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Exploiting Nonlinear Behavior in Structural Dynamics

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 536

EXPLOITING NONLINEAR BEHAVIOR IN STRUCTURAL DYNAMICS

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PREFACE

There are many physical phenomena which lead to nonlinear vibration problems. Modern structures are increasingly being built using more sophisticated materials that have a range of nonlinear material properties, some of which can be designed into the system. In some cases there are clear advantages in deliberately including nonlinear effects into the design of a structure. An obvious example are structural dampers. The most effective dampers contain highly nonlinear processes such as friction, fluids and most recently magneto-rheological fluids. Understanding and modelling the behaviour of these nonlinear effects is not a trivial process. However, there has been a dramatic increase in our understanding of nonlinear systems in the past 20 years, which has led to the realisation that beyond just modelling nonlinear effects, engineers can also use them to their advantage.

Nonlinearity arises from a range of phenomena. For example, geometric nonlinearity, including the effects of large deformations, combined stretching or compressing with vibration and nonlinear alignment of structural elements. Another source of nonlinearity is external forces acting on a linear system, such as fluid or magnetic forces. Nonlinear behaviour also comes from constraints in the system, freeplay, backlash, impact and friction.

Control forces can be added to a structural system in order to control the behaviour in some way and make it an adaptive structure. For example to reduce unwanted vibrations, detect damage, harvest energy or to shape change (morph) the structure. However, to create adaptive structures, the structure needs to have some awareness of its condition and/or the environment it is in. This is achieved by having a series of measurement sensors mounted on (or integrated into) the structure. Information from the sensors is then used by the global control system. This is where the smart (or intelligent) behaviour is generated.

This volume is a direct result of a course on "Exploiting Nonlinear Behaviour in Structural Dynamics" held at the International Centre for Mechanical Sciences (CISM) Udine, Italy Sep 13-17, 2010. Each chapter corresponds to a summary of the content of the lectures presented by each of the expert speakers who participated in the course.

Chapter 1 by Virgin & Wagg gives an overview and introduction to nonlinear phenomena in structural dynamics. The analysis of nonlinear effects using state space and bifurcation analysis is introduced, followed by an introduction to nonlinear control techniques. Chapter 2 by Neild covers material on so called approximate methods for analysing nonlinear systems where the level of nonlinearity is assumed to be relatively small. Examples include a device for harvesting mechanical energy, where the nonlinear effects are exploited to increase the useful energy which can be extracted. Chapter 3 by Virgin is devoted to the topic of vibration isolation. In particular, buckled structures and structures with large amounts of geometric nonlinearities are used as nonlinear vibration absorbers, to reduce significant unwanted vibrations in the systems under consideration. Chapter 4 by Shaw focuses attention on the mitigation of undesirable torsional vibration in rotating systems utilizing specifically nonlinear features in the dynamics. Reducing vibration in automotive motors is a problem of considerable practical relevance. Tunable vibration absorbers require careful design with the role of nonlinearities of particular relevance. Chapter 5 by Ribeiro discusses the vibration of nonlinear (beam) structures in which the motion is sufficiently large amplitude that elasto-plastic effects are induced. Both free and forced situations are analyzed including practical (numerical) solution procedures. Ultimately, more accurate modeling of these types of systems will assist in a more complete understanding of nonlinear vibration and its relationship with material failure. Finally in Chapter 6 by Wagg, structural systems with control are considered. These include nonlinear systems with active control, as well as morphing structures, where snap-through mechanisms are exploited as hinges so the structure can change its shape between two stable states.

We would like to thank all those at CISM who helped to make the course and the production of this volume such a enjoyable experience, in particular Mrs P. Agnola, Elsa Venir and Carla Toros who dealt with the administration of the meeting. In addition we would like to thank the Rectors of CISM: G. Maier, J. Salençon, W. Schneider and the Secretary General, B.A. Schrefler. Finally, we would like to thank Prof. Serafini for his assistance with the preparation of this book.

Lawrie Virgin (Duke University, USA)
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June 2011

CONTENTS

Introductory Material

Lawrie Virgin and David Wagg
Duke University and University of Bristol.

Abstract This book is based on a one-week worksop at CISM. In order to fully appreciate the benefits of nonlinearity in certain engineering systems it is important to understand the underlying behavior of linear systems, and this first chapter provides a general overview of linear dynamical systems and then begins to explore the effect of nonlinearities.

1 The Linear Oscillator

In mechanics we are primarily interested in the time evolution of systems governed by odes

$$\dot{x} = F(x, \lambda, t) \qquad x \epsilon R^n, \qquad t \epsilon R, \tag{1}$$

where x is a state vector which describes the evolution of the system under the vector field, F. Given an initial condition x_0 at time $t = 0$ we can seek to solve system 1 to obtain a trajectory, or orbit, along which the solution evolves with time. We will then seek to ascertain the stability of the system, generally as a function of the (control) parameter, λ (Guckenheimer and Holmes (1983)).

The cornerstone of dynamics in a mechanics context is, of course, Newton's second law, and thus sets of second-order, ordinary differential equation are dominant:

$$\ddot{x} + \omega_n^2 x = 0, \tag{2}$$

where an overdot represents a derivative with respect to time, and the system has two initial conditions $x(t = 0), \dot{x}(t = 0)$, from whence the dynamics develops. This is the equation of motion governing the dynamic response of the spring-mass system shown in Figure 1 with $\omega_n = \sqrt{k/m}$ and all other parameters set equal to zero. We can write the solution as

$$x(t) = Ae^{\lambda t}. \tag{3}$$

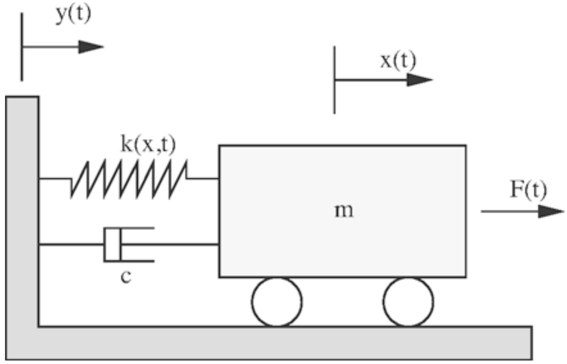

Figure 1. A spring-mass damper.

