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History of Virtual Work Laws

A History of Mechanics Prospective

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Preface

Lagrange, in the *Méchanique analitique* of 1788, identified three programs of research, or paradigms, in the history of statics: the lever, the composition of forces, and the principle of virtual work. The paradigm of the lever would have been in force from antiquity until the early XVIII century, when Varignon was asserting the parallelogram law for composition and decomposition of forces. The principle of virtual work would have become dominant since the XIX century. This picture is in my opinion quite realistic, although the final predicted by Lagrange was never fully realized because the principle of virtual work has never replaced the rule of the composition of forces, but at most has outflanked it. Also the picture is too schematic. In fact, some form of law of virtual work has always existed in mechanics, always however with limited applications.

The law of virtual work, as usually presented in modern textbooks of mechanics, says that there is equilibrium for one or more bodies subjected to a system of forces if and only if the total virtual work is zero for any virtual displacement. In Chapter 2 of this book the meaning of the terms work and virtual is described in some detail; here I will only to mention that, since Lagrange in the second half of the XVIII century, the law of virtual work had no appreciable changes in its formulation. The view on its role in mechanics is instead still varying, passing from the enthusiasm of the XIX century to a modest presence in modern rational mechanics as well as, all considered, in the engineering field, albeit with some important exceptions.

The present book starts from the first documented formulations of laws of virtual work. They usually have only a vague analogy to the modern ones and only mathematically. Attention is paid to Arabic and Latin mechanics of the Middle Ages. With the Renaissance there began to appear slightly different wordings of the law, which were often proposed as unique principles of statics. With Bernoulli and Lagrange the process reached its apex. The book ends with some chapters dealing with the discussions that took place in the French school on the role of the Lagrangian law of virtual work and its applications to continuum mechanics.

Even though the book takes a particular point of view, it presents an important slice of history of mechanics. Essential reference is made to primary sources; secondary literature is mainly used to frame the contributions of the scientists considered in their times. To allow a better understanding of the ideas of the authors studied, English translations are always accompanied by original quotations (Appendix). No pre-conceived historical hypotheses have been explicitly assumed though. The mere existence of the book suggests that I have in mind a continuous chain connecting concepts from antiquity up to now. However the nature of the chain is complex and I leave it to the reader to unveil it.

The book is the result of a twenty year study of mechanics and its history and should be of interest to historians of mathematics and physics. It should also arouse interest among engineers who are now perhaps the most important witnesses of classical mechanics, and with it, of the law of virtual work.

I want to acknowledge Giuseppe Ruta, Romano Gatto, Antonino Drago for contributing comments and suggestions to specific parts. Cesare Tocci for suggestions regarding the whole book, and finally I want to acknowledge Raffaele Pisano for his reading and the debates we have had.

Editorial considerations

Figures related to quotations are nearly all redrawn to allow a better comprehension. Symbols of formulas are always those of the authors, except in easily identifiable cases. Translations of text from French, Latin, German and Italian are as much as possible close to the original.

Rome, September 2011 *Danilo Capecchi*

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Introduction

Hereinafter, the law of virtual work is not given its contemporary meaning. To do so would be misleading in that it would attempt to write a history based on unavoidable recourse to categories of thinking that did not exist in the past. I use instead the broader meaning of law of equilibrium, where forces appear together with the motion of their points of application independently of the logical status assumed, be it a theorem, a principle or an empirical law. In this sense the *laws* of virtual work represent a particular historical point of view on mechanics.

Since the Greek origins of mechanics, there have been two alternative formulations of laws of virtual work (hereinafter VWL). The first, which dates back to Aristotle's school, today goes under the name of laws of virtual velocities, in which the effects of forces are assumed depending on the virtual velocities of their points of application. The second, which has been known at least since the Hellenistic period, today goes under the name of laws of virtual displacements, in which instead the effects of forces are assumed depending on virtual displacement of their points of application. The two approaches, though conceptually different, are mathematically equivalent.

In the early days of VWLs, virtual motions were considered primarily as possible motions, those which one would have imagined the body, or system of bodies, to assume within the respect of constraints, for example, following a disturbance induced by a small force that alters the equilibrium. If one imagines that a balance rotates around the fulcrum, at the same time one would imagine that the weights of which it is burdened move. But with this type of 'natural' conception there coexists another, though not fully conscious at the beginning, in which the virtual motion is seen as purely geometric. On the one hand one sees the balance in equilibrium under assigned weights; on the other hand one imagines the unloaded balance moving with a motion that takes place with a time flowing in a super-celestial world. This way of viewing virtual motions began to emerge from the 'subconscious' to become the 'natural' one only in the XIX century with Poinsot and Ampère.

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Below is a brief history of VWLs, almost a summary of the book, from which it is clear that the various formulations that have occurred during approximately two thousand years, from Aristotle to Galileo, showed no appreciable progress. After Galileo, there was instead an abrupt change of direction and in a few generations very sophisticated formulations were reached.

1.1 Virtual velocity laws

The reconstruction of the historical development of the laws of virtual velocities is currently very incomplete. It goes back to Aristotelian *Mechanica problemata* of the fourth century BC [12], with the law: "heavy bodies located at the end of a lever are equilibrated when, in their possible motion, velocities are in inverse ratio to weights". Its explicit formulation, however, is documented only by Galileo who introduced it especially in *Le mecaniche* [119] and in the *Discorsi sulle cose che stanno in sù l'acqua* [115]; in the latter memoir he associated explicitly the law of virtual velocities to Aristotle. The law of 'virtual velocities' of Aristotle's school took the functioning of the lever as the main reference. Velocity did not have its current quantitative meaning, but was rather the concept of the common man for which there was no well-defined measurement, and at most a formulation of a criterion of more or less. Moreover, even force – regardless of its metaphysical uncertainty – was a somewhat indefinite quantity. It could be measured by weight, and then introduced into the calculations, but uncertainties still remained. Its direction was not well defined, or rather was defined tacitly: the force applied, for example, to the end of a lever was implicitly considered orthogonal to it. The law of virtual velocities, although formulated on the basis of magnitudes not well quantified, led to correct results already in the Aristotelian school. The 'velocity' of the points of a lever that rotates around its fulcrum can be said to vary in proportion to the distance from it and this was enough to determine a quantitative relationship between the forces and the distances from the fulcrum. The idea of a VWL arose from the motion of points on a circle which rotates around its centre:

> Remarkable things occur in accordance with nature, the cause of which is unknown, and others occur contrary to nature, which are produced by skill, for the benefit of mankind. [...] It is strange that a heavy weight can be moved by a small force, and that, too, when a greater weight is involved. For the very same weight, which a man cannot move without a lever, he quickly moves by applying the weight of the lever. Now the original cause of all such phenomena is the circle; and this is natural, for it is in no way strange that something remarkable should result from some thing more remarkable [12].[1]

In their analysis of the motion of the circle, Aristotle's followers concluded that the points which tend to move more easily require less force than those which tend to move less easily. If to 'more or less easily' is given the meaning of 'more or less force', then one obtains a trivial tautology, but if it is given the meaning of 'more or less quickly', then a form of VWL is obtained.

[1] pp. 133, 135.

Nowhere in the *Mechanica problemata* did Aristotle use the word or the concept of equilibrium. At most equilibrium can be seen in dynamical key as the result of the cancellation of effects of opposing forces. Effect measured on the basis of virtual motion. The higher the virtual velocity of the point of application of a force the greater the effect. The study of equilibrium on the basis of possible motions seemed a contradiction in terms for those who could not conceive, with Aristotle, rest as motion in power. And there were many who did not share the ideas of Aristotle. But even if this metaphysical difficulty is ignored, the Aristotelian law of virtual work was too 'complex' from a logical point of view to be assumed as a principle, i.e. it had to be demonstrated.

According to the mathematicians of the time a demonstration had to be based on the existing model of geometry and had to consist of a derivation from evident propositions. The intuitive Aristotelian considerations had no probative value. For this reason in ancient Greece, the law of the lever among scientists but also among technicians, followed a different approach, based on the concept of centre of gravity. Unfortunately we have few documents relating to the mechanical studies of Greek mathematicians posterior to Aristotle. There are essentially the basic texts of Archimedes on hydrostatics and centres of gravity, and some studies on the balance atributed to Euclid. The most complete witness of Greek mechanics is contained in the *Mechanica* of Hero of Alexandria, which had an applicative character. However it can be said that Greek mechanicians assumed as their main conceptual model the lever and the law which regulates its behaviour was proved with considerations 'beyond any doubts' from principles, fixed by Archimedes (see Chapter 3) which are also 'beyond any doubts'. Here equilibrium is the key concept, while motion is not considered, except to deny it.

In the modern era Galileo was the first to assume a VWL with a dynamical connotation where equilibrium resulted from cancellation of opposing trends. The name he gave to these trends was 'momento' (moment), a term which remained long in the history of mechanics:

> Moment is the propension of descending, caused not so much by the Gravity of the moveable, as by the disposure which divers Grave Bodies have in relation to one another; by means of which Moment, we oft see a Body less Grave counterpoise another of greater Gravity Moment is the propension to go downward, caused not so much on by severity of the gravity of a mobile, but by the mutual disposition of the different heavy bodies, by the moment of which you will see many times a less heavy body counterbalance another more heavy [119].[2] (A.1.1)

Galileo was not able to combine disparate magnitudes, such as weight and velocity, and the idea of momento was expressed in the language of proportions that remains at a somewhat imprecise level. In the study of the lever, shown in *Le mecaniche*, Galileo saw virtual motion as that motion generated by altering the equilibrium with a small weight. He then retained for it a certain degree of reality.

[2] p. 159. Translation in [121].

1.2 Virtual displacement laws

The idea of virtual displacement is in principle simpler than that of virtual velocity, because a displacement could be detected unambiguously even in antiquity. So it was natural that in addition to the law of virtual velocities, also the law of virtual displacements had emerged. It had developed along two completely different paths. The first, which is generally the most emphasised, took the functioning of devices for lifting and shifting – the machines – as the main reference. The beginning is found in the writings of Hero of Alexandria, but it is present more clearly as a general law of mechanics in Thabit's *Liber karastonis* in the IX century and in Jordanus de Nemore's *De ratione ponderis* in the XIII century. Jordanus assumed the law that moving a weight p at height h is equivalent to moving a weight $q = p/k$ up to hk, whatever k. The logical status of Jordanus' virtual displacement law is still disputed: is it a principle or a theorem derived from the Aristotelian laws of motion? It had however a general character and was used in various demonstrations. Important is that of the inclined plane, which for the first time was referred to correctly. Note that Jordanus' is a law of equivalence, or conservation, but not of equilibrium. To obtain equilibrium it is necessary to present an *ad absurdum argument*. The examination of the proof of the law of the lever, reported in the *De ratione ponderis*, shows the way (see Chapter 4). Consider a lever with two weights P and Q placed at distances p and q in inverse proportion to P and Q respectively. For the law of equivalence, weight P can be replaced by a weight equal to Q placed at a distance q from the fulcrum of the lever, on the same side of P, since by hypothesis the relation $Pp = Qq$ holds true. What is obtained in this way is a lever with equal arms and equal weights and, as such, in equilibrium, thus satisfying the principle of sufficient reason. This means that the balance was in equilibrium even before the change of weight P with the weight Q.

