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Marco Lehmann-Waffenschmidt

Economic Evolution and Equilibrium

Bridging the Gap

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and Mathematical Systems 591

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With 75 Figures

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Professor Dr. Marco Lehmann-Waffenschmidt
Department of Economics
Economics, esp. Managerial Economics
Dresden University of Technology
01062 Dresden
Germany
lewaf@t-online.de

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General Introduction

> It is certainly true that an actual economy will be changing all the time.
>
> J.R. Hicks

> Change is a universal phenomenon in all systems. All equilibria are temporary.
>
> K.E. Boulding

> Life is change, and without changing it would be inexplicable.
>
> N.A. Berdjajew

> Everybody knows that life is a process. But not everybody remembers that a process will be no longer a process if it reaches an equilibrium.
>
> M. Feldenkrais

> *Πάντα ῥεῖ.*
>
> Heraklit

One of the most prominent ideas in economics undoubtedly is that of equilibrium. Even branches of economics which by their very nature are concerned with non-equilibrium states of economic systems draw on the notion of equilibrium, at least as a fundamental point of reference. Equally central to economics, however, is the idea of the evolution of an economic system over time. In fact, the understanding of an equilibrium as a final state of rest which has been borrowed from thermodynamics being prevalent in economics is obviously completely at odds with the idea of evolution. To avoid an inappropriate bipolarity of these two key concepts in economics, however, a synthesis of both seems to be desirable. Fortunately, economic theory has proposed ways to tie the two strings together. A first proposal comes from economic growth theory, which formalizes a dynamic economic system as a system of difference, or differential, equations. There equilibria mean ‘equilibrium trajectories’ of the whole evolution that, in a certain sense, are optimal. A particularly unsatisfactory feature of this conceptualization of an equilibrium, however, is the fact that the intertemporal optimizing approach completely predetermines the whole future of the economic system. This “closed loop” approach gives rise to the common reproach that economic theory is predominantly concerned with the question of ‘how the economic system ought to behave’ rather than with the question of ‘how does it behave actually’. This is the point at which the new branch of evolutionary economics has made its entrance.

In contrast to growth or business cycle theory, evolutionary economics perceives the evolution of the economic system as essentially “open” to true novelties that are unforeseeable by their very nature. This view clearly makes obsolete any conception of equilibrium that resorts to the idea of a final state of rest, or to the idea of an intertemporally optimizing trajectory which is prespecified ab initio by a system of differential equations and initial conditions. To be sure, there have been attempts to reconceptualize the notion of equilibrium from the evolutionary viewpoint. However, these proposals also appear, in one way or another, to hinge on the ideas of rest. This particularly applies to the branch of nonlinear dynamics and deterministic chaotic motion. More specifically, this approach assumes the dynamic behavior of a system as being governed by a fully deterministic process, namely by iterative application of a fixed “generator” mapping. Then ‘attractors’ are sought, i.e. a family of states that are finally run through again and again by the system under consideration. What this approach still lacks, however, is an analytical framework for the evolving economy which allows for a new and truly ‘open’ conceptualization of equilibrium.

To further the latter idea we will put forth here a new attempt to synthesize the two ideas of economic equilibrium and evolution. The basic idea of our approach is to take elaborate, but equally intuitive, models of mathematical economic equilibrium theory as our starting point and to ‘animate’ them, or, say, ‘let them evolve’.

This naturally leads us to the conception of an equilibrium as a “*transitory coordination solution*”. As we will see, this notion of equilibrium meets the requirement of an ‘open evolutionary’ equilibrium concept quite satisfactorily. Before we proceed to sketch our approach and our aims, we should, however, clarify our understanding of the term ‘evolution’ in the present study. In fact, we will adopt a broad understanding of the term ‘evolution’. This means, we do not think of any connotation of progress, or directed development in any sense (*anagenesis*) when speaking of an evolving economic system. Particularly, we may, but need not necessarily, think of an evolving economic system as being governed by ‘evolutionistic’ rules in the sense of variation, selection, and retention. Moreover, we will employ two understandings of an evolution: a temporal understanding of an evolution as a process in historical time, and an atemporal understanding of an evolution as an “artificial” evolution generated "in the mathematical economist's laboratory". In any of these two conceptualizations an evolution consists of a succession of states of the economic system under consideration. As a general remark we would like to emphasize that throughout our whole study geometrical imagination is always a good guide for intuition.

