
  - 1.2.3 Extreme Points and Extreme Directions 4
  - 1.2.4 Supporting Hyperplanes and Separation Theorems 6
  - 1.2.5 Convex Cones and Polarity 7
- 1.3 Convex Functions 10
  - 1.3.1 Algebraic Structure of the Convex Functions 13
  - 1.3.2 Composite Function 13
  - 1.3.3 Differentiable and Twice Differentiable Convex Functions 14
- 1.4 Convexity and Homogeneity 16
- 1.5 Minima of Convex Functions 17
- 1.6 Exercises 18
- 1.7 References 21

2 Non-Differentiable Generalized Convex Functions 23
- 2.1 Introduction 23
- 2.2 Quasiconvexity and Strict Quasiconvexity 23
- 2.3 Semistrict Quasiconvexity 30
- 2.4 Generalized Convexity of Some Homogeneous Functions 34
  - 2.4.1 The Cobb–Douglas Function 34
  - 2.4.2 The Constant Elasticity of Substitution (C.E.S.) Function 35
  - 2.4.3 The Leontief Production Function 35
  - 2.4.4 A Generalized Cobb–Douglas Function 35
- 2.5 Generalized Quasiconvex Functions in One Variable 36
- 2.6 Exercises 38
- 2.7 References 40

3 Differentiable Generalized Convex Functions . . . 41
3.1 Introduction . . . 41
3.2 Differentiable Quasiconvex and Pseudoconvex Functions . . . 41
3.2.1 Differentiable Quasiconvex Functions . . . 41
3.2.2 Pseudoconvex Functions . . . 43
3.2.3 Relationships . . . 48
3.3 Quasilinearity and Pseudolinearity . . . 50
3.3.1 Quasilinearity and Semistrict Quasilinearity . . . 50
3.3.2 Pseudolinearity . . . 53
3.4 Twice Differentiable Generalized Convex Functions . . . 57
3.4.1 Quasiconvex Functions . . . 57
3.4.2 Pseudoconvex Functions . . . 60
3.4.3 Characterizations in Terms of the Bordered Hessian . . . 61
3.5 Generalized Convexity at a Point . . . 63
3.6 Exercises . . . 68
3.7 References . . . 71

4 Optimality and Generalized Convexity . . . 73
4.1 Introduction . . . 73
4.2 Necessary Optimality Conditions Via Separation Theorems . . . 73
4.3 Generalized Convexity and Constraint Qualifications . . . 78
4.4 Sufficiency of the Karush–Kuhn–Tucker Conditions . . . 81
4.5 Local-Global Property . . . 82
4.6 Maxima and Generalized Convexity . . . 84
4.7 Minima, Maxima and Pseudolinearity . . . 86
4.8 Economic Applications . . . 87
4.8.1 The Utility Maximization Problem . . . 89
4.8.2 The Expenditure Minimization Problem . . . 91
4.8.3 The Profit Maximization Problem and the Cost Minimization Problem . . . 91
4.9 Invex Functions . . . 93
4.10 Exercises . . . 95
4.11 References . . . 97

5 Generalized Convexity and Generalized Monotonicity . . . 99
5.1 Introduction . . . 99
5.2 Concepts of Generalized Monotonicity . . . 99
5.2.1 Differentiable Generalized Monotone Maps . . . 103
5.3 Generalized Monotonicity of Maps of One Variable . . . 103
5.4 Generalized Monotonicity of Affine Maps . . . 105
5.5 Relationships Between Generalized Monotonicity and Generalized Convexity . . . 108
5.6 The Generalized Charnes–Cooper Transformation . . . 110
5.7 References . . . 112

6 Generalized Convexity of Quadratic Functions 115
6.1 Introduction 115
6.2 Preliminary Results 115
6.2.1 Some Properties of a Quadratic form Associated with a Symmetric Matrix Having One Simple Negative Eigenvalue 116
6.3 Quadratic Functions 119
6.4 Quadratic Functions of Non-negative Variables 126
6.5 Pseudoconvexity on a Closed Set 128
6.5.1 Pseudoconvexity on the Non-negative Orthant 130
6.5.2 Generalized Convexity of a Quadratic form on $\Re^2_+$ 131
6.6 A Special Case 132
6.7 Exercises 134
6.8 References 135

7 Generalized Convexity of Some Classes of Fractional Functions 137
7.1 Introduction 137
7.2 The Ratio of a Quadratic and an Affine Function 137
7.3 The Sum of a Linear and a Linear Fractional Function 141
7.4 Pseudoconvexity and the Charnes–Cooper Variable Transformation 147
7.5 Sum of Two Linear Fractional Functions 149
7.6 Exercises 154
7.7 References 157

8 Sequential Methods for Generalized Convex Fractional Programs 159
8.1 Introduction 159
8.2 The Linear Fractional Problem 160
8.2.1 Isbell–Marlow's Algorithm 162
8.2.2 Charnes–Cooper's Algorithm 164
8.2.3 Martos' Algorithm 166
8.2.4 Cambini–Martein's Algorithm 168
8.2.5 The Case of an Unbounded Feasible Region 172
8.3 A Generalized Linear Fractional Problem 175
8.3.1 Sequential Methods 176
8.3.2 The Sum of Two Linear Fractional Functions 182
8.4 Generalized Linear Multiplicative Programs 183
8.4.1 The Sum of a Linear Function and the Product of Two Affine Functions 183
8.4.2 The Product Between an Affine Function and the Power of an Affine Function 185
8.5 The Optimal Level Solutions Method 189
8.6 References 193

9 Solutions 195

References 213

A Mathematical Review 229
A.1 Sets 229
A.2 The Euclidean Space $\Re^n$ 230
A.3 Topological Concepts 234
A.4 Functions 235

B Concave and Generalized Concave Functions 241

Index 247

1

Convex Functions

1.1 Introduction

Convex and concave functions have many important properties that are useful in Economics and Optimization. In this Chapter the basic properties of convex and concave functions are explained, including some fundamental results involving these functions. In particular, the role of convexity and concavity in Optimization is stressed. Since a function f is concave if and only if $-f$ is convex, any result related to a convex function can easily be translated for a concave function. For this reason only the proofs related to convex functions are presented. For the sake of completeness, the corresponding results for the concave case are summarized in Appendix B.