The ideas of Jordanus found their natural successor in René Descartes, who focused on the concept of what we now call work, which he called 'force':

> The same force that can lift a weight, for example of 100 pounds to a height of two feet, can also lift 200 pounds to a height of one foot, or 400 pounds to a height of $1/2$ foot, and other [96].[3] (A.1.2)

But there was a second source of the law of virtual displacement that put equilibrium in the spotlight. This is Torricelli's principle, according to which the centre of gravity of a system of bodies in equilibrium cannot sink for any virtual displacement compatible with constraints. It is a generalisation of the ancient empirical principle that the center of gravity of a heavy body moves necessarily down when there are no obstacles that prohibit it. Torricelli's principle was already formulated by Galileo:

> Because, as it is impossible for a heavy body or a mixture of them to move naturally upward, moving away from the common centre towards which all heavy things converge, so it is impossible that it spontaneously moves, if with this motion *its own centre of gravity does not approach the common centre* [emphasis added] [118].[4] (A.1.3)

[3] vol. 2, p. 435.

[4] p. 215.

And it can be traced back to medieval times, but only with Evangelista Torricelli could it take the form of a physical law expressed in the language of mathematics. Torricelli's principle was originally formulated for only two bodies: "Two joined bodies cannot move by themselves, if their common centre of gravity does not sink", but its extension to more bodies is straightforward. It had two great advantages over the other formulations of VWLs: it was 'convincing' for it appealed to everyday experience – and therefore no particular objection can be taken to assume it as the basis of statics – and could be easily generalised to a system of bodies.

Starting from Torricelli's ideas, John Wallis reworded the principle of Torricelli, saying that the sum of the products of forces times displacements of their points of application in the direction of forces must be equal to zero. According to Varignon Wallis was the man "who went farther than any other authors [before Bernoulli]" [238].[5]

1.3 Virtual work laws as principles of mechanics

However Torricelli's principle was not received enthusiastically and was basically ignored by nearly all other mechanicians. A good number of scholars (including Pardies, Lamy, Rouhalt and Borelli), acknowledged the truth of the fact of the annulment of the virtual work of forces, but no one considered it possible to take this as a principle of statics because it was not self-evident, as the epistemology of time required for a principle. Moreover the principle, although very general, in many cases failed. It was successful for simple machines (lever, inclined plane, wedge, etc.), in which the directions of force and motion remain constant during virtual motion. It failed where this condition did not occur, such as the motion of a body on a curved profile.

René Descartes was the first to realise that, for the validity of any VWL, it was necessary to consider not the actual motion of bodies but that it would progress along straight lines or planes tangent to the constraints that limit the motion, i.e. the motion at the very beginning. This observation generalised the approach already used in statics by Galileo and Roberval, which replaced the existing constraints with equivalent ones. For example (Galileo), the inclined plane with a lever perpendicular to it. Besides this important technical improvement, Descartes claimed clearly the role of a principle of mechanics for a VWL in the formulation he gave it, that moving a weight p at height h is equivalent to moving a weight $q = p/k$ up to hk. For him it was a sufficiently clear and distinct proposition and was also enough to solve all problems of statics.

Descartes' idea of virtual motion was generalized further by Christiaan Huygens who introduced the concept of infinitesimal displacements in Torricelli's principle. His early works on the subject date back to 1667 (see Chapter 7) and concern the equilibrium of three or more ropes at the ends of which forces are applied. The memoirs of Huygens, related only to special cases, however were not published while he

[5] Preface.

was alive and it is unclear whether they came, indirectly, to the notice of Johann Bernoulli.

It was Johann Bernoulli who refined the wording of the VWL in a systematic way by introducing explicitly the concept of infinitesimal displacements. The law he formulated is today known as the *principle of virtual velocities* and is commonly considered as the prototype of the contemporary formulations of VWLs. The new edition of Johann Bernoulli's works, which also includes unpublished letters, provides a starting point for an interpretation of the origins of VWL in Bernoulli, slightly lowering the aura of mystery that until now it has been wrapped. In February 1714 Johann Bernoulli published *Manoeuvre des vaisseaux* (see Chapter 8), a book dedicated to the theory of sailing. The preparation of this book was stimulated by the publication of another book by Renau of Elissgaray, a marine engineer, on the same topic and the discussion that followed on the composition of forces. Bernoulli had recently embraced Leibniz's ideas of dead and living forces. He distinguished between the impulsive forces (living forces) and the forces that act continuously (dead forces), like the wind that pushes on the sails of ships. And the forces that act continuously are characterized by their *energies*, i.e. the product of the force by the component of the virtual infinitesimal displacement in the direction of the force, named by Bernoulli *virtual velocities*. In the end Bernoulli, as indeed did Huygens, tended to consider virtual velocities and virtual displacements essentially the same, and finally to consider the term 'virtual velocity' as a synonym for 'infinitesimal displacement'. This fact created a never-ending controversy because velocities and infinitesimal displacements are not perfectly matched, and while velocity was a well established and accepted concept in the XVIII century, the infinitesimal displacement remained shrouded in an aura of mystery.

The formulation of Bernoulli's VWL is commonly associated with a letter of Bernoulli to Pierre Varignon in 1715. Paradoxically, this letter appeared in the *Nouvelle mécanique ou statique* of 1725, a book which presented Varignon's rule of composition of forces as the fundamental principle of statics alternative to any VWLs. Bernoulli's statement affirmed that for a system of forces that maintains a point, a surface, on a body in equilibrium, the sum of positive energies equals that of negative energies, considered with their absolute value. Bernoulli was well aware of the importance of his principle. In his letter to Varignon he wrote that the composition of forces is not but a small corollary of his principle. Varignon of course did not share this enthusiasm and did not record in his book this part of Bernoulli's letter. For Varignon, Bernoulli's law is at most a theorem, to be proved case by case. Though he did not give a general proof, he devoted a large part of his *Nouvelle mécanique* [238] to prove it in 'all cases' where, using the rule of the composition of forces, it is known there is equilibrium.

Bernoulli's VWL was not immediately accepted as a possible principle of mechanics. Bernoulli himself seemed to have changed his attitude and, in his writings, referred to it only once, in 1728 in the *Discourse sur le lois de la communication du mouvement* [35]. Here he introduced again the virtual velocity, but as the velocity that each element of a body gains or loses, over the velocity already acquired, in an infinitely small time, according to its direction (see Chapter 8). The above definition is not equivalent to that contained in the letter to Varignon, because it explicitly named a velocity rather than a displacement, which is in general the variation dv of a given motion. This new point of view is due to the fact that now Bernoulli's interest is motion of bodies and not just their equilibrium. No reference or comment is made to his earlier definition of the virtual velocity, as if he had never written anything about it.

After Bernoulli, probably the most significant contribution to the development of VWLs was due to Vincenzo Riccati who introduced the *principle of action* in the *Dialogo di Vincenzo Riccati della compagnia di Gesù* of 1749 and the *De' principi della meccanica* of 1772. Vincenzo Angiulli moved in the wake of Riccati with his *Discorso sugli equilibri* of 1770. Although the idea of the principle of action was essentially Riccati's, the less original Angiulli was closer to the foundational aspects of mechanics. Angiulli tried to prove his VWL not from other mechanical principles but from 'indubitable' metaphysical principles, including the equivalence of cause and effect. He began with the Leibnizian concept of dead force, which is presented as an infinitesimal pulse, such as fds (where f is the intensity of the pulse, identified with the ordinary force, and ds the infinitesimal displacement of the point where the force is applied) continually renewed by gravity or some other cause and continuously destroyed by constraints. In the absence of constraints, the pulses can be accumulated and the action of the dead force is that of cumulative pulses; the action of the dead force generates then the living force and therefore the motion.

With the introduction of infinitesimals Angiulli could enunciate his principle of actions, which he qualifies as a theorem because it is demonstrated with his metaphysical considerations:

> The equilibrium comes from the fact that the actions of the forces which must be equilibrated, if born, would be equal and opposite, and therefore the equality, and opposition of the actions of the forces is the actual cause of equilibrium.
> […]
> The equilibrium is nothing but the impediment of the motions, that is of the effects of the forces, to which it is not surprising if the prevention of the causes, i.e. of the actions themselves is reached [4].[6] (A.1.4)

The principle of action implies the relation $\sum fds = 0$, where fds are the elementary actions emerging in the infinitesimal displacements ds, compatible with constraints. It is therefore a possible formulation of VWL. For Angiulli, the ontological status of the constraints was that of 'hard bodies', i.e. idealised bodies that absorb all the pulses and the living force. Constraints obey an economy criterion, acting only as much as it is needed. In practice Angiulli made the assumption of smooth constraints without being aware of the problematic nature of the fact. Note that the constraints have only the effect of destroying the motions and do not exert any reactive force, as this is a foreign concept to Leibniz's mechanics.

Half a century after the letter of Bernoulli to Varignon, Lagrange gave the VWL a more efficient form. Officially, he referred to Bernoulli, but its role was actually very different. When in 1764, for the first time, Lagrange exposed Bernoulli's principle

[6] pp. 16–17.

of virtual velocities, he recast it by talking about the equilibrium of bodies and not of forces and applied it considering all the motions compatible with constraints and not only rigid motions. He did not conceive the law of virtual work as a theorem, derived for example in the context of Newtonian mechanics; it was rather an alternative principle. This position should be clear from the introductory part of the *Mécanique analytique*, published more than twenty years later, where he presented the various ways of addressing the problems of equilibrium of bodies: the lever, the rule of the parallelogram and the principle of virtual velocities.

The first edition of the *Mécanique analytique* [145] of Lagrange, with the prominence it gave to his VWL, was the genesis of a wide debate on its logic status and was also the occasion for a critical analysis of the principles of statics. It was an occasion that, in the history of classical mechanics, has an equivalent only in the debate at the beginning of the XVIII century on the principles of dynamics, and of which today one no longer understands the significance. The list of scientists who became interested in the problem should make us reflect on the extent of the effort that was made and the opportunity to learn a lot by following their teachings: Lazare Carnot, Lagrange, Laplace, Poinsot, Fourier, Prony, Ampère and then also Cauchy, Gauss, Poisson and Ostrogradsky. Lobachewsky too was involved, but the content of his contribution has been lost. To have again such heated discussion of scientists on the fundaments of mechanics it will be necessary await up to the introduction of relativistic mechanics, one century later. To understand the reasons of the debate one needs to reflect that, though the logical status of dynamics was undoubtedly controversial, there was generally agreement that one could give statics a shared formulation. But Lagrangian VWL seemed to many to not meet the assumptions of epistemology of the times. Although one could say – but not everyone agreed even on that – the VWL was prior to all the laws of mechanics in the sense that these laws could be derived from it, one could not admit it was evident; in particular it seemed less simple and evident for example of the law of the lever. Lagrange also agreed and, in the second edition of the *Mécanique* [148], wrote:

> And in general I can say that all the general principles that can be discovered in the science of equilibrium, will not be but the same as the principle of virtual velocities considered differently, and from which it differs only in form. But this principle is not only itself very simple and general, it has, in addition, the precious and unique advantage of being translated into a general formula that includes all the problems that can be posed on the equilibrium of bodies. […] *As to the nature of the principle of virtual velocities, it is not so obvious that it can be claimed as a primitive principle* [emphasis added] [148].[7] (A.1.5)

Young Lagrange was attracted by the "precious and unique advantage of being translated into a general formula that includes all the problems that can be put on the equilibrium of the body" and did not hesitate to take an instrumental position. There is no doubt that he was essentially a mathematician and, in line with the times, strongly attracted by the formal aspects of Calculus. Although this position is subject to criticism, credit must be given to Lagrange for an originality that allowed him to go against the perhaps too rigid epistemology of the times. His attitude certainly con-

[7] pp. 22–23.

tributed to the development of the more liberal epistemology of the XX century, born to a large extent with the advent of non-Euclidean geometry.

