560 LECTURE NOTES IN ECONOMICS AND MATHEMATICAL SYSTEMS

Arndt von Schemde

Index and Stability in Bimatrix Games

A Geometric-Combinatorial Approach

Springer

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Library of Congress Control Number: 2005929878

ISSN 0075-8442
ISBN-10 3-540-26366-7 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-26366-1 Springer Berlin Heidelberg New York

Springer is a part of Springer Science+Business Media

springeronline.com

Typesetting: Camera ready by author
Cover design: *Erich Kirchner*, Heidelberg

Printed on acid-free paper 42/3130Di 5 4 3 2 1 0

To my parents

Preface

This work originates from my PhD thesis at the London School of Economics and Political Science. I am indebted to Bernhard von Stengel for his excellent supervision. He introduced me to the questions addressed in this work. I am thankful for the many hours of discussions, and also for his guidance and patience in times when results seemed far away.

Also, I am grateful to Srihari Govindan and Robert Wilson for useful advice and encouragement.

Furthermore, I would like to thank the members of the Mathematics Department for their general support. In particular, I thank Jackie Everid, David Scott and Mark Baltovic for their assistance, as well as Nic Georgiou and Luis Cereceda for their help on the final draft.

There were many people who supported me personally. Foremost, I would like to thank my parents and Ane S. Flaatten for always being there for me. I am also beholden to Philipp Beckmann for his advice, and to Philip Hochstrate for his inspiration.

Finally, I would like to thank the London School of Economics and Political Science (LSE), the Department of Mathematics at LSE and the UK Engineering and Physical Sciences Research Council (EPSRC) for financial support.

Oslo, July 2005 *Arndt von Schemde*

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Introduction

Since Shapley (1974) introduced the index for equilibria, its importance in the context of game theory has been increasingly appreciated. For example, index theory can be a useful tool with regards to strategic characterisations of equilibria and equilibrium components. Demichelis and Ritzberger (2003) show that an equilibrium component can only be evolutionary stable if its index equals its Euler characteristic. At the same time, most of the existing literature on the index is technically demanding, and the amount of algebraic topology required is substantial. As a consequence, this literature is difficult to access for most economists and other applied game theorists.

The contribution of this thesis can be divided into two parts. The first part concerns methods and techniques. By introducing a new geometric-combinatorial construction for bimatrix games, this thesis gives a new, intuitive re-interpretation of the index. This re-interpretation is to a large extent self-contained and does not require a background in algebraic topology. The second part of this thesis concerns the relationship between the index and strategic properties. In this context, the thesis provides two new results, both of which are obtained by means of the new construction and are explained in further detail below. The first result shows that, in non-degenerate bimatrix games, the index can fully be described by a simple strategic property. It is shown that the index of an equilibrium is $+1$ if and only if one can add strategies with new payoffs to the game such that the equilibrium remains the unique equilibrium of the extended game. The second result shows that the index can be used to describe a stability property of equilibrium components.

For outside option components in bimatrix games, it is shown that such a component is hyperessential if and only if it has non-zero index.

The new geometric-combinatorial construction, which is referred to as the *dual construction*, can be described as follows. For an $m \times n$ bimatrix game, the construction translates the combinatorial structure of the best reply regions for both players into an $(m-1)$-simplex that is divided into simplices and labelled regions (see, for example, Figure 2.6 below). The simplices in the division account for the best reply structure of player II. The simplices themselves are divided into best reply regions for player I, accounting for the best reply structure of player I.

In this representation of bimatrix games, the Nash equilibria are represented by points that are completely labelled with all pure strategies of player I. Earlier constructions required the use of all pure strategies of both players as labels. The index is simply the local orientation of the labels around a completely labelled point (Figure 2.11). The Lemke-Howson algorithm, which builds the foundation for Shapley's original index definition, can be re-interpreted as a path-following algorithm in the new construction (Figure 2.8). Since the new construction is of dimension $m-1$, both the index and the Lemke-Howson algorithm can be visualised in dimension at most 3 for every $m \times n$ bimatrix game with $m \leq 4$.