1.2 Convex Sets

From a geometrical point of view, a set $S \subseteq \Re^n$ is convex if, for any two points in S, the line segment connecting these two points lies entirely in S (see Fig. 1.1).

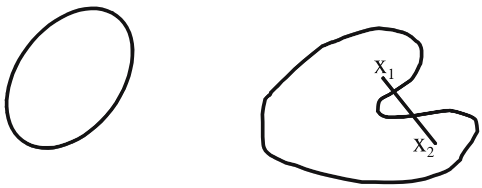

Fig. 1.1. Convex and not convex set

Formally, we have the following definition.

Definition 1.2.1. *A set $S \subseteq \Re^n$ is convex if*

$$x_1, x_2 \in S \Rightarrow \lambda x_1 + (1-\lambda)x_2 \in S,\ \forall \lambda \in [0,1]. \tag{1.1}$$

The point $x = \lambda x_1 + (1-\lambda)x_2,\ \lambda \in [0,1]$, is said to be a *convex combination* of x_1 and x_2. By $[x_1, x_2] = \{x \in S : x = \lambda x_1 + (1-\lambda)x_2,\ \lambda \in [0,1]\}$ we shall denote the closed line segment joining x_1 and x_2.

By convention, the empty set and the singleton set (a set consisting of a single point) are considered convex sets. The following are simple examples of convex sets:

- The whole set $\Re^n$;
- The line through x_0 and direction u: $r = \{x \in \Re^n : x = x_0 + tu,\ t \in \Re\}$;
- The hyperplane $H = \{x \in \Re^n : \alpha^T x = \beta\}$, $\alpha \in \Re^n, \alpha \neq 0, \beta \in \Re$;
- The closed half-spaces associated with H: $H^+ = \{x \in \Re^n : \alpha^T x \geq \beta\}$, $H^- = \{x \in \Re^n : \alpha^T x \leq \beta\}$.

Theorem 1.2.1. *The intersection of an arbitrary family of convex sets is convex.*

Proof. See Exercise 1.2. □

Definition 1.2.2. *A convex combination of finitely many points $x_i \in \Re^n$, $i = 1, ..., k$, is a point x of the form*

$$x = \sum_{i=1}^{k} \lambda_i x_i,\quad \sum_{i=1}^{k} \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1, ..., k.$$

The following theorem characterizes a convex set in terms of convex combinations of its points.

Theorem 1.2.2. *A set $S \subseteq \Re^n$ is convex if and only if every convex combination of finitely many points of S belongs to S.*

Proof. Suppose that S is convex. The proof proceeds by induction on the number k of points. For $k = 2$ the thesis is true by definition. Assuming that every convex combination of k points of S belongs to S, we must prove that every convex combination of $k+1$ points $x_1, ..., x_k, x_{k+1} \in S$ is a point of S. Let $z = \sum_{i=1}^{k+1} \lambda_i x_i,\ \sum_{i=1}^{k+1} \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1, ..., k+1$. If $\lambda_{k+1} = 0$ or $\lambda_{k+1} = 1$, then $z \in S$ by assumption. In any other case we can re-write z in the form

$z = \mu \sum_{i=1}^{k} \frac{\lambda_i}{\mu} x_i + \lambda_{k+1} x_{k+1},\ \mu = \sum_{i=1}^{k} \lambda_i = 1 - \lambda_{k+1} > 0.$

The induction assumption implies that the convex combination of k points $\bar{x} = \sum_{i=1}^{k} \frac{\lambda_i}{\mu} x_i$ belongs to S so that we have $z = \mu \bar{x} + (1-\mu)x_{k+1}$, that is z is a convex combination of two points of S and so $z \in S$.

The sufficiency follows by noting that we can consider, in particular, every convex combination of two points in S so that S is convex by definition. □

1.2.1 Topological Properties of Convex Sets

A key theorem is the following.

Theorem 1.2.3. *Let $S \subseteq \Re^n$ be a convex set with $intS \neq \emptyset$. Let $x_1 \in clS$ and $x_2 \in intS$. Then, $\lambda x_1 + (1-\lambda)x_2 \in intS$ for all $\lambda \in [0,1)$.*

Proof. The assumption $x_2 \in intS$ implies the existence of a ball $B(x_2, \epsilon)$ of radius $\epsilon > 0$ and center x_2 such that $B(x_2, \epsilon) = \{x :\| x - x_2 \|< \epsilon\} \subset S$. We prove that each point $y = \lambda x_1 + (1-\lambda)x_2,\ \lambda \in (0,1)$ is an interior point showing that the ball $B(y, (1-\lambda)\epsilon) \subset S$, i.e., every point z such that $\| z - y \|< (1-\lambda)\epsilon$ belongs to S. Set $R = \frac{(1-\lambda)\epsilon - \|z-y\|}{\lambda}$. Since $x_1 \in clS$ there exists a point $z_1 \in S$ such that $\| z_1 - x_1 \|< R$. Let $z_2 = \frac{z-\lambda z_1}{1-\lambda}$. We have
$\| z_2 - x_2 \|= \frac{1}{1-\lambda} \| z - \lambda z_1 - (1-\lambda)x_2 \|\leq \frac{1}{1-\lambda} \| z - \lambda z_1 - (y - \lambda x_1) \|\leq$
$\leq \frac{1}{1-\lambda} \| z - y \| + \lambda \| z_1 - x_1 \|< \frac{1}{1-\lambda}(\| z - y \| + \lambda \frac{(1-\lambda)\epsilon - \|z-y\|}{\lambda}) = \epsilon$.
Consequently, $z_2 \in S$. By definition of z_2 we have $z = \lambda z_1 + (1-\lambda)z_2$, i.e., z is a convex combination of two points of S and thus $z \in S$. The proof is complete. □

Theorem 1.2.4. *Let $S \subseteq \Re^n$ be a convex set with $intS \neq \emptyset$. Then, the following conditions hold:*
(i) clS is convex;
(ii) $intS$ is convex;
(iii) $cl(intS) = clS$;
(iv) $int(clS) = intS$.