Lazare Carnot reached a VWL for colliding bodies according to his laws of impact in a mechanical theory, in principle, without forces. A fundamental concept developed by Carnot which influenced the subsequent debate, Poinsot's included, is that of *geometric motion*, that is motion considered in itself independently of any force (see Chapter 11).

1.4 Virtual work laws as theorems of mechanics

Concluding, one could not accept the statement of the Lagrangian VWL as a principle and had to prove it by reduction to a theorem of another approach to mechanics. This question promoted, as already mentioned, a heated debate, especially in France where the main contributions were those of Lazare Carnot, Fourier, Ampère and Poinsot. For Italy it is worth noting the contribution of Vittorio Fossombroni. The reasons for the attention paid to Lagrange's VWL were not only scientific, however. It was no coincidence that the interest was polarised in France. Here the Cartesian tradition was still alive and national pride was still an obstacle to a full acceptance of Newtonian physics and metaphysics. VWLs seemed to offer the opportunity to develop a completely 'continental' mechanics, freed from the concept of force of a Newtonian matrix.

Joseph Fourier tried several demonstrations. In the probably most successful one, Lagrange's VWL is reduced to the law of the lever, replacing weights with active forces that exert their action by threads, rings and levers. André Marie Ampère, following Carnot, introduced the concept of virtual velocity as a vector tangent to the trajectories compatible with the constraints, where time "has nothing to do with". Vittorio Fossombroni in a memoir of 1794, demonstrated Lagrange's VWL in the case of a free rigid body starting from the cardinal equations of statics. Of some interest is Fossombroni's attempt to replace infinitesimal virtual displacements, which created some embarrassment, with finite displacements of arbitrary value. He showed that if the forces are parallel to each other and if their points of application are arranged along a line, the virtual work of these forces is zero for any finite rigid motion. This idea was generalised, to the case of forces in the space with application points lying on a plane, by Poinsot who felt the same embarrassment in the use of infinitesimal quantities.

Louis Poinsot gave in my opinion the most successful proof of a VWL. Since his mechanical theory was based on the rule of composition of forces and reduced to mathematical formulas, his proof was and is still considered by mathematicians and physicists, more interesting than the more geometric Fourier's type, based on the law of the lever, and has become a model for almost all textbooks of statics. Poinsot accepted the principle that a material point subject to a certain active force is equilibrated on a surface if and only if the force is orthogonal to it. In addition he considered other principles, among which the principle of composition of forces and the principle of solidification, according to which if one adds constraints – both internal and external – to a system of bodies in equilibrium, the equilibrium is not altered. On the basis of his principles Poinsot was able to fully characterise statics and write the equations of equilibrium in which only the constraint equations and the components of the active forces applied to various points of this system appear. To demonstrate his VWL, Poinsot gave up the virtual displacement concept, to adopt that of virtual velocity – contemporary meaning. According to Poinsot real time and virtual time run on different universes:

> It must be noted further that the system is supposed to move in any way, without reference to forces that tend to move it: the motion that you give is a simple change of position where the time has nothing to do at all [197].[8] (A.1.6)

By replacing virtual displacements with virtual velocities, it is then easy to prove a VWL in the form $\sum f dv = 0$, where v are the virtual velocities of the points of application of the active forces f. One cannot stress enough the fact that Poinsot's virtual velocity is purely geometric and his virtual work is only a mathematical definition. Poinsot thus closed the circle that had opened with Aristotle. The laws of virtual work were initially manifested as laws of virtual velocities, then the laws had split into virtual displacement and virtual velocity laws. With Bernoulli there was a partial but ambiguous reunification; Poinsot brought everything back to the baseline by eliminating the laws of virtual displacements.

1.5 Contemporary tendencies

But not everyone followed Poinsot in dealing with VWLs as theorems of mechanics and considering the virtual work as a purely mathematical concept. It is possible to identify a line of thought that instead of diminishing the mechanical meaning of the virtual work tended to enhance it. This line of thinking had its precursors in Descartes and Leibniz. Then it became precise with Lazare Carnot, who introduced the concept of work of a 'power' along an arbitrary path, named by him *moment of activity*, giving it the meaning of a physical magnitude and a key place in mechanics. A few years after Lazare Carnot's contribution, Gaspard Coriolis established definitely in 1829 the term 'work' to indicate the Bernoullian energies. This change of terminology also implied a change of the ontological status. Also referring back to the ideas of Lazare Carnot, virtual work began to take on the role of well-defined mechanical magnitudes. Coriolis adopted the molecular model of matter, where everything is reduced to material points treated as centres of force. In this mechanics, there are no constraints in the classical geometrical sense: there are 'material' constraints composed of material points carrying out repulsion actions against the particles that wish to penetrate them. Since there are no constraints, the infinitesimal displacements are not subject to any limitation and can be identified with – and indeed they were – real motions. So next to virtual work, there was room for 'real work'. Coriolis addressed for the first time the thorny problem of friction. While in a traditional formulation of VWL it was difficult to consider the reactions, without which it is impossible

[8] p. 13, part II.

to introduce friction, for Coriolis there was no difficulty. Friction is represented by the tangential components of the interaction between two bodies resulting from the superposition of forces exerted by the material points constituting the bodies. The following passage serves to illustrate the idea:

> We are led to realize that the principle of virtual velocities in the equilibrium of a machine, composed of more bodies, cannot take place without considering first the sliding friction, where the virtual displacements cause the slipping of the bodies, one on others, and finally that the rolling when bodies cannot take that virtual motion without deformation near the contact points.
> Frictions are recognized always, for experience, able to maintain equilibrium in a certain degree of inequality between the sum of the positive work and the sum of the negative work, here taking as negative the elements belonging to the smaller sum. It follows that the sum of the elements to which they give rise has precisely the value that can cancel the total sum and is equal to the small difference between the sum of the positive and negative elements [79].[9] (A.1.7)

Parallel to the discussion on the concept of virtual or real work, a new science, thermodynamics, was developing, where real work had a physical meaning in every respect.

Sadi Carnot put the work that he indicated with the term 'engine power' at the centre of his *Reflexions sur la puissance du feu* of 1824 [62]. Work moved from thermodynamics to mechanics with Rankine, Helmholtz and Duhem in the XIX century. In Duhem's mechanics, a VWL came from the principle of conservation of energy, basically in its variational version. The connection of Duhem's VWL to 'real' work or energy marked in some way a reconciliation with the principle of the impossibility of perpetual motion. Until then the two principles were kept strictly separate. Lagrange in particular, in his writings, never referred to the perpetual motion.

The role of VWLs in contemporary classical mechanics is not well defined, it is however not essential. In theoretical treatises on rational mechanics, which takes a strong axiomatic point of view, VWLs are often not even mentioned, even though the axioms upon which mechanics is erected, such as Lagrange or Hamilton equations, could be derived from them. In the applied mechanics of rigid bodies, VWLs are present but not important. They are used to solve some particular problems, in which for the presence of constraints it would be difficult to use other methods. But when considering the mechanical theory as a whole, it is generally preferred to start from the cardinal equations. The constraints are taken into account by introducing auxiliary unknowns such as the reactive forces which are then removed in the solution of the single problems. There are no conceptual difficulties in dealing with constraints friction; it is enough to provide the appropriate 'constitutive' relationships. In continuum mechanics the role of VWL is instead rather important. But this does not depend on its ability to address the various conditions of constraints, but rather on the mathematical expressions the virtual work law takes, that makes it easier for approximate solutions in many cases, for example with the finite element method.

[9] p. 117.

Table 1.1. Various versions of virtual work laws

Scholar	*Century*	*Real mot.*	*Geom. mot.*	*Logic status*	*Real work*
Aristotle	IV BC		v	p	
Hero	I	d		t	
Thabit	X	d		p	
Jordanus	XIII	d	d	t	
Galileo	XVI	v		p	
Stevin	XVI	d		t	
Dal Monte	XVI	d		t	
Torricelli	XVII	d		p	
Descartes	XVII	d		p	•
Wallis	XVII	d		p	?
Bernoulli	XVIII	d		p	•
Riccati	XVIII	d		p	•
Angiulli	XVIII	d		p	•
Lagrange	XVIII	d		p	?
Fossombroni	XVIII		d	t	
Carnot L	XVIII		v	?	
Fourier	XIX		d	t	
Ampère	XIX		v	t	
Poinsot	XIX		v	t	
Piola	XIX		d	?	?
Servois	XIX	d		p	•

In the table above the main characteristics of the various VWL formulations are reported. It is distinguished if the virtual motions are real or fully geometric, in the sense that if they run as the time of the forces or not, if displacement (d) or velocity (v) is concerned. The logic status is distinguished, i.e. the law is considered a principle (p) or a theorem (t) and if the virtual work is a physical magnitude (bullet) or instead a pure mathematical expression.

1.6 Final remarks. The rational justification of virtual work laws

The history of the various forms that VWLs have taken also focused on attempts that have been made to give them a rational justification. The degree of satisfaction achieved was different from period to period. Together with a certain agreement there was however always a tension towards overcoming the law, searching for a more powerful expression.

Aristotle seemed at first sight convincing enough to justify the law according to which the efficacy of weights placed on the arms of a balance depends on their distance from the fulcrum. He was considered persuasive by many mathematicians, but not by his contemporaries, accustomed to the high standards of rigor exemplified by Euclidean geometry. The justification of Jordanus de Nemore's proposition that what can lift p to h can also lift p/n to nh was criticized by his immediate successors. It was perfected by Tartaglia, but Tartaglia's arguments were the subject of severe criticism by Archimedean mathematicians of 1500, in particular by Guidobaldo dal Monte. Galileo did not attempt any proof, but justified his *law of moments* on an intuitive level; the same approach was followed by Johann Bernoulli with his *rule of energy*. Descartes considered self evident the simple virtual work law as expounded by Jordanus de Nemore. Riccati and Angiulli tried an external justification of Bernoulli's rule of energy, from 'certain' metaphysical principles, such as the equality between causes and effects; principles not accepted by most mathematicians. Lagrange at the beginning considered as evident a virtual work law substantially coincident with Bernoulli's rule of energy, which he called the *principle of virtual velocities*. Then he presented a very simple and elegant justification, which however did not meet completely the standards of rigor of the times according to which any recourse to geometric intuition was not allowed, that instead Lagrange had introduced, albeit without the use of figures. Fourier, Ampère, Laplace, Poinsot and many other scientists attempted to reduce their laws of virtual work, substantially coincident with the principle of virtual velocities of Bernoulli and Lagrange, to the elementary principles of statics, essentially the law of the lever and the rule of the parallelogram of forces. These attempts were followed by others who felt them as not entirely satisfactory.

From the above, must it be concluded that the laws of virtual work have never been rationally justified? Or with a term that has a more restricted, but stronger meaning, have they never been proved? The answer is not simple. To understand why just recall that the other fundamental laws of mechanics such as the law of the lever and the rule of the parallelogram of forces followed the same fate. Many explanations were proposed but always something was found to complain about.

Even contemporary scholars have dedicated themselves to attempt to justify VWLs, albeit with less passion and strength [283]. The problem has a bit shifted and transformed itself into the question: Is the mechanics L – statics and dynamics – resulting from the adoption as a founding principle the most advanced law of virtual work equivalent to the mechanics N resulting from the adoption of the most advanced version of Newtonian mechanics? The problem is a little bit easier than that to justify a VWL because the acceptance of various principles assumed for the advanced version of Newtonian mechanics can be object of a less strict scrutiny.