But the construction does not merely yield an intuitive re-interpretation of the index and the Lemke-Howson algorithm. More significantly, it can disclose relationships between the index and strategic properties. In this context, this thesis provides, as mentioned, two new results.

As for the first result, it is shown that the index of an equilibrium is $+1$ if and only if it is the unique equilibrium of an extended game. The result proves a conjecture by Hofbauer (2000) in the context of equilibrium refinement. The proof is based on the idea that one can divide an $(m-1)$-simplex such that there exists only one completely labelled point which represents the index $+1$ equilibrium (Figure 4.7). Then such a division can be achieved as the dual construction of an extended game where strategies for player II are added (Figure 4.8).

The second result solves, for a special case, a problem that was open for some time. This problem addresses the question whether and how topological essentiality and game theoretic essentiality (Wu and Jiang (1962); Jiang (1963)) are related. Govindan and Wilson (1997b) argue that the resolution of this problem is highly relevant with respect to axiomatic studies: Imposing topological essentiality as an axiom in a decision-theoretic agenda is questionable if there is a gap between topological and strategic essentiality. Hauk and Hurkens (2002) construct a game with an outside option equilibrium component that has index zero but is essential. This demonstrates that topological essentiality is not equivalent to strategic essentiality. However, their example fails the requirement of hyperessentiality, i.e. the component is not essential in all equivalent games (Kohlberg and Mertens (1986)). The follow-up question is whether hyperessentiality is the game theoretic counterpart of topological essentiality. In this thesis, it is shown that this is the case for outside option equilibrium components in bimatrix games. That is, an outside option equilibrium component in a bimatrix game is hyperessential if and only if it has non-zero index. The proof is based on creating equivalent games by duplicating the outside option. An example presented in this thesis shows that one can create an outside option equilibrium component that has index zero and is essential in all equivalent games that do not contain duplicates of the outside option. However, it can be shown that the component fails the requirement of hyperessentiality if allowing duplicates of the outside option.

The proof of this result employs the combinatorial nature of the index for components of equilibria. In the framework of the dual construction, the index for components of equilibria is defined by a combinatorial division of a boundary into labelled best reply regions. This re-interpretation of the index for components is very similar to the index in the framework of the Index Lemma, a generalisation of Sperner's Lemma. For labellings as in the Index Lemma it is shown that, if the index of a boundary triangulation is zero, then there exists a labelled triangulation such that the triangulation does not contain a completely labelled simplex. The proof extends an index-zero boundary division of a polytope into labelled regions such that no point in the interior of the polytope is completely labelled. This extension is then translated into a triangulation (Figure 6.2). The proof for outside option components works similarly. Given an index-zero component, the dual of the component can be divided into labelled regions such that no point is completely labelled. It is then shown that such a division can be achieved as the dual construction of an equivalent game in which the outside option is duplicated and perturbed (Figure 6.10).

The concept of essentiality is strongly influenced by the theory of fixed points and essential fixed point components (Fort, 1950). In a parallel and independent work, Govindan and Wilson (2004) show that, for general N-player games and general equilibrium components, a component has non-zero index if and only if it is hyperessential. Their proof is based on a well-known result from fixed point theory that shows that a fixed point component is essential if and only if it has non-zero index (O'Neill, 1953). Their proof is technically very demanding. In contrast, the proof presented here for the special case provides a geometric intuition and does not require a knowledge of fixed point theory.

There is, however, a link between the combinatorial approach of this thesis and fixed point theory. This link is established via Sperner's Lemma (Sperner, 1928). The representation of bimatrix games in form of the dual construction reveals strong analogies with Sperner's Lemma. Sperner's Lemma is a classical result from combinatorial topology and is equivalent to Brouwer's fixed point theorem. Using the parallels of the dual construction with Sperner's Lemma it is shown that the existence of Nash equilibria in a non-degenerate bimatrix game is equivalent to Brouwer's fixed point theorem. On a similar topic, McLennan and Tourky (2004) derive Kakutani's fixed point theorem using the Lemke-Howson algorithm.

An additional result of this thesis, which does not involve the dual construction, is the construction of equilibrium components with arbitrary index. It is shown that for every integer q there exists a bimatrix game with an outside option equilibrium component that has index q. The construction is purely based on the properties of the index, and does not require knowledge of algebraic topology. This result originates from Govindan, von Schemde and von Stengel (2003).