Proof. (i) Let $x_1, x_2 \in clS$ and let $z \in intS$. By Theorem 1.2.3, $\lambda x_1 + (1-\lambda)z \in intS$ for all $\lambda \in [0,1)$ so that $\mu x_2 + (1-\mu)(\lambda x_1 + (1-\lambda)z) \in intS$ for all $\mu \in [0,1)$. Taking the limit as λ approaches 1, we have $\mu x_2 + (1-\mu)x_1 \in clS$.
(ii) This follows directly from Theorem 1.2.3 by noting that the interior point x_1 is obtained for $\lambda = 1$.
(iii) Since $intS \subseteq S$, we have $cl(intS) \subseteq clS$. Consider now $z \in clS$ and let $x \in intS$. By Theorem 1.2.3, $z + \lambda(x - z) \in intS,\ \forall \lambda \in (0,1]$; consequently, $z + \frac{1}{n}(x - z) \in intS$ for all n so that taking the limit as n approaches $+\infty$, we have $z \in cl(intS)$ and thus $clS \subseteq cl(intS)$.
(iv) Since $S \subseteq clS$, we have $intS \subseteq int(clS)$. Let $z \in int(clS)$; then, there exists $\epsilon > 0$ such that the closed ball $\bar{B}(z, \epsilon) = \{x :\| x - x_2 \|\leq \epsilon\}$ is contained in clS. Let $x \in intS$ and put $y = z + \epsilon \frac{z-x}{\|z-x\|} \in B$. By simple calculations, setting $\lambda = \frac{\epsilon}{\epsilon + \|z-x\|}$, we have $z = \lambda x + (1-\lambda)y$, so that $z \in intS$ by Theorem 1.2.3. Consequently $int(clS) \subseteq intS$ and thus $int(clS) = intS$. □

Remark 1.2.1. Property (iii) of Theorem 1.2.4 implies that every boundary point of S is a limit point of a sequence of interior points of S.

The following theorem points out that every interior point of a convex set S may be expressed as a convex combination of two points of S, one of which is arbitrary.

Theorem 1.2.5. *Let $S \subseteq \Re^n$ be a convex set with $intS \neq \emptyset$. Then, $z \in intS$ if and only if for every $x \in S$ there exists $\mu > 1$ such that $x + \mu(z - x) \in S$.*

Proof. See Exercise 1.5. □

1.2.2 Relative Interior of Convex Sets

The properties stated in the previous theorems are established assuming that the set of interior points of a convex set is nonempty; sometimes such an assumption may appear to be a restrictive condition. For instance, a line on the plane or a triangle in the ordinary space or, in general, a convex set which lies entirely in a linear manifold, does not have interior points. In order to extend the previous results to every convex set it is necessary to introduce the concept of the relative interior of a convex set.

Let S be a convex set and let W be the smallest linear manifold containing S. Then, the *relative interior* of S, denoted by riS, is the set of all interior points of S with respect to the topology induced by $\Re^n$ on W; in others words, a point $x_0 \in riS$ if and only if there exists a ball B of radius ϵ and center x_0 such that $B \cap S \subset W$.

Obviously, $riS = intS$ if and only if $W = \Re^n$. In contrast to $intS$ the relative interior has the fundamental property that $riS \neq \emptyset$ for every nonempty convex set (see Exercise 1.9).

Properties stated in Theorems 1.2.3, 1.2.4 and 1.2.5 may be restated in terms of the relative interior of a convex set.

Theorem 1.2.6. *Let $S \subseteq \Re^n$ and let $x_1 \in clS$ and $x_2 \in riS$. Then, $\lambda x_1 + (1 - \lambda)x_2 \in riS$ for all $\lambda \in [0, 1)$.*

Theorem 1.2.7. *Let $S \subseteq \Re^n$ be a nonempty convex set. Then, the following conditions hold:*
(i) $riS \neq \emptyset$;
(ii) riS is convex;
(iii) $cl(riS) = clS$;
(iv) $ri(clS) = riS$;
(v) $z \in riS$ if and only if for every $x \in S$ there exists $\mu > 1$ such that $x + \mu(z - x) \in S$.

1.2.3 Extreme Points and Extreme Directions

A point x belonging to a convex set $S \subseteq \Re^n$ is said to be an extreme point of S if it is not possible to express x as a convex combination of two distinct points of S.

The following example points out that the set of extreme points may be empty, finite, or infinite.

Example 1.2.1.
• A line does not have extreme points while a closed half-line has only one extreme point;
• A rectangle has four extreme points in its vertices;
• Every boundary point of a ball is an extreme point of the ball.

Regarding the existence of an extreme point, we have the following theorem (see [234]).

Theorem 1.2.8. *The set of all extreme points of a compact convex set S is nonempty. Furthermore, every $x \in S$ may be expressed as a convex combination of finitely many extreme points of S.*

The last statement of Theorem 1.2.8 cannot be extended to an unbounded convex set. For instance, a point of a closed half-line starting from x_0 cannot be expressed as a convex combination of its only extreme point x_0. This motivates the introduction of the concepts of recession direction and extreme direction. Consider, firstly, the following theorem.