However the problem absorbs more philosophers of science than specialists of classical mechanics. To the latter the problem seems not difficult to solve and in a positive way, following the reasoning of Chapter 2 of the present text. On the other hand, Poinsot in his *Mémoire sur la théorie générale de l'équilibre et du mouvements des systèmes* of 1806 had given the problem a fully satisfactory response according to the modern standards of evidence, by proving the equivalence between a VWL and a Newtonian mechanics enlarged with a series of principles to take into account the presence of constraints.

Concluding from the historical path and also from a modern logical analysis it can be concluded that the laws of virtual work have been considered justified in a fairly satisfactory way in the past and today, no less than many other fundamental laws of mathematical-physics.

2

Logic status of virtual work laws

I use the term *law of virtual work* to mean any rule of equilibrium that includes both forces and possible displacements of their points of application. In this chapter I will give a more restricted meaning by referring only to the most (modern) advanced formulations and I will consider their logical status, i.e. whether they are principles of an autonomous mechanics or theorems of 'another' mechanics. To demonstrate a law, a proposition, means in the broadest sense, to bring it to laws 'assumed as known', with a sense that varies with the epistemological frame of reference. Until the development of modern axiomatic theories and their application to physical science by the neopositivists of the XX century, a principle was considered as acceptable if it had an intuitive nature of evidence, possibly established a priori. Currently there is a more liberal view and one does not require evidence of the principles, only that they must have sufficient strength and do not lead to logical contradictions.

The problem of provability of a VWL clashes immediately with the fact that even today there does not exist a reference theory of mechanics that is fully defined and universally accepted. This is true even for systems of material points, although there do exist some axiomatizations [382, 360, 390]. A major difficulty encountered in various formulations of VWL and mechanics concerns the ontological status attributed to constraints and reactive forces. Before the XVIII century, constraints had been treated only as passive elements not able to act. Only after studying elasticity and accepting models of matter based on particles considered as centres of forces, have researchers begun to think of constraints as capable of administering active forces. In dynamics, according to Lagrange, the first scholars to assimilate constraint reaction to active forces were the Bernoullis, Clairaut and Euler, in the period 1736 to 1742. "The use of these forces dispensed from taking account of the constraints and allows one to make use of the laws of motion of free bodies" [145].[1] In statics, constraint reactions are less problematic; they can be considered as the forces necessary to maintain the constraint. The first to introduce them in calculations was probably Varignon in his *Nouvelle mécanique ou statique* of 1725 [238].

[1] p. 178–179.

Capecchi D.: History of Virtual Work Laws. A History of Mechanics Prospective.
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It was simply the difficulty of incorporating the reactive forces in a consistent mechanical theory that led Johann Bernoulli to the formulation of an effective law of virtual work, known after Lagrange as the *principle of virtual velocities*, which provides a criterion of equilibrium without intervention of these undesirable ‘beings’. But his statement that "the sum of power each multiplied by the distance traveled from the point where they are applied, in the direction of this power will always be zero", always questioned its nature; in this regard the following comment by Fossombroni is of interest:

> That common faculty of primitive intuition, so everyone is easily convinced by a simple axiom of geometry, as for example, that the whole is greater than the part, certainly does not need to agree on the aforementioned mechanical truth, which is much more complicated than that one of the common axioms, as the genius of the great Men who have admitted the axiom, exceeds the ordinary measure of human intelligence, and it is therefore necessary for those who are not satisfied to obtain a proof resting on foreign theories [...] or to rest on the faith of chief men despising the usual reluctance to introduce the weight of authority in Mathematics [109].[2] (A.2.1)

The success of Lagrange’s *Mécanique analytique* [145] which assumed Bernoulli’s principle of virtual velocities, suitably reformulated, as the source of all mechanics, opened a heated debate on its plausibility. In this chapter I do not refer to these efforts, nor to the preceding others, but try mainly to clarify in what sense one can prove a law of virtual work, be it Lagrange’s or otherwise. Attempts to demonstrate can be divided into two categories. In the first, which I refer to as *foundational*, one tries to deduce a VWL without reference to existing criteria of equilibrium; in the second category, which I refer to as *reductionist*, one tries to deduce a VWL by a pre-existing criterion of equilibrium of a pre-existing mechanics. It should be said however that there is a certain arbitrariness in this dichotomy, because any reductionistic attempt can be reformulated as a foundational one, as will be clear in the following.

Attempts were made in the first direction by Vincenzo Riccati and Vincenzo Angiulli, Johann Bernoulli, Lazare Carnot and Lagrange himself. The first two thought they could demonstrate the law of virtual work with metaphysical considerations, using a reference mechanics of Leibnizian type but without a pre-existing criterion of equilibrium. Carnot tried to reach a VWL starting from the law of impact using a mechanics of reference in principle without force; Lagrange made use of the law of the pulley. Attempts in the second direction were made by French scientists of the École polytechnique. Summing up and using the categories of Lagrange’s mechanics, they assumed as reference mechanics those derived from the law of the lever and the rule of the parallelogram. The demonstration of Poinsot was the one that most influenced subsequent treatises of statics.

In the recent scientific literature, the problem of the logic status of VWL is only addressed in the manuals of statics, where the author gives his idea in a few pages on the subject, usually referring to a limited number of ‘basic versions’ [283]. Given the predominantly teaching character of the manuals, problematic aspects tend to be hidden to provide greater certainty. To my knowledge, there are no recent theoretical

[2] pp. 13–14.

works on basic aspects of VWLs, as there are no recent theoretical works on the foundation of the mechanics of a material point, even if the argument is far from exhausted. It seems further that there are only a few recent studies on the history of VWL [197], and that it is treated marginally in the numerous monographs on Lagrange, Laplace, etc.

In this chapter I will try to highlight the logical status of VWLs focusing mainly on the reductionist approach that is more widespread. Just because there is no generally accepted formulation of classical mechanics I will not consider the situation in its generality and assume only what is more consolidated, in particular I will assume a mechanics of material points and forces applied to them – i.e. corpuscular mechanics. The wording of the VWL will emerge in a natural way along the line of least resistance, avoiding inessential complications. I will simplify the constraint conditions, limiting myself to dealing with holonomic, bilateral and independent-of-time constraints, that can be represented mathematically by an algebraic equation only of the position variables, because I think they equally capture the essence of the problem and an extension to more general constraints is possible involving only technical complications. The reductionist approach to VWLs examined, assuming an already given mechanics, presupposes the concept of force. However, it is possible to tackle the problem from a different point of view, in which it is not necessary to posit the concept of force, giving as a primitive the concept of work. This view will be discussed briefly at the end of the chapter. According to this approach also, the VWL will be stated as a principle or deduced from the most fundamental laws, always related to the concept of work, which will now be virtual in a different way.

2.1 The theorem of virtual work

A constrained system S of a finite number n of material points is a system the configuration of which is defined by a number m of degrees of freedom less than the $3n$ that would be needed to describe the configuration of the system as supposedly free. I will indicate with M the space, or better the *manifold*, of the possible configurations of dimension m for S and with N the space of configurations of dimension $3n$ in the absence of constraints. Each space of configuration M and N is associated with a space of tangent vectors indicated below with M^T and with N^T respectively. For example, for a material point P constrained to move on a surface, N has dimension 3 and M has dimension 2. The vector space N^T is the space associated with the ordinary three-dimensional vector space; the space M^T is the set of vectors that lie on the tangent plane to the surface in the position occupied by P. In the case of two material points, constrained to keep a constant distance, the space of all configurations N has dimension 6, corresponding to the 6 degrees of freedom of two free material points in three dimensional space. The space M of compatible configurations has instead dimension 5, because the degrees of freedom are reduced by a constraint equation which expresses the invariance of the distance between the two points. The vector space N^T consists of pairs of vectors representing the displacements of the two material points which can be any, and the space M^T is represented by the pairs of vectors

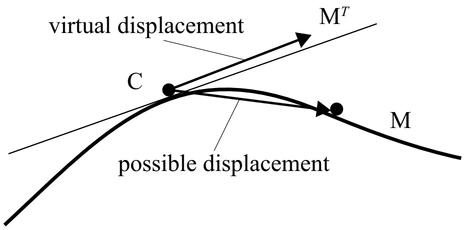

Fig. 2.1. Tangent manifold

that have the same component in the line joining the two material points. The space M^T of vectors u tangent to M is called the *tangent manifold* of M and vectors u are called *virtual displacements*. It is clear from the definition that the virtual displacements in general are not possible displacements, which are motions taking place on M. The difference between possible and virtual displacements is shown in Fig. 2.1. The virtual displacements coincide with the possible motion only for infinitesimal values. If it is considered that the virtual displacements occur in the direction tangent to the constraints and that the possible velocities are tangent to the constraints, it is instead possible to identify the virtual displacements with possible velocities, considering time as an arbitrary parameter.

There are essentially two ways to study the equilibrium of a constrained system. In the first way it is assumed that there are known external forces, named *active forces*, and forces due to the constraints, named *reactive forces* or constraint reactions, the presence of which should be inferred indirectly from the empirical evidence that motions of the material points of a constrained systems are different from those registered without constraints. The value of constraint reactions is not given, depending on the geometry of constraints and the active forces. In the second mode there are only active forces while the constraints are characterized exclusively by their geometry; in this paragraph I will examine the first situation.

On the system S of material points there are active forces f_i, with a given law of variation in time and space and reactive forces r_i, associated to the constraints, a priori unknowns. Collecting the active forces in the vector f and the reactive forces in the vector r, assume the following principle of equilibrium:

> P_1. A system of material points constrained to a manifold M starting with zero velocity, is in equilibrium in a given configuration C if and only if the following relation is satisfied at any time:
>
> $$f + r = 0. \tag{2.1}$$

Notice that the sufficient part of the principle (i.e. if $f + r = 0$ then there is equilibrium) calls for this other principle:

> P_1^*. If constraints can furnish reactive forces r such that $f + r = 0$ then they actually furnish them.

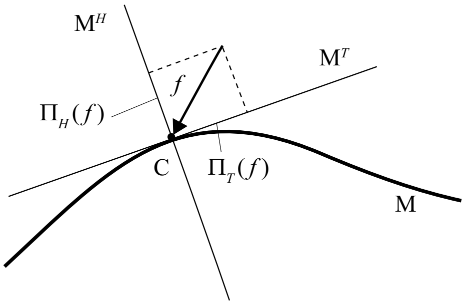

Fig. 2.2. Orthogonal projections

In the following, in all considerations relating to equilibrium, I will assume implicitly a certain configuration C and any instant of time while the initial condition is rest.

Denote by Π_T the projection operator from N^T to M^T, and by Π_H the projection operator from N^T to M^H, the complementary vector space of N^T orthogonal to M^T. Then equation (2.1) is equivalent to the two relations:

$$\begin{aligned} \Pi_T(f+r) &= \Pi_T(f) + \Pi_T(r) = 0 \\ \Pi_H(f+r) &= \Pi_H(f) + \Pi_H(r) = 0. \end{aligned} \tag{2.2}$$

Fig. 2.2 clarifies the meaning of (2.2) on a two-dimensional space. The space of admissible configurations is defined by the curve M, the tangent space M^T is the line tangent to M in C – the position occupied by the material point P – the orthogonal space is the line M^H orthogonal in C to M^T.

Define now the *virtual work* of forces acting on S as the linear form on N^T: $L(u) = (f+r) \cdot u$, where u is a vector of N^T and dot denotes the inner, or scalar, product. Then consider the two other linear forms $L_a(u) = f \cdot u$ and $L_r(u) = r \cdot u$, respectively called virtual work of active forces and reactive forces. Note that the virtual work coincides with the classical definition of work but it refers to a virtual displacement and not to a possible displacement. If the virtual displacements are identified with velocities, then the virtual work has the mechanical significance of power. The following theorem of virtual work can easily be proved:

> T_1. A system of material points on a manifold M is equilibrated if and only if $L_f(u) + L_r(u) = 0$ for any u in N^T.