Placing 3 in to 2 we find that $\lambda = i\omega_n$ and thus the general form of the solution is given by

$$x(t) = ae^{i\omega_n t} + be^{-i\omega_n t}. \tag{4}$$

Alternatively, using Euler's identities we can write:

$$x(t) = c\cos(\omega_n t) + d\sin(\omega_n t). \tag{5}$$

In order to determine a and b, (or c and d), we make use of the initial conditions to get

$$x(t) = x(0)\cos(\omega_n t) + \frac{\dot{x}(0)}{\omega_n}\sin(\omega_n t). \tag{6}$$

This system can be converted in to a pair of coupled, first-order ordinary differential equations (in state variable format) by introducing a new variable

$$y = \dot{x} \tag{7}$$

and substituting in equation 2 gives:

$$\dot{x} = y, \qquad \dot{y} = -\omega_n^2 x \tag{8}$$

and in matrix notation:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{9}$$

The solutions to this type of equation are harmonic, with oscillation occurring about the origin (the unique equilibrium position), Inman (1994). The form, and frequency, of the resulting motion is independent of the initial conditions.

2 A Nonlinear Spring

Now suppose we have a spring whereby the applied force and corresponding deflection are related via a cubic expression:

$$F = k(x) = Ax + Bx^3. \tag{10}$$

Adding a small amount of inertia and a little damping we obtain a standard nonlinear oscillator known as Duffing's equation (Virgin (2000)):

$$\ddot{x} + 0.1\dot{x} + Ax + Bx^3 = 0. \tag{11}$$

We still have a spring that initially responds the same way in compression and extension, but now the nonlinearity depends on the signs of A and B. Superposition no longer holds. Suppose we have $A = 1$ and $B > 0$. In this case we have a hardening spring, i.e., it becomes disproportionately stiffer as the deflection increases. This is shown schematically in Figure 2. Also shown is the softening case $A = 1, B < 0$, indicated by the dashed line. Furthermore, if $A = -1$ and $B > 0$ we get a (still symmetrical, and

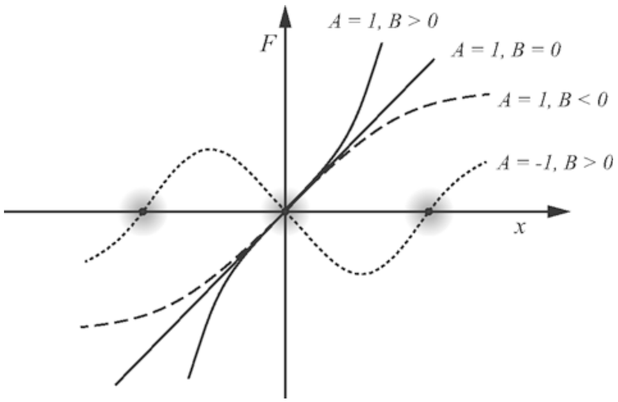

Figure 2. Force-deflection relations for some typical springs.

shown by the dotted line) system in which the origin is now unstable (to be shown later), and the spring, given a typical load will take up one of two available equilibrium positions. The force-deflection characteristic need not be symmetric, and in fact, this will typically be the case under some pre-loading.

When incorporated into the dynamics context (equation 11) we find that the frequency as well as the amplitude depends on the initial conditions. Equation 11 is of course a nonlinear ordinary differential equation and does not submit to standard methods. However, we can gain considerable insight into the behavior of such systems, and we start by ignoring damping. As such, the total mechanical energy is now conserved and using $\ddot{x} = \dot{x}\mathrm{d}\dot{x}/\mathrm{d}x$ we can separate variables, integrate, and write

$$\dot{x} = \pm\sqrt{-Ax^2 - (B/2)x^4 + C}, \tag{12}$$

in which C is determined from the initial conditions. For $A = 1$ and $B = 0$ we obtain ellipses in the phase space corresponding to simple harmonic motion. However, for other combinations of A and B we obtain behavior that may be very different from simple harmonic.

We can also obtain the natural period of motion in a related way. It can be shown that the period is equal to four times the time it takes to move from the maximum amplitude ($\bar{x}$) to zero, and rearranging equation 12 we can evaluate the period of motion, T, (and hence natural frequency) from Jordan and Smith (1977); Stoker (1992)

$$T = 4\int_0^{\bar{x}} \frac{\mathrm{d}x}{\sqrt{-Ax^2 - (B/2)x^4 + C}}. \tag{13}$$

However, this integral is not easy to solve, but whereas the linear oscillator has a natural frequency that is independent of the amplitude of motion, the nonlinear oscillator will have a period that depends on the initial conditions and hence the amplitude of motion. In Figure 2 was shown typical hardening and softening spring systems. The natural frequencies, often referred to as 'backbone curves' corresponding to these two case are shown schematically in Figure 3. These mildly nonlinear cases are centered on an equilibrium position. In some cases, e.g., when $A = 1$ and $B = -1$ and the motion exceeds $x = 1$ the behavior can become unbounded. In general, this type of behavior has to be investigated numerically, and we shall see that these backbone curves have a profound resonance effect when incorporated into the context of (harmonically) forced oscillators.

2.1 Linearization

With no force applied the 'rest length' of the linear spring is the unique position of equilibrium, which we consider without loss of generality to be the origin. For a nonlinear spring, for example, the dotted line in Figure 2

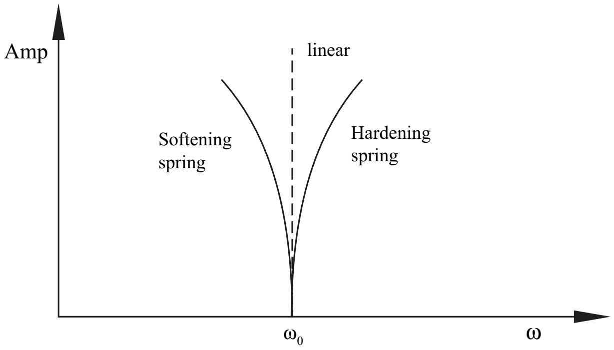

Figure 3. The frequency relation for mildly nonlinear oscillators.

we see two 'remote' equilibrium positions. These happen to be stable: their local slopes are positive and any deviation from the position will be opposed by the (restoring) force. In the vicinity of equilibrium (indicated by the three shaded regions) we see that the force-deflection relation is practically linear. More formally, we can take a Taylor series expansion about equilibria. The equilibria are found from $Ax+Bx^3=0$, from which have $x_e=0$ (sometimes referred to as the trivial equilibrium) but there may also be other roots. For example with $A=-1, B=+1$ we get two more real roots at $x_e=\pm 1$, but if A and B have the same sign then the origin is the only real root. Consider a small perturbation, δ about an equilibrium, x_e:

$$x = x_e + \delta. \tag{14}$$

Placing this into equation 10 and assuming $A=-1, B=+1$ we obtain:

$$F = -x_e - \delta + x_e^3 + 3x_e^2\delta + 3x_e\delta^2 + \delta^3. \tag{15}$$

Since δ is small we neglect terms in δ higher than linear, and $-x + x_e^3 = 0$ to satisfy equilibrium, and thus we have

$$F = \delta(3x_e^2 - 1), \tag{16}$$

which is valid in the vicinity of the equilibria. For $x_e=0$ we have a locally negative slope and the force tends to move the system further away with deflection. For $x_e=\pm 1$ we have a locally positive slope and a restoring force. If the spring undergoes 'large' deflections then the system becomes progressively more nonlinear.