Theorem 1.2.9. *Let $S \subseteq \Re^n$ be a closed convex set. Then, S is unbounded if and only if there exists a half-line contained in S. Furthermore, if the half-line $x = x_0 + td, t \geq 0$ is contained in S then, for every $y \in S$, the half-line $x = y + kd, k \geq 0$ is contained in S.*

Proof. Obviously, the existence of a half-line contained in S implies the unboundedness of S. Viceversa, the unboundedness of S implies the existence of a sequence $\{x_n\} \subset S$ such that $\lim_{n \to +\infty} \| x_n \| = +\infty$. Let $x_0 \in S$; without loss of generality we can suppose that the sequence $\left\{ \frac{x_n - x_0}{\|x_n - x_0\|} \right\}$ converges to a point $d \in \Re^n \backslash \{0\}$. In order to reach the thesis it is sufficient to prove that the half-line $x_0 + td, t \geq 0$, is contained in S. The convexity of S implies $x_0 + \lambda(x_n - x_0) \in S$ for all $\lambda \in [0, 1]$. For any fixed $t > 0$ choose $\lambda_n = \frac{t}{\|x_n - x_0\|}$; the sequence $\left\{ x_0 + \frac{t}{\|x_n - x_0\|}(x_n - x_0) \right\}$ is contained in S so that its limit, given by $x_0 + td$, belongs to the closure of S for all $t \geq 0$. The thesis is achieved since $clS = S$.
The last statement of the theorem still needs to be proven. Since the half-line $x = x_0 + td,\ t \geq 0$, is contained in S, we have $x_n = x_0 + nd \in S$ for all n. The convexity of S implies that, for every $y \in S$, $y + \lambda(x_n - y) = y + \lambda(x_n - x_0) + \lambda(x_0 - y) = y + \lambda nd + \lambda(x_0 - y) \in S$ for all $\lambda \in [0, 1]$. For any fixed $t \geq 0$ choose $\lambda_n = \frac{t}{n}$; the sequence $y + \frac{t}{n}nd + \frac{t}{n}(x_0 - y)$ converges to $y + td$ so that $y + td \in clS = S$ for all $t \geq 0$. The proof is complete. □

A direction $d \in \Re^n$ such that for every $y \in S$, the half-line $x = y + kd, k \geq 0$ is contained in S, is called a recession direction.

Theorem 1.2.9 establishes that the set of recession directions of a closed convex set S is nonempty if and only if S is unbounded.
A recession direction d is said to be an extreme direction if it is not possible to express d as a convex combination of two distinct recession directions.
Regarding the existence of an extreme point and extreme direction for an unbounded closed convex set, we have the following theorem (see [234]).

Theorem 1.2.10. *An unbounded closed convex set containing no lines has at least one extreme point and one extreme direction.*

The following fundamental representation theorem holds (see [234]).

Theorem 1.2.11. *Let $S \subseteq \Re^n$ be a closed convex set containing no lines. Then, $x \in S$ if and only if x can be expressed as the sum $x = y + d$, where y is a convex combination of extreme points of S and d is a positive linear combination of extreme directions.*

A polyhedron, defined as the intersection of finitely many closed half-spaces, is a special convex set having a finite number of extreme points and extreme directions. The extreme points of a polyhedron are also called vertices of the polyhedron. A bounded polyhedron is called a polytope.

1.2.4 Supporting Hyperplanes and Separation Theorems

Theorems of separation play a fundamental role in Optimization. We will limit ourselves to presenting some basic results which will be utilized later.
Let S be a convex subset of $\Re^n$ and let x_0 be a boundary point of S.
A supporting half-space to S at x_0 is a closed half-space containing S.
A supporting hyperplane to S at x_0 is the boundary of a supporting half-space to S at x_0.
In other words, the hyperplane $H_{x_0} = \{x \in \Re^n : \alpha^T x = \alpha^T x_0\}$ is a supporting hyperplane to S at x_0 if either $S \subseteq H_{x_0}^+ = \{x \in \Re^n : \alpha^T x \geq \alpha^T x_0\}$ or else $S \subseteq H_{x_0}^- = \{x \in \Re^n : \alpha^T x \leq \alpha^T x_0\}$.
Without loss of generality we can assume that $S \subseteq H_{x_0}^+$ by replacing α with $-\alpha$ if necessary.

Definition 1.2.3. *Let S, T be two subsets of $\Re^n$.*
A hyperplane $H = \{x \in \Re^n : \alpha^T x = \beta\}$ is said to separate S and T if $\alpha^T x \geq \beta,\ \forall x \in S$, and $\alpha^T x \leq \beta,\ \forall x \in T$.

In Fig. 1.2, a supporting hyperplane and a separating hyperplane are depicted. Some fundamental results related to the existence of a supporting hyperplane and to the existence of a separation hyperplane are found in the following theorems whose proofs can be found in any text-book (see references at the end of this Chapter).

Theorem 1.2.12. *(Separation of a convex set and a point)*
Let S be a closed convex subset of $\Re^n$ and let $y_0 \notin S$. Then, there exist $\alpha \in \Re^n \backslash \{0\},\ x_0 \in S$ such that $\alpha^T x \geq \alpha^T x_0$ for all $x \in S$ and $\alpha^T y_0 < \alpha^T x_0$.

Theorem 1.2.13. *(Existence of a supporting hyperplane at a boundary point)* *Let S be a closed convex subset of $\Re^n$ and let x_0 be a boundary point of S. Then, there exists $\alpha \in \Re^n \backslash \{0\}$ such that $\alpha^T x \geq \alpha^T x_0$ for all $x \in S$.*

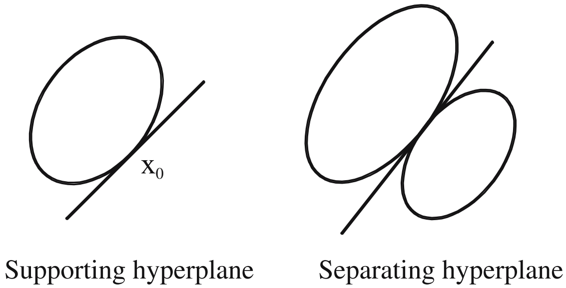

Fig. 1.2. Separating hyperplanes

Theorem 1.2.14. *(Separation of two sets)*
Let S_1 and S_2 be nonempty convex sets in $\Re^n$. Then, there exists a hyperplane which separates S_1 and S_2 if and only if $riS_1 \cap riS_2 = \emptyset$.