Indeed $L_f(u) + L_r(u) = (f+r) \cdot u = 0 \;\; \forall u \in N^T \leftrightarrow f + r = 0$, for the same definition of scalar product.

To check the balance, with theorem T_1, it is necessary to specify the manner in which the reactive forces vary on the manifold M. A traditional way to characterize the reactive forces is to introduce the concept of smooth constraint, which can be expressed as:

> D_1*. A system of constraints associated to a manifold M and a system of material points S is smooth if and only if it is able to furnish reactive forces r such that $\Pi_T(r) = 0$.

That is in a system of smooth constraints the reactive forces belong to the space M^H orthogonal to the tangent space M^T in C. This mean that if $\|\Pi_T(f)\| > 0$, i.e. if there is at least a force f_i that has a non-zero component in the direction of the displacement u_i allowed by constraints, the equilibrium is not possible and the system moves, however small is the force f_i. This corresponds to the intuitive concept of smooth constraints as constraints without friction.

The characterization of smooth constraint can also be given, equivalently, referring to the linear form $L_r(u)$, reaching the definition:

D_1. A system of constraints associated to a manifold M and a system of material points S is smooth if and only if $L_r(u) = 0$ for any u in M^T.

Or, alternatively, in a less formal way, using the definitions of virtual displacement and work:

D_1. A system of constraints associated to a manifold M and a system of material points S is smooth if and only if the virtual work of reactive forces is zero for any virtual displacement.

Usually the characterization of smooth constraints assumes only the condition that r belongs to M^H. But it is equally important to stress that constraints are able to exercise all the forces belonging to M^H regardless of their intensity. So if constraints are smooth their reactions could be any values in a known direction, and it is possible to apply the criteria of balance P_1 or T_1, to state the two theorems:

T_2^*. If the constraints are smooth, a system of material points on the manifold M is equilibrated if and only if $\Pi_T(f) = 0$.
T_2. If the constraints are smooth, a system of material points on the manifold M is equilibrated if and only if $L_f(u) = 0$ for any u in M^T.

Proof of T_2^* is simple and is implicitly contained in equations (2.2). Necessary part: if a system of material points is in equilibrium for P_1 it is $f + r = 0$, then equations (2.2) hold, and from the first of them, because constraints are smooth and $\Pi_T(r) = 0$, it is $\Pi_T(f) = 0$. Sufficient part: assume $\Pi_T(f) = 0$, because for smooth constraints $\Pi_T(r) = 0$, the first relation of (2.2) is satisfied. The second relation $\Pi_H(f) + \Pi_H(r) = \Pi_H(f) + r = 0$, is also satisfied because the constraints (smooth), by definition, can provide all the reactions orthogonal to M^T, and therefore also $r = -\Pi_H(f)$. It follows that $f + r = 0$, and then the system of material points is in equilibrium. The demonstration of T_2 immediately follows from T_2^* for the properties of scalar product.

It is worth noting that in the case of constraints that are not smooth, to check the equilibrium may not be easy. As an example consider the material point of Fig. 2.3 in which there is also a tangential component of the constraint reaction, due to friction. If the point is in equilibrium it is certainly $f = -r$, but for an arbitrary value of f it is not said that there will be equilibrium because for example the friction is not enough and the constraint is not able to provide $r = -f$.

T_2 is a theorem of virtual work as is T_1; although commonly only T_2 is called *theorem of virtual work*. It would thus appear to have solved the problem of the logic

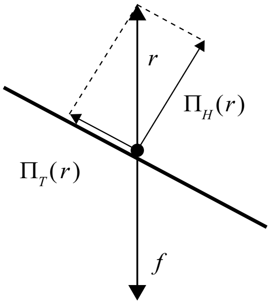

Fig. 2.3. Not smooth constraint

status of the law of virtual work, at least as formulated above: if properly formulated, it is a theorem of statics. Unfortunately, such a conviction is no longer anything but an illusion, disguised in the words with which the concept of smooth constraints has been introduced. In fact, it is given only a definition but it does not provide any 'decisions' criterion and the definition leads to circularity: if the constraint is smooth, reactive forces are orthogonal to virtual motion and if the reactive forces are orthogonal to virtual motion then the constraint is smooth.

To justify the usefulness of the theorem of virtual work, and then the opportunity of referring to T_2 as a VWL, an operating criterion is necessary to determine in advance whether a constraint is smooth or not, and this criterion cannot exist because the constraints are usually defined analytically only by the variety M and are not observable, i.e. they are not entities on which to have a priori reasoning. The only way to use T_2 (and T_2^*) it seems is to assume the following principle:

P_2. All constraints are smooth.

Then from P_2, by applying *modus ponens* to T_2^* and T_2, two theorems are obtained:

T_3^*. A system of material points on the manifold M is equilibrated if and only if $\Pi_T(f) = 0$.

T_3. A system of material points on the manifold M is equilibrated if and only if $L_f(u) = 0$ for any u in M^T.

Today theorem T_3 is usually called *principle of virtual work*; for historical reasons even here it is a theorem. It derives from a principle of the mechanics of material points (P_1) and a principle (P_2) that seems external to it. Given the critical role of P_2 in the proof of T_3 it itself is often called the principle of virtual work. In the following I will not accept this use and with the term virtual work principle I always refer to T_3.

T_3 may be a theorem of the reference mechanics only if P_2 holds good. It is then clear that the problem of provability of the virtual work principle is closely related to the problem of the characterization of constraints and, ultimately, of the reference mechanics, so that it be complete. If in the reference mechanics there are no assumptions about the constraints it does not make sense to think seriously about the provability of the virtual work principle.

Is it possible to say anything more about the constraints within a reference particle mechanics in which ‘solids’ are assumed to consist of material points which act as centres of forces that underlie the cohesion? Rather than the locus of points expressed by an algebraic equation a constraint can be associated, and it normally is, with a body sufficiently ‘hard’ to be considered impenetrable. When a particle approaches the body that acts as a constraint, forces awake – the reactive forces – which are opposed to opening up the parts of that body. Knowing the laws of forces as functions of distance of the centres, the laws of interaction between the body and the particle could be determined, at least in principle. In practice this is not possible and recourse to an approximate description is necessary with an empirical character, in the broadest sense, which will provide the necessary characterization of constraints.

In this way there would be no problem to determine whether a particular constraint system is smooth or not on the basis of its constitutive relationships and definitions D_1 and D_1^* and the theorem of virtual work T_2 then would make sense because there is an operational criterion to be applied case by case to decide on the basis of empirical observations if T_2 can be applied or not. Note however that in mechanics, in fact, one tends to apply the principle of virtual work T_3; generally, the assumption of smooth constraints is not object to scrutiny because it is not practically possible to do so.

Assuming that the constraints are formed of bodies, in the past it was thought, and sometimes it is still thought, to prove the principle P_2 and the theorem T_3 showing first a seemingly weaker assumption, namely that:

> P_3. All the surfaces of the material bodies are smooth constraints for material points.

It is clear that this principle expresses ideals; in practice constraints are never smooth and there are horizontal forces, or friction. Ignoring this fact and accepting the ideal nature of P_3, is it possible to accept it? It seems doubtful that criteria of symmetry and sufficient reason – in the sense that the reaction forces must be ‘always’ orthogonal because there is no reason that they are not – can be applied in the particle model in which the very concept of the surface of a body presents difficulties.

But even by accepting P_3, P_2 cannot be proved without any other assumption. In fact, P_3 does not say anything about the internal constraints between material points, where the attribute smooth appears unintuitive. So, if the reference mechanics is not changed, even assuming P_3, P_2 is not certified and therefore the virtual work principle is not a theorem.

Before concluding this section I would like to briefly refer to the extension of VWLs to dynamics, extension that was made for the first time by Lagrange in 1763 (see Chapter 10) [142]. Though VWLs can be extended quite easily to dynamics, of course they will no longer provide a criterion for equilibrium, but a criterion of balance that leads to the equations of motion. Including among forces also the forces of inertia equal to $-ma$, the theorem takes the form T_4:

T_4. For smooth constraints a motion of a system of material points moving on the manifold M is such that the virtual work of all forces, the inertial ones included, is zero for any virtual displacement.

The proof is immediate, because the equation of motion can be written as $-ma + f + r = 0$ and by assuming $I = -ma$ in the form $(I + f) + r = 0$, the same assumed in the proof of theorem T_2.

2.1.1 Proofs of the virtual work theorems in the literature

In the technical and teaching literature the provability of the virtual work theorems, T_1, T_2 or T_3 is addressed some ways differently than the one reported above. The difference in presentation depends on the audience to which it refers. There are the 'proofs' described in treatises of physics and mechanics, those of specific texts of statics and those of texts of continuum mechanics. For the last area, the virtual work principle is usually presented for systems that are either unconstrained or subject to constraints that require simply the vanishing of the displacements of certain points, so it becomes a theorem which can be easily derived from the principles of continuum mechanics, generally described by partial differential equations. More difficult and interesting is the approach in the other two types of texts.

2.1.1.1 Physics and rational mechanics treatises

As regards the presentation of the classic texts on mechanics the work by Capriglione and Drago [283, 301] which I sum up briefly, seems conclusive. The goal is usually to demonstrate the virtual work principle, the hypothesis of smooth constraints is implicit and there is always the consciousness of the author of the manual that he is proving the theorem of virtual work in the form T_2 but not the virtual work principle T_3. It can be said that although there is no complete agreement on how to define the virtual displacement, in most cases the infinitesimal displacement du instead of velocity are adopted, but the 'degree of virtuality' of this displacement is not always clear. This problem did not appear at the beginning of the present chapter where it was assumed that the motions have only a virtual geometric characterization, in which time does not intervene. The concept of virtual displacement has, however, developed historically with reference to a magnitude that evolves over time; the virtual velocities are obtained by the derivatives of displacements with respect to time. Traces of this historical development have remained in the demonstrations in physics textbooks. Sometimes there is distinction between the time with which the forces change – referred to as the 'real time' – and the time with which the motion varies – referred to as the 'virtual time' – flowing independently of each other. In this case it may be that the real time is frozen and only the virtual time flows; virtual displacements have in this case only a purely geometric characterization, consistent with the presentation in § 2.1.

The proof of the necessary part of the virtual work principle, that is if a system is in equilibrium then the virtual work of active forces is zero, takes place essentially as presented in the previous pages where the criterion of equilibrium was provided by the annulment of the forces acting on individual material points. If the system is in equilibrium, then the balance of forces $f+r=0$ subsists. Multiplying both sides by the virtual motion du, it is:

$$L=(f+r)\cdot du=0. \tag{2.3}$$

If the constraints are smooth $r\cdot du=0$ = 0, so (2.3) provides the necessary part of the virtual work principle, $L=L_f=f\cdot du=0$.

The proof of the sufficient part, i.e. the virtual work of active forces is always zero, then the system is in equilibrium, is usually treated in a manner substantially different from what has been done in previous paragraphs:

> The proof is by reduction to the absurd. It is assumed that despite being valid (*) [$L_f=0$], the system is put in motion, namely that at least one of its points, say the i-th, is affected, in the time dt subsequent to t, by a displacement dr_i, compatible with constraints. Since the material point under consideration starts from rest, it is necessarily: $F_i dr_i>0$, then the sum of all partial work relating to other parts of the system that actually moves, it is also:
>
> $$\sum F_i dr_i>0, \tag{1}$$
>
> since the sum is made up entirely of non-negative terms and at least one of them, by assumption, is not null.
> But $F_i=F_i^{(a)}+R_i$, for which we rewrite (1) as:
>
> $$\sum(F_i^{(a)}+R_i)\cdot dr_i>0. \tag{2}$$
>
> At this point one makes the assumption of smooth constraints and the absurd is obtained $\sum F_i^{(a)}\cdot dr_i>0$ [$L_f>0$], because against the hypothesis [283].[3] (A.2.2)

The demonstration is taking place assuming that true motions exist, then considering these as virtual motions and assuming that the constraints are smooth. Rather than to show that $L_f=0$ for all virtual displacement is equivalent to $f+r=0$, and then an existing equilibrium criterion is fulfilled, it is shown that to admit the motion is in contradiction with $L_f=0$ for all virtual displacements, using an argument of dynamic type.