In terms of dynamic response we can still use the approach of the first part of this chapter, but now the motion local to the equilibrium at the origin is described by

$$\ddot{x} - \omega_n^2 x = 0. \tag{17}$$

The solution again has the form $x(t) = Ae^{\lambda t}$ where $\lambda = \pm\omega_n$, and thus

$$x(t) = ae^{\omega_n t} + be^{-\omega_n t}, \tag{18}$$

and using the definition of hyperbolic functions we also have

$$x(t) = x(0)\cosh \omega_n t + (\dot{x}(0)/\omega_n)\sinh \omega_n t. \tag{19}$$

In this case we do not have a periodic solution: the positive exponential indicates that typically $x \to \infty$ as $t \to \infty$. Hence, this behavior is unstable (Virgin (2007)). However, we also observe that we can choose very specific initial conditions (unlikely but nevertheless important cases), where the trajectory will end up at the origin, i.e., where the positive exponential term is completely suppressed, as well as the case where the negative exponential term in equation 18 dominates for a short time before the trajectory is swept away. For all practical purposes, i.e., arbitrary initial conditions, the motion is clearly unstable.

Thus, the key stability information is contained in (the sign of) λ in equations 3. If it is negative then the motion will decay with time, otherwise it will grow (Thompson and Stewart (1986)). It is convenient to introduce a more general (state) matrix of the form:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{20}$$

and extract the determinant, Δ, and trace, T:

$$\Delta = (ad - bc), \quad T = a + d, \tag{21}$$

and recast stability given the fact that these relate to the system eigenvalues:

$$\lambda_{1,2} = \tfrac{1}{2}\left(T \pm \sqrt{T^2 - 4\Delta}\right), \quad \Delta = \lambda_1\lambda_2, \quad T = \lambda_1 + \lambda_2. \tag{22}$$

Hence for stability (negative eigenvalues) we require T negative and Δ positive (Strogatz (1994)). In the case of the two systems considered earlier both times the trace is zero (Newton's laws provide certain restrictions for typical mechanical systems), but for the system in equation 2 we have a positive Δ and hence stability, and a negative Δ for the system described by equation 17. We will now refine this to take account of damping.

2.2 Damping

Most real systems undergo a form of energy dissipation. With the inevitable presence of damping the question of stability becomes less ambiguous. Typical motion will then consist of a transient followed by some kind of recurrent long-term behavior, e.g. the motion will die out and the mass position is maintained at equilibrium.

Suppose we now allow for some energy dissipation in the form of linear viscous damping, i.e., $c \neq 0$ in Figure 1. The equation of motion is now

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0, \tag{23}$$

in which a nondimensional damping ratio, $\zeta = c/(2m\omega_n)$ has been introduced. Solutions to this equation now also depend on the value of ζ. For lightly damped systems we have $\zeta < 1$ and solutions of the form

$$x(t) = e^{-\zeta\omega_n t}\left(\frac{\dot{x}(0) + \zeta\omega_n\dot{x}(0)}{\omega_d}\sin\omega_d t + x(0)\cos\omega_d t\right) \tag{24}$$

where the damped natural frequency ω_d is given by

$$\omega_d = \omega_n\sqrt{1-\zeta^2}. \tag{25}$$

A typical underdamped response ($\zeta = 0.1$) is shown in Figure 4(a) and (b) as a time series and phase portrait. The origin in Figure 4 (b) indicates the position of asymptotically stable equilibrium. The trajectory gradually spirals down to this rest state: we can imagine a family of trajectories forming a flow as time evolves. Since this equilibrium is unique, the whole of the phase space is the attracting set for all initial conditions and disturbances. Damping in this range, e.g., $\zeta \approx 0.1$, is quite typical for mechanical and structural systems.

For a heavily (or overdamped) system $\zeta > 1$, and in this case the form of the solution is

$$x(t) = Ae^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + Be^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} \tag{26}$$

where

$$A = \frac{\dot{x}(0) + (\zeta + \sqrt{\zeta^2-1})\omega_n x(0)}{2\omega_n\sqrt{\zeta^2-1}} \tag{27}$$

and

$$B = \frac{-\dot{x}(0) - (\zeta - \sqrt{\zeta^2-1})\omega_n x(0)}{2\omega_n\sqrt{\zeta^2-1}}. \tag{28}$$

The motion is a generally monotonically decreasing function of time and may take a relatively long time to overcome relatively heavy damping forces

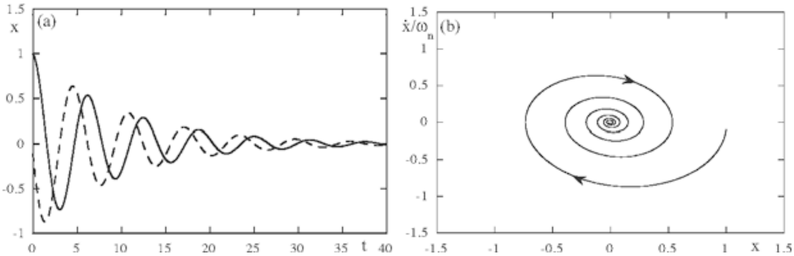

Figure 4. Time series and phase portraits for underdamped (oscillatory) motion. $x(0) = 1.0; \dot{x}(0) = 0.0; \zeta = 0.1$.

on the way to equilibrium. The boundary between these two cases is the critically damped case, i.e., $\zeta = 1$ when the eigenvalues are equal.

Returning to the state variable matrix format of the linear oscillator and adding damping we therefore have

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{29}$$

We can also write the solution in terms of the eigenvalues of the state matrix, i.e., the roots of the characteristic equation

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0. \tag{30}$$

Now, the only difference with the expressions given in equation 9 is that the trace, T, becomes $-2\zeta\omega_n$, i.e., negative.