The structure of this thesis is as follows. Chapter 1 introduces notations and conventions used throughout this work (Section 1.1). Sections 1.2 and 1.3 contain reviews of the Lemke-Howson algorithm and index theory. Section 1.4 shows how equilibrium components of arbitrary index can be constructed. Chapter 2 introduces the dual construction (Sections 2.1 and 2.2) and gives a re-interpretation of the index and the Lemke-Howson algorithm (Sections 2.3 and 2.4). Chapter 3 describes the parallels between the dual construction, Sperner's Lemma, and Brouwer's fixed point theorem. In Chapter 4, it is shown that the index for non-degenerate bimatrix games can be fully described by a strategic property. In Chapter 5, the dual construction is extended to outside option equilibrium components (Section 5.2). It also contains a review of the Index Lemma (Section 5.1). Finally, Chapter 6 investigates the relationship between the index and hyperessentiality. Section 6.1 considers index-zero labellings in the context of the Index Lemma. In Section 6.2, it is shown that an outside option equilibrium component is hyperessential if and only if it has non-zero index. A list of symbols is given at the end. Proofs and constructions are illustrated by figures throughout this work.

1

Equilibrium Components with Arbitrary Index

This chapter describes a method of constructing equilibrium components of arbitrary index by using outside options in bimatrix games. It is shown that for every integer q there exists a bimatrix game with an outside option equilibrium component that has index q. The construction is similar to the one used in Govindan, von Schemde and von Stengel (2003). That paper also shows that q-stable sets violate a symmetry property which the authors refer to as the *weak symmetry axiom*. The construction of equilibrium components of arbitrary index is the main result of this chapter.

The structure of this chapter is as follows. Section 1.1 introduces notational conventions and definitions that are used throughout this work. Section 1.2 gives a brief review of the classical Lemke-Howson algorithm that finds at least one equilibrium in a non-degenerate bimatrix game. Although the Lemke-Howson algorithm does not play a role in the construction of equilibrium components of arbitrary index, it can be used in the index theory for non-degenerate bimatrix games. Shapley (1974) shows that equilibria at the ends of a Lemke-Howson path have opposite indices. The Lemke-Howson algorithm also plays an important role in subsequent chapters when it is interpreted in a new geometric-combinatorial construction (see Chapters 2 and 3). Section 1.3 reviews the concept of index for Nash equilibria in both non-degenerate bimatrix games and general N-player games. Using basic properties of the index for components of Nash equilibria, Section 1.4 shows how equilibrium components of arbitrary index can be constructed as outside options in bimatrix games. It is shown that for every integer q there exists a bimatrix game with an equilibrium component that has index q (Proposition 1.6).

1.1 Preliminaries

The following notations and conventions are used throughout this work. The k-dimensional real space is denoted as $\mathbb{R}^k$, with vectors as column vectors. An $m \times n$ bimatrix game is represented by two $m \times n$ payoff matrices A and B, where the entries A_{ij} and B_{ij} denote the payoffs for player I and player II in the i-th row and j-th column of A and B. The set of pure strategies of player I is denoted by $I = \{1,\dots,m\}$, and the set of pure strategies of player II is represented by $N = \{1,\dots,n\}$. The rows of A and B are denoted a_i and b_i for $i \in I$, and the columns of A and B are denoted A_j and B_j for $j \in N$. The sets of mixed strategies for player I and player II are given by

$$X = \left\{x \in \mathbb{R}^m \mid \mathbf{1}_m^\top x = 1,\ x_i \geq 0\ \forall\, i \in I\right\},$$
$$Y = \left\{y \in \mathbb{R}^n \mid \mathbf{1}_n^\top y = 1,\ y_j \geq 0\ \forall\, j \in N\right\},$$

where $\mathbf{1}_k \in \mathbb{R}^k$ denotes the vector with entry 1 in every row. For easier distinction of the pure strategies, let $J = \{m+1,\dots,m+n\}$, following Shapley (1974). Any $j \in N$ can be identified with $m+j \in J$ and vice versa. A *label* is any element in $I \cup J$. For notational convenience, the label j is sometimes used to refer to the pure strategy $j-m$ of player II if there is no risk of confusion.