The asymmetry between the demonstration of the necessary and sufficient condition is not convincing for me. Besides, the use of virtual displacements taking place in real time does not permit the direct extension of the proof to the case of time-dependent constraints and the case of an impulsive force. This form of proof of the sufficient part of the virtual work principle was probably introduced for the first time by Poisson in his *Traité de mécanique* of 1833 [200]. The central point of the proof lies in taking the dynamic assumption that the resultant forces F_i and displacements dr_i, which are generated by the absurd, necessarily have the same sense, and this assumption is not at all obvious, as will be explained further in Chapter 16.

2.1.1.2 Statics handbooks

Generally theorem T_1 or at most the necessary part of T_2, is proved, which is sufficient for applications. Instead of a system of particles, reference is made to a system

[3] pp. 331–348.

of rigid bodies connected to each other and with the outside by a system of rigid hinged rods, thus tacitly admitting that any kind of constraint can be reproduced by an appropriate system of rods. Normally the following additional assumptions are made:

1) constraints (connecting rods) exert forces R_h (reactive forces) that have the same ontological status of the active forces F_k;
2) the direction of the reactive forces R_h is that of the rods;
3) for each rigid body the equilibrium is defined by the satisfaction of the cardinal equations of statics between R_h and F_k;
4) an infinitesimal displacement field is assumed, i.e. virtual displacements are coincident with virtual velocities (unless an inessential constant);
5) all forces and displacements are independent of time.

Assumption 2 is a principle analogous to P_2 because the rod imposes the point in touch with a body to move on a sphere and then the reactive force, being collinear to the rod, is orthogonal to the surface.

The proof below differs a little from those normally presented in statics handbooks [285], because it avoids any recourse to matrix calculus which, making the proof automatic, the proof hides the nature of assumptions, implicit and explicit.

As first, at least in many handbooks, it is shown that the work of a system of forces applied to a rigid body can be estimated from the resultants F and static moment M_O – about a point O fixed – of forces, in the form:

$$L = F u_O + M_O \theta, \tag{2.4}$$

where u_O is the virtual displacement of a reference point O and θ the rigid body rotation, according to Fig. 2.4.

Having proved (2.4), indicating respectively with F^a and M_O^a the resultant of forces and moments of active forces and with F^r and M_O^r the corresponding quantities of reactive forces, it is easily shown that there is equilibrium for the body if and only if the relation holds true:

$$L = (F^a + F^r) u_O + (M_O^a + M_O^r)\theta = 0, \tag{2.5}$$

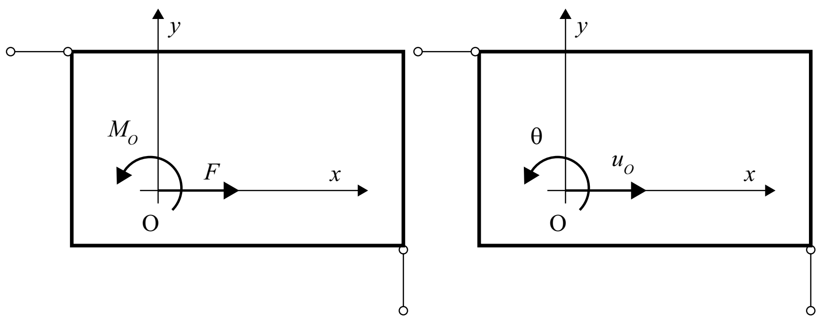

Fig. 2.4. Force and virtual displacement for a rigid body

for any virtual displacement u_O, θ, i.e. that a generalization of theorem T_1 is true, as can be proved into two steps:

a) the condition $L = 0$ is necessary for the equilibrium. Indeed if there is equilibrium the cardinal equations of statics are satisfied:

$$\begin{aligned} F^a + F^r &= 0 \\ M_O^a + M_O^r &= 0 \end{aligned} \tag{2.6}$$

and then $L = 0$;
b) the condition $L = 0$ is sufficient for the equilibrium. Indeed if $L = 0$ for any values of u_O, θ, from (2.5) it is easy to see that the cardinal equation of statics (2.6) are satisfied.

In the case of smooth constraints and inextensible rods, the virtual work of the reactive forces is zero and from (2.5) and (2.6), verified at equilibrium, one can derive the necessary part of the theorem T_2:

$$L = F^a u_O + M_O^a \theta = 0. \tag{2.7}$$

Usually nothing is said on the demonstration of the sufficient part of the theorem T_2, which is a bit more complicated.

2.1.1.3 Poinsot's proof

The examination of Poinsot's proof, which will be discussed in more detail in Chapter 14, has presently a considerable interest because it influenced much of the demonstrations of statics handbooks and is considered as the best one ever given. The reasons for this success are essentially two: (a) Poinsot considers a reference mechanics where the equilibrium is determined by the balance of forces which can be expressed by simple analytical relations – equations of equilibrium of a particle – and (b), primarily, he uses virtual velocities, obtained considering a non-physical time, instead of virtual infinitesimal displacements.

Poinsot takes for granted the assumption P_3, or more precisely its modified version, that does not require the concept of constraint reaction, whereby a particle is in equilibrium on a surface if and only if the applied force is perpendicular to it. In the words of Poinsot:

> Indeed, it is shown that if a point has no freedom in space other than to move on a fixed surface or line, there may not be equilibrium unless the resultant of forces which press it is perpendicular to the surface or the curved line [197].[4] (A.2.3)

In fact he does not prove what he says and does not even have the opportunity to do so because his constraints have the ontological status of relations between the positions of points and are not bodies. Poinsot is not the only one to think it logically necessary that a constraint cannot resist tangential forces. Laplace is also convinced of this:

[4] p. 467.

> The force of pressure of a point on a surface perpendicular to it, could be divided into two, one perpendicular to the surface, which would be destroyed by it, the other parallel to the surface and under which the point would have no action on this surface, which is against the supposition [156].[5] (A.2.4)

The reasoning is not conclusive, in fact it reduces to the trivial tautology that the constraint is not acting in a tangential direction because it does not act in the tangent direction. Lagrange also expressed similar ideas:

> Now if one ignores the force *P*, and assuming that the body is forced to move on this surface, it is clear that the action or rather the resistance that the surface opposes to the body cannot act but in a direction perpendicular to the surface [149]. [6] (A.2.5)

But he seems to realize the problematic nature of the concept of smooth constraint because, often, in his writings he associates the constraints, expressed by mathematical relations, to constraints made of inextensible wires deprived of bending stiffness. In this case the orthogonality of the reaction to the surface – for example the spherical surface described by a material point with a wire connected to a fixed point – is more convincing, even if this evidence has in fact an empirical rather than logic basis, making reference to our everyday experiences.

In addition to P_3, Poinsot considers other principles. The first principle is that of *solidification*, for which if one adds constraints – both internal and external – to a system of bodies in equilibrium, the equilibrium is not altered. The second principle, presented by Poinsot as the fundamental property of the equilibrium, states that necessary condition for the equilibrium of an isolated system of particles is that all the forces applied at various points can be reduced to any number of pairs of equal and opposite forces. The third principle is required by the second, even if not explicitly, and concerns the possibility to decompose a force into other forces using the rule of the parallelogram. A fourth principle concerns the mechanical superposition for constraints, for which if on a system of points there act more constraints, they are able to absorb the sum of the forces that each constraint is able to absorb separately. Based on these principles, he quite convincingly proves the principle P_2, with reference to the definition D_1^*; more precisely he proves that if $L = 0$, $M = 0$, etc. are the relations among the coordinates x, y, z, x', y', z', etc., which define constraints, the reactive forces are orthogonal to the resulting constraints. The demonstration of the virtual work principle follows the same reasoning of the first part of § 2.1 and in my opinion is free from any criticism.

2.2 The principle of virtual work

The term *principle* has a meaning not entirely unique, as is the case with all important concepts. It is the foundation of a science, which in turn may allow more than one set of principles. Among the principles, even at the time of Lagrange, there

[5] vol. 1, p. 9.

[6] vol. 9, p. 378.

were axioms, principles and theorems otherwise proved. Today this distinction no longer holds and there is the tendency to group all the principles under the term *axioms*. Compared to the views of the XIX century, essentially Aristotelian, now the premises, i.e. the *explicans* of the theories, are neither required to be certain nor the premises of the conclusions, i.e. the *explicandum*, to be better known. The first requirement is ignored because it would exclude almost all scientific laws, for which instead of a certain knowledge plausible conjectures are considered. The second requirement is usually ignored, and there are theories – such as atomic and quantum, for example – that explain relatively well-known phenomena with beings of indecipherable nature. Moreover, the boundary between what seems known and what does not, is largely the result of metaphysical and epistemological conceptions of the times and does not reside only in the object. With the use, things that were not known or obvious, become of public domain, an example of this are the concepts of force, atom, energy that sparked diffuse controversy when they were introduced.

According to the modern epistemology therefore the principle of virtual work may also be accepted as a principle of mechanics if it proves to have sufficient logic strength to describe all the mechanics – of course combined with other axioms – and to produce results in agreement with the experimental evidence. In the following I will try to analyze in more detail the consequences of taking a VWL as a principle without any reference to another mechanical theory.

There can be considered essentially three formulations, which gradually move away from what was presented as a theorem in the previous pages. It is posssible:

a) to assume forces as the primitive quantities and virtual displacements that take place in a virtual time;
b) to assume forces as the primitive quantities and virtual displacements that take place in the real time (in the same time with which the forces vary);
c) to assume work as the primitive quantity that takes place in real time.

In the following I will refer only to cases a) and c), being easy to extrapolate to case b) the considerations valid for the first two. With the usual ambiguity, which that should not bother us, the virtual displacements are treated as virtual velocities and virtual work as virtual power.

2.2.1 Force as a primitive concept

2.2.1.1 Equilibrium case

Assuming some concept of force, even a pre-Newtonian one, that in principle can always be replaced by a weight attached to a rope. I introduce only the active forces, while the reactive forces do not appear explicitly, in the limit the concept can also be missed, which avoids many problems of both logical and ontological type. In the case of a system of n material points, with the symbols used in the preceding paragraphs, if f is the vector of the active forces and u the vector of the virtual velocities/virtual displacement – the virtual work is defined as the product $f \cdot u$, and the principle of virtual work could be enunciated in the form of T_3:

PP_1. A system of particles constrained to move on a manifold M is in equilibrium if and only if the virtual work of the active forces is zero for any virtual displacement.

Note however that now, because there is no reference mechanics and a priori criterion of equilibrium, *equilibrium* is not intended as a balance of forces, but simply as rest. The principle can also be seen with Poincaré, as a methodological principle, a stipulation. If the applications to an empirical case do not work, it is always possible to say: it is because there are hidden forces, for example frictions.