Given the scenario of a system losing stability we can usefully view all the response possibilities of this type of linear system according to the location of the roots in the complex plane. For example, having two complex roots with negative real parts corresponds to an exponentially decaying oscillation. Summarizing these outcomes in terms of the trace and determinant leads to Figure 5. In general we will have a system with positive stiffness and damping and thus a root structure corresponding to the lower right quadrant. Critical damping corresponds to the dashed parabola, and phase portraits and eigenvalues are indicated for various combinations of (T, Δ) and hence the natural frequency and damping.

3 A Nonlinear Damper

Another basic nonlinearity sometimes occurring in mechanical vibration is the appearance of energy dissipation in which the damping force is not

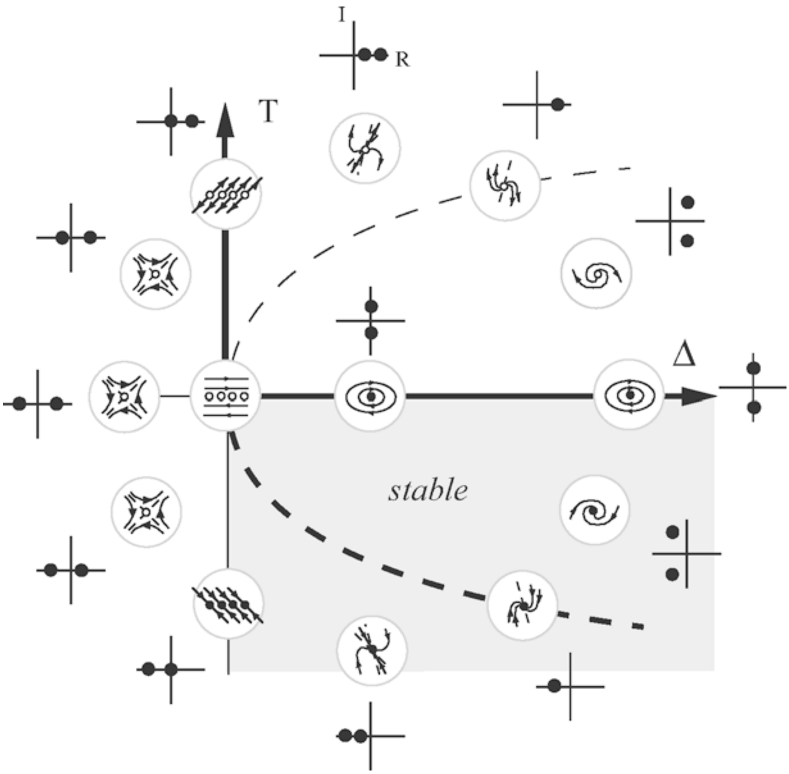

Figure 5. The root structure of a linear oscillator.

necessarily proportional to the velocity of motion. The adoption of linear viscous damping in our spring-mass-damper model is partly motivated by relevant damping processes, for example, the mechanism by which a dashpot (and other related devices) utilizes this type of energy dissipation. However, it is also used because of its relative analytic simplicity, i.e., assumption of linear viscous damping does not violate the rules of linearity, and this also has certain advantages in the study of dynamics in continuous dynamical systems.

3.1 Coulomb Damping

Friction commonly occurs in mechanical systems in which rubbing, or contact, between two dry surfaces causes the dissipation of energy (often in the form of heat). The assumption here is that the damping force is equal to the product of the normal force and a material-dependent coefficient of friction, Thomson (1981). The free response consists of a linear (as opposed to an exponential) decay of motion (with a frequency of oscillation the same as the underlying undamped system), and the mass may come to rest with a slight static offset if the static force of friction is greater than the restoring force of the spring. A typical example is shown in Figure 6. This type of

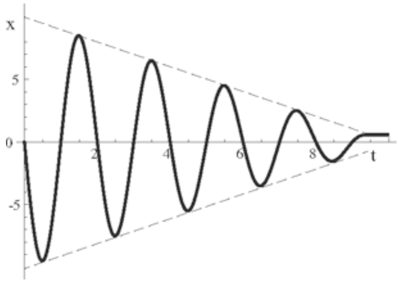

Figure 6. Time series for a mass subject to Coulomb damping.

energy dissipation can lead to a variety of interesting behavior especially in the context of forced vibrations. For example, stick-slip occurs in many mechanical systems.

3.2 Motion-dependent Damping

Another type of nonlinear energy dissipation is the mechanism underlying the appearance of certain types of limit cycle. The classic example is the van der Pol equation (van der Pol (1934)):

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0. \tag{31}$$

The parameter μ has a profound effect on the behavior of this system. For positive μ, we see that if $x^2 > 1$ then the damping term is positive and energy is dissipated. However, again for positive μ the damping becomes negative when $x^2 < 1$. Using the same linearization process as detailed in section 2.1 we consider small perturbations about equilibrium (the origin) which leads to:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & \mu \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{32}$$

The eigenvalues of this system are

$$\lambda = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4}. \tag{33}$$

The determinant $\Delta = 1$ and the trace $T = \mu$. Thus, we have real negative roots when $\mu \leq 0$ indicating stable behavior. If $0 < \mu < 2$ the roots are complex with positive real parts indicating an unstable spiral. Thus, we locate these possible responses within Figure 5. This behavior is only valid in the vicinity of equilibrium. Solving equation 31 numerically for various