Intuition, Scope, and Aims of the Book

Our study consists of three main Parts. In **Part I**, the concept of an evolution of economies is formalized analytically. This will be done on the basis of nine different general equilibrium models, which are henceforth refered to as the "basic models". They have been partly adopted from the literature, partly they are new. The necessary mathematical tools are introduced in the 'Mathematical Preliminaries' following this Introduction. They are mainly intuitive concepts from geometry, general topology, homotopy theory, algebraic geometry, and differential topology. At the heart of our analytical formalization of evolution lies the notion of a "continuous one-parametrization of states of the economy". This way of analytically formalizing evolutions is not only intuitive, but it also appears to be the only reasonable one for our purposes. To aid the reader's intuition, the single states of the evolution correspond to the single shots of a movie, if one compares an evolution of economies to a movie. The roots of this conceptualization as well as of further analytical treatment can be traced back to early publications by Lehmann-Waffenschmidt (1983, 1985, special aspects have been analysed by the author in 1987, 1994, 1995, 2005, 2006). Moreover, continuously one-parametrized economies have also been analyzed for instance by A. Mas-Colell in his comprehensive monograph from 1985 (Chapters 5 and 8). Indeed, both approaches have originated in complete independence of each other. The reader should note, however, that the study by Mas-Colell only provides an *analytical* treatment of one-parametrized economies, but gives no further economic applications. Nevertheless, Mas-Colell's contribution will be an important point of reference for our formal analysis in Part II of the present study. But there is a clear distinction from the mathematical viewpoint: Our constructions primarily draw on algebraic parametrized fixed point theory, whereas Mas-Colell's constructions come from the field of differential topology. It is noteworthy that our approach nowhere resorts to differentiability assumptions. All our constructions and results are solely based on assumptions of continuity.

The main task in Part I is to formalize the concept of evolutions in our nine basic set-ups and to fit these formalizations into a unifying analytical setting. This is done in order to make them accessible for an application of a crucial result from one-parametrized algebraic fixed-point theory. In Chapter 4 evolutions in three basic models from the Walrasian exchange framework, one of which is a model of large exchange economies will be formalized. In fact, this model is similar to the one used by Mas-Colell as a basis for one-parametrizations (1985, Section 5.8).

In Chapter 5, evolutions are formalized in two basic models which relax the usual assumptions of Walras' law (the budget identity) and of homogeneity of degree zero of the excess demand functions. These models are inspired by a former model by N. Schulz (1985) the purpose of which has been to model a subsystem of the system of all conceivable markets in an economy. Nevertheless, the relaxation of these two standard assumptions will prove to be of great help later in our study when a new formalization of an economy evolving in historical time is developed (Section 19.2.2).

In Chapter 6, evolutions are formalized in two models with production, tax, and subsidy schemes originally developed by T. Kehoe (1985b). Finally, in Chapter 7 evolutions are formalized in two models from the quantity constrained equilibrium framework. More precisely,a micromodel with effective demand of the Benassy type is employed, which we have slightly adapted for our purposes. Furthermore, a new model is designed with many productive sectors on a medium level of aggregation.

In **Part II** of our study, the main analytical results which will provide the basis for later applications in Part III are derived. Any proofs in Part II which employ advanced mathematical results are relegated to the appendices at the end of this monograph.

The central analytical results of this study are given in Chapter 10. Using a certain core result from parametrized algebraic fixed point theory it is shown that for any evolution of each one of the nine types introduced in Part I, there is a certain structural property of its equilibrium set. This structural property ensures the existence of what we call 'near-equilibrium paths'. This result is certainly not at all clear from the outset since even for simple examples a total indeterminacy of the equilibrium set of one-parametrizations can be observed. The intuitive geometrical meaning of a near-equilibrium path is that of a polygonal path, which lies in the graph of the Walras correspondence of the given evolution of economies. For the pure exchange framework a related result has been formerly shown by Lehmann-Waffenschmidt (1983, 1985). Another related result for a basic model of a large exchange economy has been provided by Mas-Colell (1985, 5.8.24).