The following corollary shows that every closed convex set can be represented as the intersection of closed half-spaces.

Corollary 1.2.1. *Let S be a closed convex subset of $\Re^n$. Then, S is the intersection of all its supporting half-spaces, i.e., $S = \bigcap_{x_0 \in S} H^+_{x_0}$.*

Proof. Obviously, S is contained in the intersection of all half-spaces $H^+_{x_0}$. Let $y \in \bigcap_{x_0 \in S} H^+_{x_0}$ and suppose that $y \notin S$. Then, from Theorem 1.2.12, there exists a supporting hyperplane at a point x_0 belonging to the boundary ∂S of S such that $y \notin H^+_{x_0}$ and this is a contradiction. □

1.2.5 Convex Cones and Polarity

A cone (with vertex at zero) in $\Re^n$ is a nonempty set C satisfying the following property:

$$x \in C,\ k \geq 0 \Rightarrow kx \in C.$$

A convex cone is a cone which is convex as a set.

Half-lines, lines, subspaces, and half-spaces through the origin are examples of convex cones. The union of disjoint closed convex cones generates non-convex cones.
A cone is convex if and only if it is closed under the operations of addition and multiplication by a non-negative scalar as is shown in the following theorem.

Theorem 1.2.15. *A set $C \subseteq \Re^n$ is a convex cone if and only if the following properties hold:*
(i) $x \in C,\ k \geq 0 \Rightarrow kx \in C$;
(ii) $x, y \in C \Rightarrow x + y \in C$.

Proof. If C is a convex cone, (i) follows by definition of a cone. Furthermore, if $x, y \in C$, the convexity of C implies $\frac{1}{2}x + \frac{1}{2}y \in C$ and thus $2 \cdot (\frac{1}{2}x + \frac{1}{2}y) = x + y \in C$, so that (ii) holds.
Assume now the validity of (i) and (ii) and let $x, y \in C$. Then, from (i) C is a cone so that $\lambda x \in C,\ \lambda \geq 0,\ (1-\lambda)y \in C,\ \lambda \leq 1$, and from (ii) $\lambda x + (1-\lambda)y \in C,\ 0 \leq \lambda \leq 1$, i.e., C is convex. □

Corollary 1.2.1 may be specified in the case where S is a closed convex cone obtaining the following corollary.

Corollary 1.2.2. *Let C be a closed convex cone in $\Re^n$. Then, C is the intersection of all its supporting half-spaces at the origin.*

Proof. Taking into account Corollary 1.2.1, it is sufficient to prove that a supporting hyperplane H_{x_0} to C at $x_0 \in \partial C$ passes through the origin. We have $\alpha^T x \geq \alpha^T x_0, \forall x \in C$. Since $kx_0 \in C$ for all $k > 0$ we have $k\alpha^T x_0 \geq \alpha^T x_0, \forall k > 0$, that is $(k-1)\alpha^T x_0 \geq 0, \forall k > 0$ and this last inequality holds if and only if $\alpha^T x_0 = 0$. □

In some problems we are interested in the existence of a strict supporting hyperplane to C at the origin, i.e., in the existence of a supporting hyperplane H such that $H \cap C = \{0\}$. In order to fully illustrated this important aspect, we shall first introduce, the notion of polarity.

Definition 1.2.4. *Let C be a cone in $\Re^n$. Then, the positive polar of C, denoted by C^+, is given by $C^+ = \{\alpha \in \Re^n : \alpha^T c \geq 0,\ \forall\ c \in C\}$.*

The opposite of C^+ is referred to as the negative polar of C and is denoted by C^-. Equivalently, $C^- = \{\alpha \in \Re^n : \alpha^T c \leq 0,\ \forall\ c \in C\}$.

Remark 1.2.2. It follows immediately from the definition that polarity is order-inverting, i.e., if $C_1 \subset C_2$ then $C_1^+ \supset C_2^+$.

The following theorem states the structure of C^+ and, in addition, it characterizes the elements of a closed convex cone in terms of its positive polar and viceversa.

Theorem 1.2.16. *Let C be a closed convex cone in $\Re^n$. Then:*
(i) C^+ is a closed convex cone;
(ii) $c \in C$ if and only if $\alpha^T c \geq 0$ for all $\alpha \in C^+$;
(iii) $c \in intC$ if and only if $\alpha^T c > 0$ for all $\alpha \in C^+ \backslash \{0\}$;
(iv) $C = C^{++}$, where C^{++} is the positive polar of C^+;
(v) $\alpha \in intC^+$ if and only if $\alpha^T c > 0$ for all $c \in C \backslash \{0\}$.