One might ask whether a mechanical theory based on PP_1 is acceptable as follows: does it provide satisfactory results from an empirical point of view? Has it something different from Newtonian mechanics? The first thing that catches the eye is the extraordinary simplicity of the principle in the case of constrained systems. The idea of constraint reaction that creates difficulties should not be formulated. All breaks down in the examination of only the active forces. All the rules of simple machines and the rule of the parallelogram, which can be used for alternative formulations of mechanics, become simple theorems.

The necessary part of the principle is falsified by every experience, however. That is, if a system is in equilibrium under a system of active forces f, it is not true that the virtual work is zero for every possible virtual displacement; it could be both positive and negative. An example of this is obtained by considering the equilibrium of a heavy object on an inclined plane. It is found empirically that, for a very rough surface there is equilibrium for a material point even when the plane is tilted several degrees, but the weight force can make a positive virtual work – for downward virtual displacements – and a negative virtual work – for upward virtual displacements. The sufficient part of the virtual work principle PP_1, i.e. if the virtual work is zero for each value of virtual velocities, then the system of particles is in equilibrium, seems instead to be always empirically verified. This is somehow a consequence of the principle P_1^*, in the sense that if the reactive forces r associated with smooth constraints are sufficient to maintain in equilibrium a system of points ($L = 0$), then the actual non-smooth constraints furnish effectively the reaction r and the system is in equilibrium.

To treat the case with friction it is necessary to reformulate the virtual work principle by involving the forces of friction, that is the forces in the direction of the virtual displacement:

PP_2. A system of particles constrained to move on a manifold M is in equilibrium if and only if the virtual work of the active and reactive forces is zero for any virtual displacement.

2.2.1.2 Motion case

The question of applicability of the virtual work principle to motion arose also when it was considered as a theorem of a mechanics of reference, but in that case, just because there is a mechanics of reference, one could think that motion would be otherwise faced within this mechanics. Adopting the virtual work principle as a proper principle one should instead investigate whether it is possible to study the motion of bodies and how. To use the virtual work principle in the case of motion it is necessary to operate in a mechanics where it is at least possible to define the mass and also to talk about inertial reference frames; in fact it seems necessary to assume most of the concepts of Newtonian mechanics. Supposing that these difficulties have been overcome, a possible statement of the virtual work principle could be the following:

> PP_3. The virtual work of the active and inertia forces for a system of material points constrained on a manifold M is always zero in any inertial reference system.

(Here the term inertia forces must be considered merely a nominal definition of $-ma$.) Statement PP_3 may seem asymmetrical with that represented by PP_1. In fact it can be divided into two parts, one necessary and the other sufficient. To calculate the virtual work in a factual situation, with a and f 'real', one gets $L = 0$; vice versa if one requires $L = 0$ for any virtual displacements, one obtains the real value of a.

The falsification of the virtual work principle in the presence of constraints that are not smooth was not taken very seriously by scientists from the XVI to the XIX century. The problem of the correspondence of scientific theories to physical reality is probably as old as science, but it was put into evidence by Galileo Galilei. Guidobaldo dal Monte's polemic on the law of isochronism of the pendulum proposed by Galileo is well known. Dal Monte, along with his contemporaries, argued that the law of isochronism was not verified in practice; Galilei argued instead that the ideal pendulum would obey the law. Then Dal Monte replied that physics must relate to the real world and not an imaginary ideal world, a 'paper world' [86].[7] Today the position of Galileo is generally accepted, but it is also clear that it is not possible to abstract from accidents the essence of phenomena, mainly because one cannot always tell in advance which attributes are accidental and which ones are substantial. If in many situations, the friction is presented as an accident, which masks the substantial reality of the problem, in other situations this is no longer true. And the justification that the cases in which the presence of friction are important only to technology and not to science is senseless.

Without entering the merits of theories for the study of the motion of constrained body systems, think of how strange would appear a world in which the virtual work principle holds true strictly; in this world without friction, life itself would be impossible. And though in some calculations friction can be neglected, in describing the substance of the world it must be considered. A theory that does not have the conceptual tools to address important issues must be considered unsatisfactory and its basic principles as incomplete. Then a mechanics with a rigid axiomatic structure with PP_1 incorporated, cannot be the 'mechanics'. The only solution to solve the aporia that appears – PP_1 should be taken as an axiom but it cannot be taken as an axiom – is to adopt a liberal epistemology, not rigidly axiomatic, which may allow acceptance of PP_1 in some cases and non-PP_1 in others, without being able to decide which option to choose. The choice whether to apply PP_1 or non-PP_1 is so delegated to 'moral' considerations, in charge of the i, though not explicitly verbally.

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[7] Preface.

2.2.2 *Work as a primitive concept*

The interpretation of work as a primitive concept is perhaps the most interesting interpretation of the virtual work principle that has been revived in recent years in the international literature [328]. Taking, perhaps without a clear understanding of the historical reference, considerations of the late XIX century, it involves a complete revision of classical mechanics where the concept of work, which is taken as a scalar quantity susceptible of measurement, is accepted as primitive while the force is simply a definition. So there is a reversal from what happens in Newtonian mechanics, in which it is the force that is primitive and the work is a defined magnitude.

This possibility is not completely counterintuitive. The idea of work and fatigue as physical magnitudes surely already appeared in Galileo [119] though it was only the energetic movement of the late XIX century that captured the attention of physicists. The mathematical formulation of this idea could take the following line: consider a system of material points S and a vector space V that contains the virtual motions (velocities or displacements) of S that are considered to be in real time. The imposition of an element of V to S gives the virtual work L, to be treated as a physical quantity that is in principle measurable. The assumption, empirical in nature, is that L is a linear form defined on V. The vector space F dual of V is the space of the forces. In other words, the components of the forces acting on a system of material points, whose motions are defined by the virtual components u_k with respect to a fixed coordinate system, are those numerical values f_k that determine the virtual work, according to the relation:

$$L = \sum f_k u_k.$$

This formulation is valid for both discrete and continuous systems. In the case of a discrete system, in which V is a finite dimensional space, the foregoing considerations are substantially equivalent to those developed for the first time in Lagrange's *Mécanique analytique*, when the generalized forces are introduced, with the important difference that there the generalized forces are defined in terms of other forces, regarded as primitive and known quantities.

2.2.2.1 Equilibrium case

In this new formulation of mechanics the virtual work principle can be expressed either directly, without reference to forces, or indirectly, considering the forces that now are defined quantities. Moreover it can be a principle or can be obtained from a more fundamental law and therefore to appear as a theorem. In the following I treat only the first point of view, postponing the second to Chapter 18, dedicated to energetism. Considering the work of friction, it is possible to formulate the principle:

> PP_4. The equilibrium is possible if and only if the virtual work of all forces applied to the system is never positive, whichever the virtual displacement assumed.

Here, as for PP_1, there is no reference to an a priori equilibrium criterion, and *equilibrium* is simply rest. In this formulation of the virtual work principle, the possibility of also considering the virtual work done by the reactive forces, i.e. the frictions, is not ruled out, even though in practical application it has to rely on the assumption of smooth constraints. The same difficulties in studying the problem of constrained bodies in the context of Newtonian mechanics are found but with an important difference: the virtual work is a primitive magnitude and therefore it seems more natural to characterize smooth and non-smooth constraints:

PP_5. The virtual work made by the reaction forces cannot be positive.

This characterization is implicit in the commonly accepted principle of the impossibility of perpetual motion.

Consider, for example, a heavy material point that is bilaterally constrained to move on a rough plane, slightly sloped, and which is in equilibrium. The work L_r made by the reactive forces (frictions) is always negative, the work L_a of the weight can be both positive and negative. For a virtual downward motion it is $L_r < 0$ and $L_a > 0$, so it is possible to have $L = L_r + L_a = 0$; for a virtual upward motion it is $L_r < 0$ and $L_a < 0$ so that $L = L_a + L_r < 0$.

2.2.2.2 Motion case

It is not entirely clear how to generalize principle PP_4 to dynamics. In [328] it is suggested to consider the forces of inertia as ordinary forces. This suggestion, however, is controversial because, if work is the primitive concept, the concept of force, albeit of inertia, should be eliminated. Probably the only satisfactory way, from a logical and epistemological point of view, is to define kinetic energy and to add kinetic power (the time derivative of kinetic energy) to static power. This is what Duhem did, as in discussed in Chapter 18, somehow solving also the difficulty of introducing the concept of mass. With this addition it is possible to enunciate the following principle (only valid for smooth constraints):

PP_6. In each moment and for any system of material points the virtual work, measured as the sum of that of all the efforts applied to the system and the kinetic power, is zero whatever is the considered virtual displacement.

3 Greek origins

Abstract. This chapter explains the meaning of the partition of Greek mechanics into Aristotelian and Archimedean. In the first part the Aristotelian mechanics is considered that identifies as a principle the following VWL based on virtual velocity: *The effectiveness of a weight on a scale or a lever is the greater the greater its virtual velocity*. In the central part the Archimedean mechanics is considered where there is no reference to any VWL. In the final part devoted to the late Hellenistic mechanics, the VWL of Hero of Alexandria is considered for which the possibility of raising a weight is determined by the ratio of its virtual displacement and that of the power. The law is presented not as a principle but as a simple corollary of equilibrium.

In ancient Greece, mechanics was the science that dealt primarily with the study of equipment or machines (in Greek *μηχανή*), to transport and lifting weights, also as a response to other technological problems of the times. The search for equilibrium was not of practical interest – excluding the case of weighing by means of a balance – and mechanics, at least at the beginning, did not take care of it. From this point of view mechanics was very different from modern statics which is instead seen as the science of equilibrium.

Pappus wrote that "the mechanician Heron and his followers distinguished between the *rational* part of mechanics (involving knowledge of geometry, arithmetic, astronomy, and physics) and its *manual* part (involving mastery of crafts such as bronze-working, building, carpentry, and painting)" [181].[1] In the following, mechanics is only used in the sense of rational mechanics with Pappus' meaning; as such its status was neither well-defined for its social appreciation nor for its epistemology.

Regarding the social appreciation there were contrasting valuations. Mechanics concerned problems of everyday life, connected to manual work and as such considered negligible by the intellectual aristocracy. But its applications gave rise to wide interest. To cite the famous attribution to Archytas of Tarentum of a dove "which flew according to the rules of mechanics. Obviously it was kept suspended

[1] p. 447.

Capecchi D.: History of Virtual Work Laws. A History of Mechanics Prospective.
DOI 10.1007/978-88-470-2001-6_3, © Springer-Verlag Italia 2012

by weights and filled with compressed air" [201].[2] Aristotle appreciated the activity of mechanicians, as is clear from the prologue to his *Mechanica problemata*. To the contrary Plutarch (46–127 AD) in his *Vitae parallelae*, wrote that Archimedes felt ashamed for his interest in mechanics and wanted to be remembered only for his mathematical works. Even though Plutarch attributes his own conception to Archimedes, his opinion indicates the low consideration mechanics held in some circles [192].[3]

As far as the epistemological status is concerned, mechanics was considered by Aristotle as a mixed science:

> These are not altogether identical with physical problems, nor are they entirely separate from them, but they have a share in both mathematical and physical speculations, for the method is demonstrated by mathematics, but the practical applications belong to physics [12]. [4]

This classification was used through all the Middle Ages. Archimedes probably did not share this opinion and considered mechanics as a branch of pure mathematics.

The mechanics of Greek, as has happened in many areas of Western knowledge, is the basis of modern conceptions. Available sources are not numerous, but they are important. The earliest references are to the pythagorean Archytas of Tarentum (c 428–350 BC). For sufficiently precise written documentation, however, reference to Aristotle (384–322 BC), Euclid (fl 365–300 BC), Archimedes (287–312 BC), Hero (I century AD) and Pappus of Alexandria (fl 320 AD) is needed.