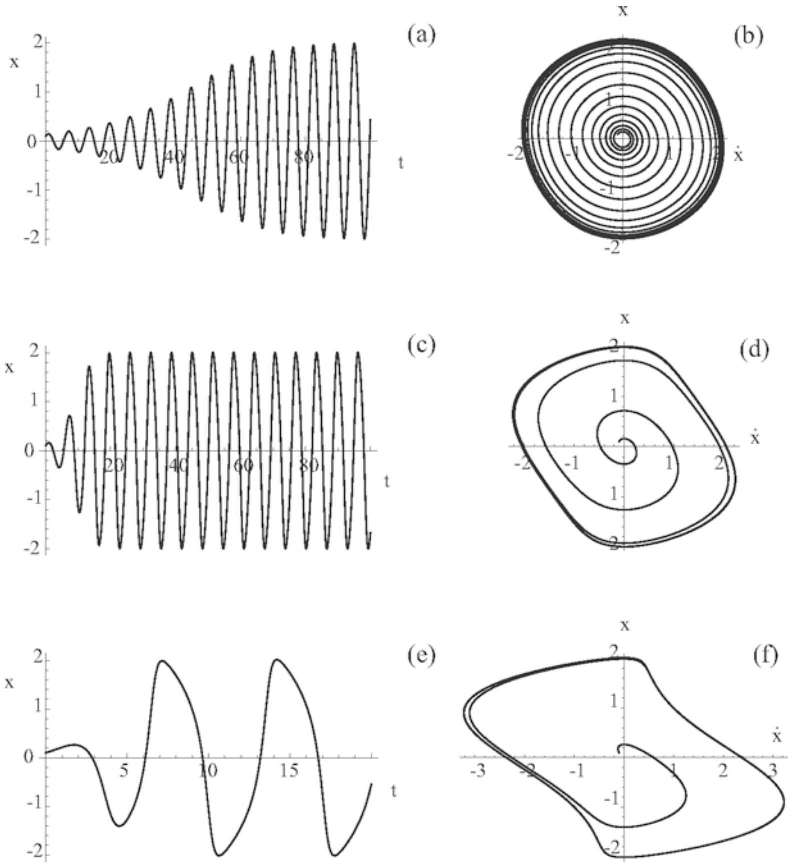

Figure 7. Time series and phase portraits for van der Pol's equation, (a) and (b) $\mu = 0.1$; (c) and (d) $\mu = 0.5$; (e) and (f) $\mu = 1.5$

positive values of μ leads to the results shown in Figure 7 as time series and phase projections. These are sometimes called relaxation oscillations, and since transients are attracted from within and without the oscillation this type of behavior is called a limit cycle oscillation (LCO). A perturbation approach (see chapter 2) and Jordan and Smith (1977), assuming μ is relatively small, can be used to show that the amplitude of the LCO remains close to 2 and that the frequency of oscillation is approximately $\omega = 1 - (1/16)\mu^2$.

We conclude this section by showing a couple of flows in phase space. An unforced Duffing system of the form $\ddot{x} + 0.1\dot{x} - x + x^3 = 0$ is shown for a variety of initial conditions in phase space in Figure 8(a). Likewise, the

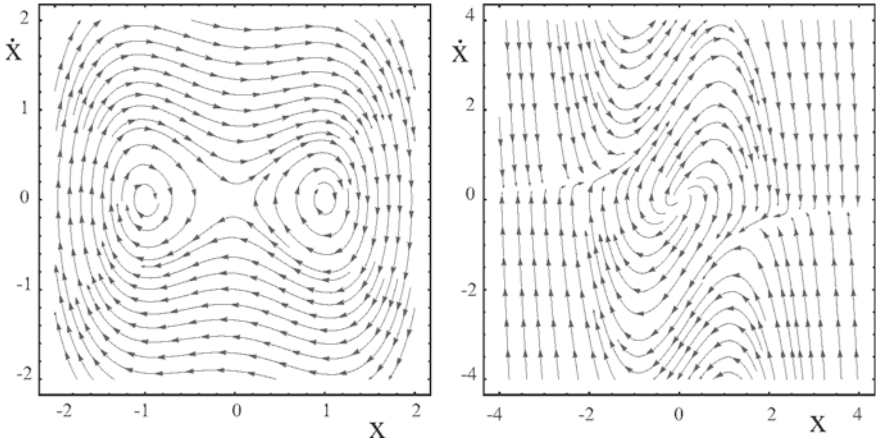

Figure 8. Flows in phase space: (a) Duffing's equation, (b) van der Pol's equation.

behavior of van der Pol's equation of the form $\ddot{x} - 1.5(1 - x^2)\dot{x} + x = 0$ is displayed in part (b). In both cases, for these parameter values, we see an unstable origin. In part (a) we have $(\Delta, T) = (-1, -0.1)$ indicating a saddle point and motion is swept away (and ultimately settles about one of the two stable equilibria at $x_e = \pm 1$), whereas in part (b) we have $(\Delta, T) = (+1, +1.5)$ and motion spirals away from the origin and settles onto the stable periodic orbit.

We thus observe what will typically happen when the stiffness or damping of the system changes, and especially where one of these parameters drops to zero, corresponding to an instability. The important issue here is that linear scenarios occur naturally within the context of nonlinear oscillators. The geometric view afforded by a consideration of the root structure and phase portraits of families of solution about equilibrium points is very useful.

4 Bifurcations

In many practical situations the forces acting on a system change. For example, the spring force in equation 10 might be subject to changing values of A and B, and this of course has a fundamental impact on the nature of solutions. Bifurcation theory (Doedel (1986); Seydel (1994)), classifies the generic ways in which an equilibrium loses its stability. Under the action of a single control parameter, for example, the stiffness or damping in an oscillator, we have already seen how the instability corresponds to an eigenvalue moving into the positive half-plane. The behavior of the linear oscillator provides an informative local view of behavior, but in a practical situation we might expect nonlinear effects to influence, or limit, the response in some way. As a parameter is varied the response of a system changes, and often gradually, but it is the qualitative change in the dynamics that is classified as a bifurcation. The elementary bifurcations are essentially one-dimensional but since the focus here is dynamics, we embed these (four elementary) bifurcations within the context of oscillations (Guckenheimer and Holmes (1983)).

4.1 Bifurcations from a Trivial Equilibrium

There are some systems in which some kind of initial symmetry is present, e.g., Euler buckling (Virgin (2007)). They represent an important class of instability in structural mechanics: super- and sub-critical pitchfork bifurcations. For the super-critical pitchfork bifurcation we can consider the oscillator:

$$\ddot{x} + 0.1\dot{x} + x^3 - \mu x = 0. \tag{34}$$

Again we observe the fundamental $x_e = 0$ solution, which is stable for $\mu < 0$. We immediately see how the example of Duffing's equation described earlier is a specific example with positive μ. At $\mu = 0$ a secondary equilibrium intersects the fundamental and it can be shown that the two (symmetric) non-trivial solutions are stable (see Figure 9(a)). The stability of equilibrium is determined using the (linearization) approach of section 2.1. This corresponds to the classic 'double-well' potential which is also shown superimposed for a specific (positive) value for μ.