Proof. (i) Let $\alpha_1, \alpha_2 \in C^+$. Then, we have $\alpha_1^T c \geq 0$, $\alpha_2^T c \geq 0$ for all $c \in C$, so that $(\alpha_1 + \alpha_2)^T c \geq 0$ and $k\alpha_1^T c \geq 0$, for all $k \geq 0$. It follows, from Theorem 1.2.15, that C^+ is a convex cone. Consider now a sequence $\{\alpha_n\} \subset C^+$ converging to an element α. By means of the continuity of the scalar product, and by taking the limit in $\alpha_n^T c \geq 0$, we obtain $\alpha^T c \geq 0$ for all $c \in C$. Consequently, C^+ is a closed cone.
(ii) Taking into account the definition of C^+, we must prove that the condition $\alpha^T c \geq 0,\ \forall \alpha \in C^+$ implies $c \in C$. If not, by Theorem 1.2.12 and by Corollary 1.2.2, there exist $\gamma \in \Re^n \backslash \{0\},\ c_0 \in \partial C$, such that $\gamma^T x \geq \gamma^T c_0 = 0,\ \forall x \in C$ and $\gamma^T c < 0$. The former inequality implies $\gamma \in C^+$ contradicting the latter which implies $\gamma \notin C^+$.
(iii) Let $c \in intC$. From (ii) we have $\alpha^T c \geq 0$ for all $\alpha \in C^+$. Assume, by contradiction, the existence of $\alpha \in C^+, \alpha \neq 0$, such that $\alpha^T c = 0$. Since c is an interior point, there exists $\epsilon > 0$ such that $c + \epsilon\ d \in C$ for every direction d of unitary norm. It follows that $\alpha^T(c + \epsilon\ d) = \epsilon\alpha^T d \geq 0$ for all d and this is absurd since, by choosing $d^* = -\frac{\alpha}{\|\alpha\|}$, we have $\epsilon\alpha^T d^* < 0$. Assume now $\alpha^T c > 0$ for all $\alpha \in C^+ \backslash \{0\}$. We must prove that $c \in intC$. If not, taking into account (ii), we have $c \in \partial C$ so that, by Theorem 1.2.13, there exists $\gamma \in \Re^n \backslash \{0\}$ such that $\gamma^T x \geq \gamma^T c = 0,\ \forall x \in C$, which contradicts the assumption.
(iv) By applying (ii) to the polar cone C^+, we have $\alpha \in C^+$ if and only if $z^T \alpha \geq 0$ for all $z \in C^{++}$. By comparing this last inequality with (ii), the thesis is achieved.
v) This follows by applying (iii) to the polar cone C^+, taking into account that $C^{++} = C$.
□

Remark 1.2.3. The proof of (i) of Theorem 1.2.16 points out that C^+ is a closed convex cone even if C is not closed and/or convex.

From (v) of Theorem 1.2.16, the existence of a strict supporting hyperplane to C at the origin is equivalent to the condition $intC^+ \neq \emptyset$. As we will see, this last condition is strictly related to the non-existence of lines contained in C. A closed cone which does not contain lines, i.e., $c \in C$ implies $-c \notin C$, is called a pointed cone. Equivalently, C is a pointed cone if and only if $C \cap (-C) = \{0\}$. The set $C \cap (-C)$ is called the lineality space of C and it is denoted by $\ell(C)$. The following theorem holds, where $dim\ C^+$ denotes the dimension of C^+, i.e., the maximum number of linearly independent vectors contained in C^+ or, equivalently, the dimension of the smallest subspace containing C^+.

Theorem 1.2.17. *Let C be a closed convex cone in $\Re^n$. Then:*
(i) $\ell(C)$ is the largest subspace contained in C;
(ii) $dim\ \ell(C) + dim\ C^+ = n$;
(iii) $intC^+ \neq \emptyset$ if and only if $\ell(C) = \{0\}$.

Proof. (i) Let $c \in C \cap (-C)$; $c \in C$ implies $kc \in C$ for all $k \geq 0$, while $c \in -C$ implies $-kc \in C$ for all $k \geq 0$, so that $kc \in C \cap (-C)$ for all $k \in \Re$.

Furthermore, Theorem 1.2.15 implies that $C \cap (-C)$ is closed with respect to the addition. It follows that $\ell(C)$ is a subspace. Let $W \subset C$ be a subspace; since $w \in W$ implies $-w \in W$, we have $w, -w \in C$ or equivalently, $w \in C \cap (-C)$. Consequently $W \subseteq \ell(C)$ so that $\ell(C)$ is the largest subspace contained in C.
(ii) Let $c \in \ell(C)$ and $\alpha \in C^+$. Since $c, -c \in C$, we have $\alpha^T c \geq 0$, $\alpha^T(-c) \geq 0$, so that $\alpha^T c = 0$ for all $c \in \ell(C)$. It follows that $\alpha \in [\ell(C)]^\perp$, i.e., $C^+ \subseteq [\ell(C)]^\perp$. Let V be the smallest subspace containing C^+. If $V \subset [\ell(C)]^\perp$, from $C^+ \subseteq V \subset [\ell(C)]^\perp$ we have $C^{++} = C \supseteq V^+ = V^\perp \supset \ell(C)$ and this contradicts (i). Consequently, $V = [\ell(C)]^\perp$ so that $dim\ C^+ = dim\ V = dim\ [\ell(C)]^\perp$ Since $dim\ \ell(C) + dim\ [\ell(C)]^\perp = n$, (ii) follows.
(iii) It follows from (ii) by noting that $intC^+ \neq \emptyset$ if and only if $dim\ V = dim\ C^+ = n$ or, equivalently, if and only if $dim\ \ell(C) = 0$, i.e., $\ell(C) = \{0\}$. □

The following corollary, which is a direct consequence of (v) of Theorem 1.2.16 and of (iii) of Theorem 1.2.17, states a necessary and sufficient condition for the existence of a strict supporting hyperplane to a cone at the origin.

Corollary 1.2.3. *Let C be a closed convex cone in $\Re^n$. Then, there exists $\alpha \in \Re^n$ such that $\alpha^T c > 0$ for all $c \in C, c \neq 0$, if and only if C is pointed.*

By noting that Theorem 1.2.17 implies $riC^+ = intC^+$ with respect to the topology induced by $\Re^n$ on the subspace $[\ell(C)]^\perp$, we have the following theorem which generalizes (v) of Theorem 1.2.16.

Theorem 1.2.18. *Let C be a closed convex cone in $\Re^n$. Then $\alpha \in riC^+$ if and only if*
(i) $\alpha^T c = 0$ for all $c \in \ell(C)$;
(ii) $\alpha^T c > 0$ for all $c \in C \backslash \ell(C)$.