X is a standard $(m-1)$-simplex that is given by the convex hull of the unit vectors $e_i \in \mathbb{R}^m$, $i \in I$, and Y is a standard $(n-1)$-simplex given by the convex hull of the unit vectors $e_{j-m} \in \mathbb{R}^n$, $j \in J$. The terms "$(m-1)$" and "$(n-1)$" refer to the dimension of the simplex. In general, an $(m-1)$-simplex is the convex hull of m affinely independent points in some Euclidian space. These points are the *vertices* of the simplex, and the simplex is said to be *spanned* by its vertices.

An *affine combination* of points $z_1,\dots,z_m$ in an Euclidian space can be written as $\sum_{i=1}^m \lambda_i z_i$ with $\sum_{i=1}^m \lambda_i = 1$ and $\lambda_i \in \mathbb{R}$, $i = 1,\dots,m$. A *convex combination* is an affine combination with the restriction $\lambda_i \geq 0$, $i = 1,\dots,m$. A set of m points $z_1,\dots,z_m$ is *affinely independent* if none of these points is an affine combination of the others. This is equivalent to saying that $\sum_{i=1}^m \lambda_i z_i = 0$

and $\sum_{i=1}^{m} \lambda_i = 0$ imply that $\lambda_1 = \ldots = \lambda_m = 0$. A convex set has *dimension* d if it has $d+1$, but no more, affinely independent points. A *k-face* of an $(m-1)$-simplex is the k-simplex spanned by any subset of $k+1$ vertices. The standard $(m-1)$-simplex spanned by the unit vectors in $\mathbb{R}^m$ is denoted by Δ^{m-1}. So $X = \Delta^{m-1}$ and $Y = \Delta^{n-1}$.

For a mixed strategy $x \in X$, the support of x are the labels of those pure strategies that are played with positive probability in x. The support for $y \in Y$ is defined similarly. So

$$\mathrm{supp}(x) = \{i \in I \mid x_i > 0\}, \quad \mathrm{supp}(y) = \{j \in J \mid y_{j-m} > 0\}.$$

The strategy sets X and Y can be divided into best reply regions $X(j)$ and $Y(i)$. These are the regions in X where $j \in J$ is a best reply and the regions in Y where $i \in I$ is a best reply, so

$$X(j) = \left\{x \in X \mid B_j^\top x \geq B_k^\top x \; \forall\, k \in J\right\}, \quad Y(i) = \{y \in Y \mid a_i y \geq a_k y \; \forall\, k \in I\}.$$

The regions $X(j)$ and $Y(i)$ are (possibly empty) closed and convex regions that cover X and Y. For a point x in X the set $J(x)$ consists of the labels of those strategies of player II that are a best reply with respect to x. The set $I(y)$ is defined accordingly, so

$$J(x) = \{j \in J \mid x \in X(j)\}, \quad I(y) = \{i \in I \mid y \in Y(i)\}. \tag{1.1}$$

For $i \in I$, the set $X(i)$ denotes the $(m-2)$-face of X where the i-th coordinate equals zero. For $j \in J$, the set $Y(j)$ is defined as the $(n-2)$-face of Y where the $(j-m)$-th coordinate equals zero.

$$X(i) = \left\{(x_1, \ldots, x_m)^\top \in X \mid x_i = 0\right\}, \; Y(j) = \left\{(y_1, \ldots, y_n)^\top \in Y \mid y_{j-m} = 0\right\}.$$

Similar to (1.1), the sets $I(x)$ and $J(y)$ are defined as

$$I(x) = \{i \in I \mid x \in X(i)\}, \quad J(y) = \{j \in J \mid y \in Y(j)\}. \tag{1.2}$$

The labels $L(x)$ of a point $x \in X$ and the labels $L(y)$ of a point $y \in Y$ are defined as

$$L(x) = \{k \in I \cup J \mid k \in X(k)\}, \quad L(y) = \{k \in I \cup J \mid k \in Y(k)\}. \tag{1.3}$$