The corresponding sub-critical pitchfork bifurcation is given by:

$$\ddot{x} + 0.1\dot{x} - x^3 - \mu x = 0, \tag{35}$$

and is shown in Figure 9(b). In this case, starting from a negative value of μ the trivial equilibrium is again stable but becomes completely unstable at the critical point, i.e., there is no local stable equilibrium to gradually

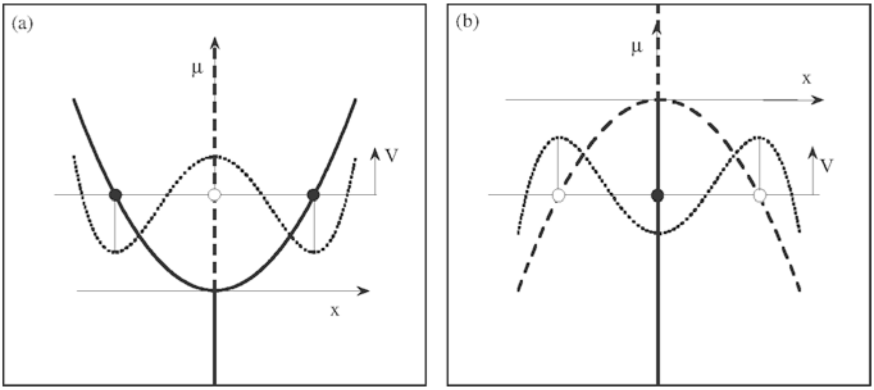

Figure 9. (a) A super-critical pitchfork bifurcation, (b) A sub-critical pitchfork bifurcation.

move onto. Furthermore, as the critical point is approached, the adjacent saddle points (associated with the unstable equilibria) start to erode the size of allowable perturbations. Although these two bifurcations have the same stable trivial equilibrium and critical point they have quite different consequences if encountered in practice. Hence, they are sometimes characterized as 'safe' or 'unsafe' according to whether a local post-critical stable equilibrium is available.

Another elementary bifurcation is the transcritical, or asymmetric, bifurcation:

$$\ddot{x} + 0.1\dot{x} + x^2 - \mu x = 0, \tag{36}$$

and illustrated in Figure 10(a). Here, a fundamental (trivial) equilibrium for negative μ loses stability as μ passes the through the origin. The other equilibrium becomes stable at this point and deflection occurs in the positive x direction.

Figure 10(b) shows the final example of an elementary bifurcation. It is also perhaps the most fundamental, since later we will show that the symmetry of the bifurcations already described is unlikely to be exactly observed in practice. The saddle-node bifurcation is characterized by the control parameter μ and coordinate x linked quadratically:

$$\ddot{x} + 0.1\dot{x} + x^2 - \mu = 0. \tag{37}$$

Equilibrium corresponds to the rest state and thus

$$x_e = \pm\sqrt{\mu}. \tag{38}$$

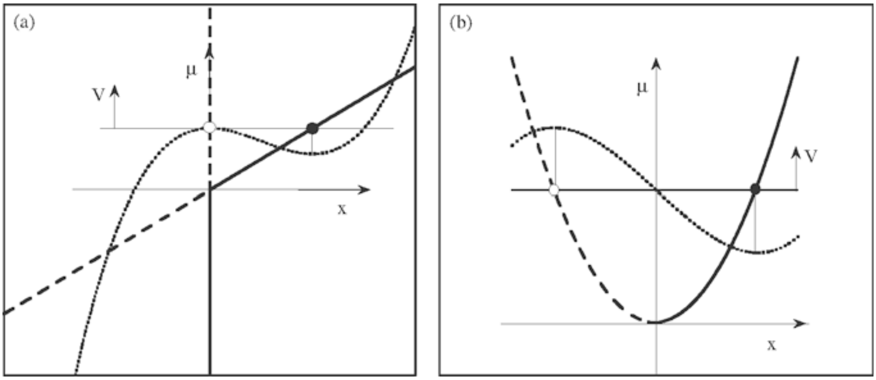

Figure 10. (a) A transcritical bifurcation, (b) A saddle-node bifurcation.

and we see either two co-existing solutions (one stable and the other stable) or no (real) solutions, depending on the sign of μ. The fundamental path is nonlinear, rather than the trivial initial path exhibited by the other bifucations.

We conclude this section by relating these situations to the changing potential energy (Bazant and Cedolin (1991)). For example, the potential energy associated with the saddle-node can be written as

$$V = \frac{x^3}{3} - \mu x + C, \tag{39}$$

and equilibrium from

$$\frac{\mathrm{d}V}{\mathrm{d}x} = x^2 - \mu = 0. \tag{40}$$

The sign of the curvature of the potential energy governs stability:

$$\frac{\mathrm{d}^2 V}{\mathrm{d}x^2} = 2x, \tag{41}$$

which is evaluated about equilibrium. When $x_e = \sqrt{\mu}$ the second derivative of the potential energy function is positive indicating that this is a minimum and hence is stable. The opposite conclusion can be drawn from the other equilibrium branch thus confirming the results of the stability properties based on the decay or growth of local perturbations. As the value of μ is reduced the two equilibria come together (the frequency of small oscillations will decrease and effective damping increases) as the potential surface flattens out. Just prior to coalescence the stable equilibrium can be thought of as a node, and the unstable equilibrium is a saddle. Hence their approach (at the critical point) is called a saddle-node bifurcation. No equilibria exist for negative μ and trajectories would simply be swept away. This instability is also sometimes referred to as a fold or limit point. The potential energy is shown in Figures 9 and 10 (shown dotted) for a given value of the control parameter.

4.2 Initial Imperfections

Initial geometric imperfections or load eccentricities that tend to break the symmetry may have a relatively profound effect on stability (Virgin (2007)). We shall consider this type of effect and its influence on the sub-critical pitchfork. Incorporating a small offset causes equation 35 to be altered to

$$\ddot{x} + 0.1\dot{x} - x^3 - \mu x + \epsilon = 0, \tag{42}$$

where ϵ is a small parameter which breaks the symmetry. Figure 11 shows how the instability transition is changed. We see that for large negative μ

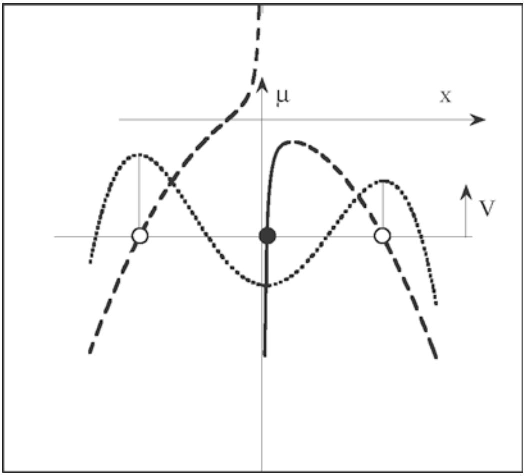

Figure 11. A perturbed sub-critical pitchfork bifurcation.

we have an equilibrium slightly offset from $x = 0$, and this grows as μ approaches the underlying critical value for the perfect geometry, but then falls off and the system completely loses stability. There is also a complementary (remote) solution for negative x but this wouldn't ordinarily be accessed as μ is monotonically increased (and it is unstable in any event). However, the fundamental solution does posses a critical point, and this is actually a saddle-node bifurcation. We also note the small tilt in the potential energy function. This behavior is termed 'imperfection-sensitive' (Thompson and Hunt (1973)) since the maximum value of the control parameter diminishes with the magnitude of the initial imperfection. The saddle-node and super-critical bifurcations are not imperfection sensitive.