1.3 Convex Functions

From a geometrical point of view, a function f is convex provided that the line segment connecting any two points of its graph lies on or above the graph. The function f is strictly convex provided that the line segment connecting any two points of its graph lies above the graph (see Fig. 1.3).

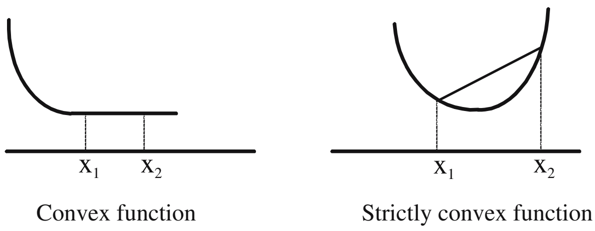

Fig. 1.3. Examples of convex functions

From an analytical point of view, we have the following definitions.

Definition 1.3.1. *Let f be a function defined on a convex set $S \subseteq \Re^n$.*
(i) The function f is said to be convex on S if for every $x_1, x_2 \in S$

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2), \quad \forall \lambda \in [0,1]. \tag{1.2}$$

(ii) The function f is said to be strictly convex on S if for every $x_1, x_2 \in S$

$$f(\lambda x_1 + (1-\lambda)x_2) < \lambda f(x_1) + (1-\lambda) f(x_2), \quad \forall \lambda \in (0,1). \tag{1.3}$$

A function f defined on a convex set $S \subseteq \Re^n$ is concave if and only if $-f$ is convex on S. It follows that all the results related to convex functions that we are going to establish can be easily stated in terms of concave functions. For the sake of completeness, and also taking into account that concave functions are more common in Economics, in Appendix B we shall give a summary of the main properties of concave functions.
Simple examples of convex and concave functions are given below.

Example 1.3.1.
1. An affine function $f(x) = a^T x + b,\ x \in \Re^n$ is both convex and concave (not strictly);
2. The function $f(x) = x + \mid x \mid,\ x \in \Re$ is convex (not strictly);
3. The function $f(x) = ax^2 + bx + c,\ x \in \Re$ is strictly convex if $a > 0$ and it is strictly concave if $a < 0$.

Obviously, a strictly convex function is convex, too; the converse statement is not true as it follows from (i) or (ii) of Example 1.3.1.
The inequalities (1.2), (1.3), may be extended to any weighted average of its values at a finite number of points as is shown in the following theorem.

Theorem 1.3.1. *(Jensen's inequality)*
(i) A function f is convex on S if and only if for every $x_1, ..., x_n \in S$,

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i), \lambda_i \geq 0,\ i = 1, ..., n,\ \sum_{i=1}^{n} \lambda_i = 1. \tag{1.4}$$

(ii) A function f is strictly convex on S if and only if for every $x_1, ..., x_n \in S$,

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) < \sum_{i=1}^{n} \lambda_i f(x_i), \lambda_i \geq 0,\ i = 1, ..., n,\ \sum_{i=1}^{n} \lambda_i = 1. \tag{1.5}$$

Proof. (i) Suppose that f is convex. The proof proceeds by induction on the number n of points. For $n = 2$ the thesis is true by definition. By assuming that (1.4) is verified for every convex combination of n points, we must prove that

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) \leq \sum_{i=1}^{n+1} \lambda_i f(x_i), \lambda_i \geq 0,\ i = 1, ..., n+1,\ \sum_{i=1}^{n+1} \lambda_i = 1.$$

If $\lambda_i = 0$ for some i, the thesis follows by means of the induction assumption, otherwise we have (see the proof given in Theorem 1.2.2)

$$\sum_{i=1}^{n+1} \lambda_i x_i = \mu\bar{x} + (1-\mu)x_{n+1},\ \mu = \sum_{i=1}^{n} \lambda_i,\ \bar{x} = \sum_{i=1}^{n} \frac{\lambda_i}{\mu} x_i.$$

By taking into account the convexity of f and the induction assumption, we have

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = f(\mu\bar{x} + (1-\mu)x_{n+1}) \leq \mu f(\bar{x}) + (1-\mu)f(x_{n+1}) \leq$$

$$\leq \mu\left(\frac{1}{\mu}\sum_{i=1}^{n} \lambda_i f(x_i)\right) + \lambda_{n+1} f(x_{n+1}) = \sum_{i=1}^{n+1} \lambda_i f(x_i).$$

The reverse statement follows by noting that if (1.4) is verified for every n of points it is in particular verified for $n = 2$, so that by definition f is convex.
(ii) This follows analogously. □

Associated with a convex function are the epigraph and the lower level sets defined, respectively, as follows:

$$epi f = \{(x,z) : x \in S,\ z \geq f(x)\};\ S_{\leq\alpha} = \{x \in S : \quad f(x) \leq \alpha\}.$$

A convex function is characterized by the convexity of its epigraph as is shown in the following theorem.

Theorem 1.3.2. *Let f be a function defined on a convex set $S \subseteq \Re^n$. Then:*
(i) f is convex if and only if $epi f$ is a convex set;
(ii) f is strictly convex if and only if $epi f$ is a convex set and it does not contain any line segment.