In the following I first present the ideas of Aristotle, who is usually credited for a mechanics based on a law of virtual work. Then I introduce the principles of Archimedes's mechanics, alternative to the Aristotelian and where there is no use of any virtual work laws. Finally, a hint of the mature Hellenistic mechanics, that although influenced more by Archimedes is also influenced by Aristotle.

3.1 Different approaches to the law of the lever

3.1.1 Aristotelian mechanics

The principal Aristotelian treatises on mechanical arguments are the *Physica*, *De caelo* and the *Mechanica problemata*. They were largely studied and commented upon, both for their philological aspects and for their content. In the following I will give a very concise summary. Firstly I will consider the *Physica* and *De caelo* which describe motion according to nature (free motion) and motion against nature (forced motion) with both qualitative (causes) and quantitative (mathematical laws of motion) considerations. We could say that the context is quite general, it concerns all kinds of forces and bodies and can be defined as 'theoretical physics'. I shall then consider the *Mechanica problemata*. In this treatise, which can be defined as 'engineering based', the approach is less systematic than the *Physica* and *De caelo*.

[2] vol. I, p. 483.

[3] Marcellus.

[4] p. 331.

3.1.1.1 *Physica* and *De caelo*

The first Aristotelian thesis developed in the *Physica* and *De caelo* refers to the motion according to nature of a heavy body: it is downward along the line connecting the body with the centre of the world. The space traversed in the fall, in a given time, is directly proportional to the weight and inversely proportional to the resistance of the medium:

> Further, the truth of what we assert is plain from the following considerations. We see the same weight or body moving faster than another for two reasons, either because there is a difference in what it moves through, as between water, air, and earth, or because, other things being equal, the moving body differs from the other owing to excess of weight or of lightness [14].[5]
> A, then, will move through B in time Γ, and through Δ, which is thinner, in time E (if the length of B is equal to Δ), in proportion to the density of the hindering body. For let B be water and Δ air; then by so much as air is thinner and more incorporeal than water, A will move through Δ faster than through B. Let the speed have the same ratio to the speed, then, that air has to water [14].[6]
> A given weight moves a given distance in a given time; a weight which is as great and more moves the same distance in a less time, the times being in inverse proportion to the weights. For instance, if one weight is twice another, it will take half as long over a given movement [13].[7]

The Aristotelian law on motion according to nature is assumed by most scholars with the mathematical relation $v = p/r$, where v is the velocity, p the weight and r the resistance of the medium. There are however some objections to this view [287],[8] [349].[9] Perhaps the main objection is that of regarding velocity as a definite kinematic quantity, summarizing space and time with their ratio, which to a modern is just velocity. This assumption is clearly anachronistic. Not only because in Greek mathematics there was no sense in the ratio between two heterogeneous quantities, like space and time, but also because there was no use for the quantification of velocity, which was, in fact introduced only as an intuitive concept, something that allowed one to say something is greater or lesser [14][10] [287].[11]

To the community of mechanics scholars, the first known writings on the quantification of velocity, which were presented within the mathematics of proportions, and on the systematic use of this quantification is commonly considered that by Gerardus de Brussel, in the first half of 1200, moreover expressed in a form not completely explicit as ‘petitiones’ of his famous book *Liber de motu*, where it is said the velocity (motus) is measured by the space traversed in a given time [127]. After Gerardus de Brussel many medieval and all Renaissance scholars read Aristotle as most modern scholars do. It must however be said that Aristotle in some places con-

[5] IV, 8, 215a.
[6] IV, 8, 215b.
[7] I, 6, 274a.
[8] Chapter 7.
[9] Chapter 9.
[10] VI, 2; VII, 4.
[11] Chapter 3.

ceives of velocity as a well-defined kinematical quantity. This occurs, for example, in the preceding quotation of *Physica*, where velocity and resistance are considered as inversely proportional to each other, and in the following passage:

> For since the distinction of quicker and slower may apply to motions occupying any period of time and in an equal time the quicker passes over a greater length, it may happen that it will pass over a length twice, or one and a half times, as great as that passed over by the slower: *for their respective velocities may stand to one another in this proportion* [emphasis added] [14].[12]

where it is said quite clearly that velocity can be measured with space covered in a given time. But Aristotle does not develop this reasoning and any time he refers to mathematical laws he speaks about space and time separately and not about velocity.

The second Aristotelian thesis on motion, clearly stated in the *Physica*, concerns the motion against nature of a heavy body: it occurs along a straight line and the space covered, in a given time, is directly proportional to the 'force' applied to the body and inversely proportional to its weight:

> Then, A the movent have moved B a distance Γ in a time Δ, then in the same time the same force A will move 1/2B twice the distance Γ, and in 1/2 Δ it will move 1/2B the whole distance for Γ: thus the rules of proportion will be observed. Again if a given force move a given weight a certain distance in a certain time and half the distance in half the time, half the motive power will move half the weight the same distance in the same time. Let E represents half the motive power A and Z half the weight B: then the ratio between the motive power and the weight in the one case is similar and proportionate to the ratio in the other, so that each force will cause the same distance to be traversed in the same time [14].[13]

The difficulty for the modern reader in the interpretation of Aristotle's writing lies mainly in giving sense to 'force'. Aristotle, at the beginning of book VII of the *Physica* explains which are precisely the kinds of forces to consider, but this is not enough to achieve a complete understanding:

> The motion of things that are moved by something else must proceed in one of four ways: for there are four kinds of locomotion caused by something other than that which is in motion, viz. pulling, pushing, carrying, and twirling [14].[14]
> […]
> Thus pushing on is a form of pushing in which that which is causing motion away from itself follows up that which it pushes and continues to push it; pushing off occurs when the movent does not follow up the thing that it has moved: throwing when the movent causes a motion away from itself more violent than the natural locomotion of the thing moved, which continues its course so long as it is controlled by the motion imparted to it [14].[15]

Most scholars maintain that 'force' had the same meaning as it has today, though not in the Newtonian sense of cause of motion variation, but in the less demanding sense of muscular activity [287, 171, 305]. Expressed in modern terms, and synthesizing space and time into velocity, this position assumes the direct proportionality between

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12 VI, 2, 233b.

13 VII, 5, 249b.

14 VII, 5, 243a.

15 VII, 5, 243b.

force (f), velocity (v) and weight (p); with a formula $f = pv$. Other scholars consider the modern concept of work as being the closest to 'force' [370, 295, 136]. This interpretation is strongly advised in the case of a thrown object, where it is difficult to see a force in the preceding sense. Even the fact that a force must act together against a resistance seems to confirm this position. Finally, there are those who think that to interpret Aristotle's writings it is enough to make references to the common sense of an uneducated person. Some scholars indeed think that the learning process of a person resumes the historical process (the ontogenesis resumes the phylogenesis) and the scientific conceptions of classical Greece could be understood by assuming the identity of a youth who has not yet studied Newtonian mechanics [373, 359, 369]. This last position has the merit of averaging the two preceding, because sometimes it is more straightforward to translate 'force' with force, sometimes with work, and sometimes more with static moment.

Before taking a position it must be said that the precise differentiation between force and work will occur only in the XVIII century, and as late as the XIX century 'force' will ambiguously be used to mean both force and work [129]. Moreover, it must be noticed that the various interpretations cannot be decided upon empirically. The experimental context needed to verify the Aristotelian laws of motion is different from that foreseen by the modern paradigm, the Newtonian for example. With Newton one has to observe the motion of a material point in a void space under a force with assigned direction and intensity. With Aristotle one has to study the motion of an extended body, which moves against resistances of the external medium which tends to oppose the applied force and maintains the body in a status of constant velocity. For Aristotle it is implicit that a resistance opposes a force, otherwise there would be no motion, or an infinity velocity motion would occur, which is impossible.

The causes of resistance to a body's motion are not made explicit by Aristotle; in a passage of *De caelo*, he attributes them to weight:

> Again, a body which is in motion but has neither weight nor lightness, must be moved by constraint, and must continue its constrained movement infinitely. For there will be a force which moves it, and the smaller and lighter a body is the further will a given force move it. Now let A, the weightless body, be moved the distance CE, and B, which has weight, be moved in the same time the distance CD. Dividing the heavy body in the proportion CE : CD, we subtract from the heavy body a part which will in the same time move the distance CE, since the whole moved CD: for the relative speeds of the two bodies will be in inverse ratio to their respective sizes. Thus the weightless body will move the same distance as the heavy in the same time. But this is impossible. Hence, since the motion of the weightless body will cover a greater distance than any that is suggested, it will continue infinitely [13].[16]

In other passages resistances are attributed even to the medium [14][17] and one would not be mistaken by assuming friction too as a resistance.

When results furnished by Aristotelian laws of motion are compared with those of modern mechanics (Newtonian or Lagrangian), one sees that the Aristotelian laws are 'true' whatever the interpretation of 'force', when the parameter time is not considered. They are in general 'false' when this parameter is considered. For example,

[16] III, 2, 301b.

[17] IV, 8, 215b.

by interpreting 'force' as force one gets agreement only for equal intervals of time; by interpreting 'force' as work one gets agreement only if the parameter time is excluded completely.

In what follows, for both motions, according to nature and against nature (natural and violent motions), I will refrain as much as possible from adopting a preconceived position on the ontological status of the various mechanical concepts, because often there is no need to do this and different interpretations do not necessarily conflict. When I have to choose I will opt for the (pre-Newtonian) common sense.

3.1.1.2 *Mechanica problemata*

In the *Mechanica problemata* (known also as *Mechanical problems*, *Mechanica*) there is no explicit affirmation of the general theoretical principles contained in the *Physica* and *De caelo*, in particular no reference to the laws of natural and violent motion. Also for this reason and for its practical contents, the attribution of this treatise to Aristotle is still debated.[18] In the following I will not enter into the merit of this attribution and, for the sake of simplicity, I will talk about *Mechanica problemata* as an Aristotelian work, instead of, as frequently seen, a pseudo-Aristotelian one.

The writing is largely dedicated to the solution of problems, some mechanical in nature; they are referred to in Table 3.1 [296].[19] The object of the mechanical problems is mainly the study of the shifting of heavy bodies. Nowhere in the text do the concept and the word *equilibrium* (*ισορροπειν*) occur. The functioning of machines or devices, among them the wedge, pulley and winch, is reconnected to the lever. The validation of the law of the lever is suggested and may be the first in the history of mechanics. In the following I will comment on this validation.

At the beginning Aristotle refers all the mechanical effects to the properties of the circle:

> Remarkable things occur in accordance with nature, the cause of which is unknown, and others occur contrary to nature, which are produced by skill for the benefit of mankind. Among the problems included in this class are included those concerned with the lever. For it is strange that a great weight can be moved by a small force, and that, too, when a greater weight is involved. For the very same weight, which a man cannot move without a lever, he quickly moves by applying the weight of the lever. Now the original cause of all such phenomena is the the circle; and this is natural, for it is in no way strange that something remarkable should result from something more remarkable, and the most remarkable fact is the combination of opposites with each other [12].[20]

The cause of the farthest points of a circle moving more easily than the closest under a given force is identified by Aristotle in the fact that in circular motion the compo-

[18] During the Middle Ages and Renaissance the attribution of the *Mechanica problemata* to Aristotle was substantially undisputed. For the attributions in more recent periods see [15]. It is worth noticing that Fritz Kraft considers the *Mechanica problemata* to be an early work by Aristotle, when he had not yet fully developed his physical concepts [348, 378]. A recent paper by Winter considers Archytas of Tarentum as the author of *Mechanica problemata* [398].

[19] pp. 136–137.

[20] pp. 331–333.