4.3 Hopf Bifurcation

The other way in which an equilibrium can lose its stability under the operation of a single control parameter is the Hopf bifurcation (Thompson and Stewart (1986)). We have already seen this in van der Pol's equation, in which a complex conjugate pair of eigenvalues changes from having negative real parts to positive. That is, given a positive value of the determinant, the trace becomes positive (see figure 5). This instability is inherently dynamic, and is the main mechanism by which limit cycle oscillations occur. This also occurs in both the sub- and super-critical forms.

Figure 12 shows some typical transitions through these elementary bifurcations, in which the control is made a linear (ramp) function of time. Part (a) is a sub-critical pitchfork bifurcation in which the control parameter evolves with time according to $\mu = 0.01t - 1$ and thus the (quasi-static) critical point is reached after 100 time units. A slight delay in the realization is observed since the system remains somewhat in the vicinity of the unstable equilibrium after the critical point, before losing stability completely. The initial conditions for this case are $x(0) = 0.2, \dot{x}(0) = 0.0$. In part (b) is shown the corresponding super-critical case where the post-critical path follows one of the two available non-trivial (but stable) equilibrium paths. Part (c) is the saddle-node. Since there is no trivial equilibrium in this case the simulation was initiated at $x(0) = 1.2$, i.e., not far from equilibrium at $x_e = 1$ when $\mu = -1$. Finally part (d) illustrates a realization of a super-critical Hopf bifurcation. In this case $\mu = 1 - 0.5t$ and thus the quasi-static critical point is reached after approximately 20 time units, and again a delay is observed. Also seen in this figure is the lengthening of the period for larger t (and hence μ) anticipated from the initial post-critical approximation $\omega = 1 - (1/16)\mu^2$, as well as the motion becoming gradually less sinusoidal.

5 Forced (Linear) Oscillators

This section will focus on externally-excited systems, i.e., where $F(t) \neq 0$ in Figure 1. An important class of forcing function is harmonic excitation:

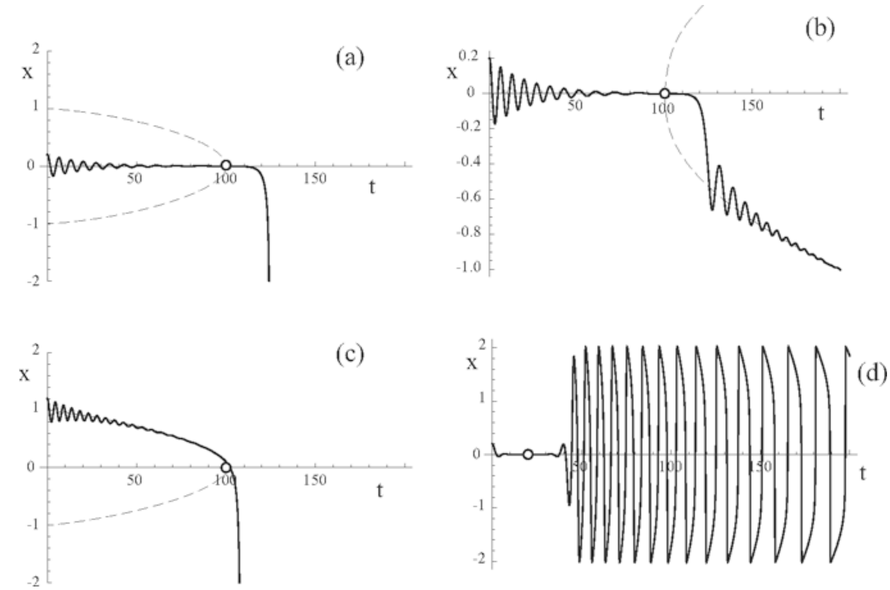

Figure 12. Examples of transitions through generic instabilities, (a) sub-critical pitchfork, (b) super-critical pitchfork, (c) saddle-node, (d) Hopf.

$F(t) = F_0 \sin \omega t$, or $y(t) = Y_0 \sin \omega t$, where this latter expression relates to a base movement that transmits motion to the mass via the support system (Thomson (1981); Inman (1994)). This latter situation is an important practical aspect of vibration and underlies the concept of vibration isolation to be considered from a nonlinear perspective in chapter 3.

For the case when the force is applied directly to the mass we have a governing equation of motion of the form

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t, \tag{43}$$

or in nondimensional terms

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f_0 \sin \omega t, \tag{44}$$

where $f_0 = F_0/m$. The solution of equation 44 consists of the summation of two parts: a homogeneous solution, obtained from the free vibration (obtained in the previous section); and the particular solution, which is related primarily to the forcing. Its general solution has the form:

$$x(t) = X_1 e^{-\zeta\omega_n t}\sin(\sqrt{1-\zeta^2}\omega_n t + \phi_1) + \frac{f_0}{k}\frac{\sin(\omega t - \phi)}{\sqrt{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2}}, \tag{45}$$

where trigonometric identities have been used to combine the harmonic terms from equation 24, and X_1 and ϕ_1 depend on the initial conditions.

The first (transient) part of the solution decays with time leaving the second part as the steady-state oscillation. Some sample responses are shown in Figure 13 in which the (lightly damped) system is started from rest at three different forcing frequencies. Parts (a) and (b) show that for a forcing frequency, $\omega = 0.3$, which is less than the system natural frequency, $\omega_n = 1.0$, the transient is relatively mild compared with the steady-state response and is quickly attracted to the harmonic oscillation. When the forcing frequency is equal to the natural frequency, as in parts (c) and (d) resonance occurs, i.e., a significant magnification effect (the denominator in the second term in equation 45 becomes small for $\omega \approx \omega_n$). Note the much larger amplitude of the response. In parts (e) and (f) the forcing frequency is increased to a value of 1.6, and now the transient solution is on the same order of magnitude as the steady-state, and the steady-state amplitude is back down to a lower level. Thus we observe that both parts of the solution depend quite strongly on the frequency ratio. The rate, and hence duration, of the transient decay is primarily a function of the damping. In all these cases the final steady-state motion is independent of the initial conditions (the choice of the origin in Figure 13 is arbitrary). This will not necessarily be the case for nonlinear systems, and indeed transients may be repelled by an unstable solution, as for example one of the cases shown in Figure 2.