Proof. (i) Let f be convex. If $(x_1, z_1), (x_2, z_2) \in epi f$, then $z_1 \geq f(x_1)$, $z_2 \geq f(x_2)$, so that, for every $\lambda \in [0,1]$, we have $\lambda(x_1, z_1) + (1-\lambda)(x_2, z_2) = (\lambda x_1 + (1-\lambda)x_2, \lambda z_1 + (1-\lambda)z_2) \in epi f$ since $(\lambda z_1 + (1-\lambda)z_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2)$.
Assume now the convexity of $epi f$ and let $x_1,\ x_2 \in S$.
Since $(x_1, f(x_1)) \in epi f,\ (x_2, f(x_2)) \in epi f$, we have $\lambda(x_1, f(x_1)) + (1-\lambda)(x_2, f(x_2)) = (\lambda x_1 + (1-\lambda)x_2, \lambda f(x_1) + (1-\lambda)f(x_2)) \in epi f$, $\forall \lambda \in [0,1]$.
On the other hand $(\lambda x_1 + (1-\lambda)x_2, \lambda f(x_1) + (1-\lambda)f(x_2)) \in epi f$ if $\lambda f(x_1) + (1-\lambda)f(x_2) \geq f(\lambda x_1 + (1-\lambda)x_2)$, i.e., if f is convex.
(ii) The proof is similar to the one given in (i). □

Regarding the lower level sets of a convex function we have the following theorem.

Theorem 1.3.3. *Let f be a convex function defined on a convex set $S \subseteq \Re^n$. Then, $S_{\leq\alpha}$ is convex for every $\alpha \in \Re$.*

Proof. The thesis is true by convention if $S_{\leq\alpha} = \emptyset$ or $S_{\leq\alpha}$ is a singleton set. The points x_1, x_2 belong to $S_{\leq\alpha}$ if and only if $f(x_1) \leq \alpha$, $f(x_2) \leq \alpha$ so that, by means of the convexity of f, we have $f(\lambda x_1 + (1-\lambda)x_2) \leq (\lambda f(x_1) + (1-\lambda)f(x_2) \leq \lambda\alpha + (1-\lambda)\alpha) = \alpha$. It follows that $\lambda x_1 + (1-\lambda)x_2 \in S_{\leq\alpha}$. □

Remark 1.3.1. The necessary condition for a function to be convex stated in Theorem 1.3.3 is not generally sufficient. For instance, any increasing nonlinear concave single-variable function has convex lower sets but it is not convex. This fact has led to the introduction of a new class of functions, as we will see in the next chapter.

1.3.1 Algebraic Structure of the Convex Functions

The class of convex functions defined on a convex set is closed with respect to the addition and with respect to the non-negative scalar multiplication. More precisely, we have the following theorem.

Theorem 1.3.4. *Let $f_1, f_2, ..., f_m$ be functions defined on a convex set $S \subseteq \Re^n$ and set $f(x) = \sum_{i=1}^{m} \alpha_i f_i(x), \;\; \alpha_i \geq 0$. Then:*
(i) If $f_i, \; i = 1, .., m$, are convex on S, then f is convex on S.
(ii) If $f_i, \; i = 1, .., m$, are strictly convex on S, then f is strictly convex on S.

Proof. See Exercise 1.24. □

1.3.2 Composite Function

Another important property is related to the composition product.

Theorem 1.3.5. *Let $f : S \to \Re$ be a convex function defined on a convex set $S \subseteq \Re^n$ and let $g : A \to \Re$ be a non-decreasing convex function, with $f(S) \subseteq A$. Then the composite function $h(x) = g(f(x))$ is convex on S. Furthermore, if f is strictly convex and g is an increasing convex function, then h is strictly convex.*

Proof. See Exercise 1.25. □

Let us note that the requirement of the convexity of g is essential to guaranteeing the convexity of the composite function. For instance, the function $h(x) = x$ is convex, the function $g(x) = x^3$ is an increasing non-convex function and the composite function $f(x) = g(h(x)) = x^3$ is not convex.
Theorems 1.3.4, 1.3.5 and the analogous ones for concave functions are sometimes useful in constructing convex or concave functions.

Example 1.3.2.
1. The function $f(x) = e^{a^T x+b}$, $x \in \Re^n$ is convex since the affine function $a^T x+b$ is convex and the exponential function is an increasing convex function.
2. The function $f(x) = (a^T x + b)^2$ is convex on $S = \{x \in \Re^n : a^T x + b > 0\}$ since the affine function $a^T x + b$ is convex and the square function is an increasing convex function on the set of positive real numbers.

Example 1.3.3.
1. The power $f(x) = x^\alpha,\ x \geq 0$, is strictly concave for $0 < \alpha < 1$ and it is strictly convex for $\alpha < 0$ and for $\alpha > 1$;
2. $f(x) = \log x,\ x > 0$, is strictly concave.

Example 1.3.4. If f is a positive concave function, then $z(x) = \log f(x)$ is concave since the logarithm function is increasing and concave.

Example 1.3.5. If f is a positive concave function, then $\frac{1}{f}$ is convex. In fact, $z(x) = \log \frac{1}{f(x)} = -\log f(x)$ is convex as the opposite of the concave function $\log f(x)$. It follows that $e^{z(x)} = \frac{1}{f(x)}$ is convex.

1.3.3 Differentiable and Twice Differentiable Convex Functions

A convex function is continuous on the interior of its domain but not necessarily differentiable. For instance, the convex function $f(x) = |x|$ is continuous on $\Re$ but it is not differentiable at $x = 0$.
From a geometrical point of view, a differentiable function is convex if and only if its graph lies on or above the tangent in any point of the graph; it is strictly convex if its graph lies above the tangent in any point of the graph. From an analytical point of view, the convexity of a function of one variable may be characterized by means of its first and second derivatives, according to the following properties:
• Let I be an open interval of the real line. A differentiable function f is convex on I if and only if for every $x_0 \in I$ we have

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0),\ \forall x \in I. \tag{1.6}$$

• Let I be an open interval of the real line. A twice differentiable function f is convex on I if and only if

$$f''(x) \geq 0,\ \forall x \in I. \tag{1.7}$$

The extensions of (1.6) and (1.7) to functions of more variables are given below.

Theorem 1.3.6. *Let f be a differentiable function defined on a nonempty open convex set $S \subseteq \Re^n$. Then, f is convex on S if and only if for every $x_0 \in S$*

$$f(x) \geq f(x_0) + (x - x_0)^T \nabla f(x_0), \quad \forall x \in S. \tag{1.8}$$