It is useful to summarize how the maximum amplitude of the (steady-state) response ($A = x_{max}k/F_0$) varies with the frequency ratio Ω, where $\Omega = \omega/\omega_n$. The normalized amplitude of response can also be written as $A = x_{max}\omega_n^2/f_0$ (Inman (1994)). The response scales linearly with the forcing amplitude f_0. Figure 14 (a) shows a typical amplitude response diagram for four different damping values. The phenomenon of resonance is apparent, i.e., a significant amplitude magnification when the forcing frequency is close to the natural frequency (i.e., $\Omega \approx 1$). In fact we see for zero damping a growth to infinite amplitudes. The resonant peak is thus very sensitive to damping ($A_{res} \approx 1/2\zeta$, for light damping), and since many of the nonlinearities of interest are related to larger amplitude motion, we might anticipate interesting behavior in the vicinity of resonantly forced, lightly damped systems.

When the system is subject to $y(t) = Y\sin\omega t$ (a displacement applied

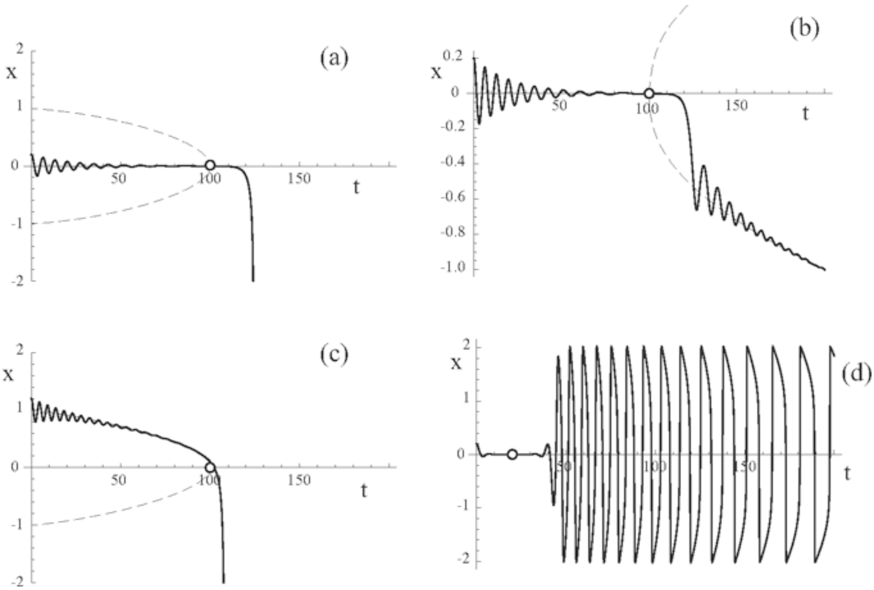

Figure 12. Examples of transitions through generic instabilities, (a) sub-critical pitchfork, (b) super-critical pitchfork, (c) saddle-node, (d) Hopf.

$F(t) = F_0 \sin \omega t$, or $y(t) = Y_0 \sin \omega t$, where this latter expression relates to a base movement that transmits motion to the mass via the support system (Thomson (1981); Inman (1994)). This latter situation is an important practical aspect of vibration and underlies the concept of vibration isolation to be considered from a nonlinear perspective in chapter 3.

For the case when the force is applied directly to the mass we have a governing equation of motion of the form

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or in nondimensional terms

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where $f_0 = F_0/m$. The solution of equation 44 consists of the summation of two parts: a homogeneous solution, obtained from the free vibration (obtained in the previous section); and the particular solution, which is

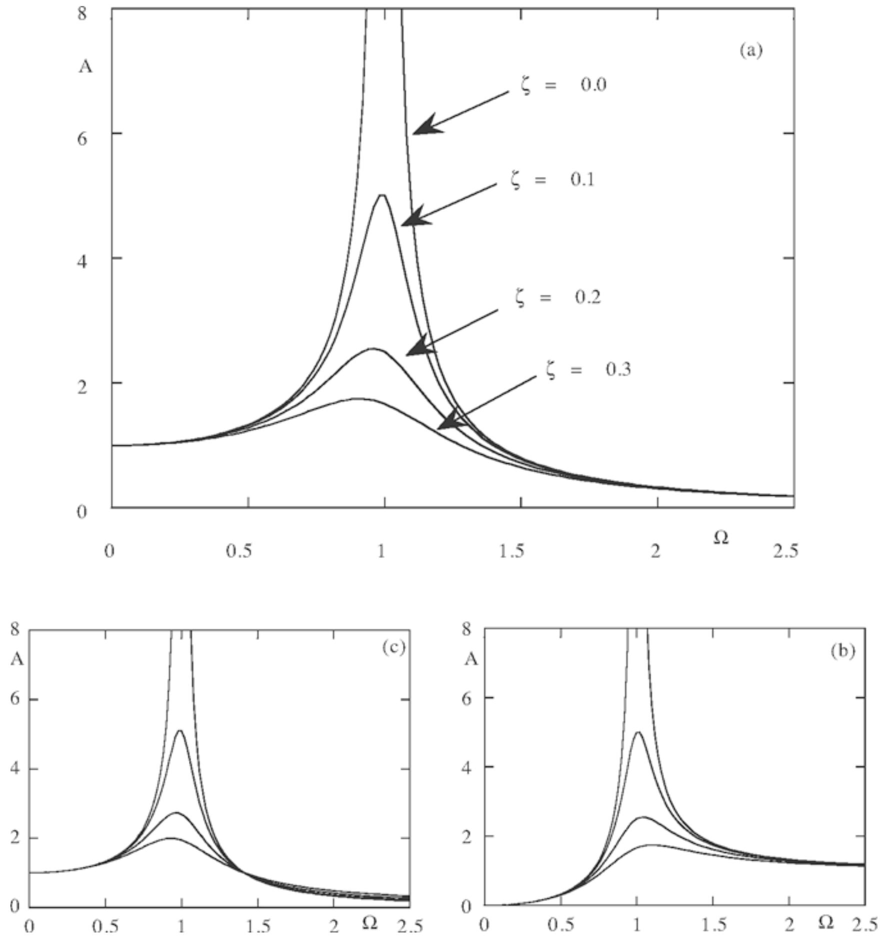

Figure 14. Amplitude response diagrams for linear oscillators, (a) direct mass excitation, (b) support motion (relative response), (c) support motion (absolute response). The same damping values as used in (a).