A mathematical criterion is provided for checking which points lie on near-equilibrium paths. In Chapter 11, it is shown how any evolution can be approximated so that there even exists a geometrically, nicely behaved equilibrium path, i.e., a path consisting only of true equilibrium points. To our knowledge so far there is no precursor in the literature of our class of well-behaved paths and our approximating evolutions. From Mas-Colell's extension of the regular theory to the one-parametrized case merely follows the existence of approximating evolutions in the basic exchange framework. We will come back to this below. To achieve our aims, we have to accomplish three tasks. First, we must design a general class of paths that deserve the qualification "well-behaved". Second, we must provide a general construction of approximating evolutions for each of our basic models, and third we have to verify that our approximating evolutions always possess an equilibrium path from the designed well-behaved class. While this makes some analytical efforts necessary, in our eyes they are fully justified by the achievements that become possible with their help.

Actually, it is the notion of a (near-)equilibrium path which will later provide the basis for our new concept of a 'homeostatic equilibrium' of an evolution. In Section 11.1 two alternative methods of approximating evolutions of exchange economies are developed. The first one is based on piecewise linear functions, whereas the second one is based on polynomial approximation. Both methods have advantages. While the first one is completely constructive, the second one can easily be generalized to other basic frameworks.

It is noteworthy that as a byproduct of our constructions it can be shown that the graphs of the equilibrium correspondence of each of the nine basic models from Part I are "maximally well-connected". This result significantly extends the related global results on the arc-connectedness of the graph of the Walras correspondence by Y. Balasko and others (see Balasko 1988, 1996 for surveys, see also Balasko, Lang 1998 and Bonnisseau, Cayupi 1999). At this point it is also natural to examine the relationship of our results in this monograph to the results of the so-called law of demand (Hildenbrand 1989, 1994, 1998, 1999a, b). Actually, the validity of the law of demand would ensure uniqueness of the equilibrium set of any single state economy of an evolution. Then the existence of geometrically well-behaved (near)-equilibrium paths of our type would directly follow from the continuity of evolutions. Unfortunately, all theoretical and empirical results supporting the validity of the law of demand pertain to special static equilibrium model types different from any one of the nine basic models developed here. What's more, the law of demand cannot hold true for the exchange model, as simple computation shows.

Chapter 12 provides further natural interpretations and extensions of the general concept of an economic evolution developed here. Obviously our conceptualization of economic evolution by one-parametrizations gives room for two economic interpretations. On the one hand, one may emphasize the aspect that a one-parametrization connects its initial state with its terminal state. In this case, we speak of a "*connection evolution*". On the other hand, one may understand an evolution in this context as starting from its initial state and openly evolving in some continuous way. In this case we speak of a "*course evolution*". A particularly interesting question is whether for each of the basic models there is always a connection evolution for any two given economies. Fortunately, it can not only be shown that the correct answer is "yes", but also general standard constructions of connection evolutions for each basic model can be provided.

Whether one adheres to the understanding of an evolution as the performance of the economic system in historical time, or one employs the formal understanding of evolution as any succession of states, be it chronological, or artificial, it seems to be desirable to admit both cases of 'new comodities' appearing on markets and of 'old commodities' disappearing from markets during the evolution. This will be our theme in Section 12.2 where we provide the analytical constructions that are necessary to realize this in each of the nine basic models.

The structure results of the existence of (near)-equilibrium paths from Chapters 10 and 11 raise the following natural question: Is this the only structural property of the equilibrium set of evolutions that generally holds? In Chapter 13 the answer to this question will be given as affirmative for the basic models from the exchange framework in Chapters 4 and 5. Moreover, the one-parametrized extension of Mas-Colell's famous result from 1977 which extended the celebrated decomposition result of market excess demand functions by Sonnenschein, Debreu, and Mantel will be achieved. More precisely, our result shows that in the one-parametrized case of an evolution of economies there is a structural property of the equilibrium set, whereas Mas-Colell's result has verified the total lack of restriction on the equilibrium price set of a static exchange economy. As a notable corollary of our result, any two non-empty compact subsets of the price domain can be realized as the equilibrium sets of two arbitrarily close exchange economies.

Our results are also closely related to the results on the local surjectiveness of the graph of the Walras correspondence by B. Allen (1981). As we will see in Chapter 13 our results and those by B. Allen neither extend, nor contain each other, but are complementary in their characterization of the graph of the Walras correspondence. Together with the above mentioned global results by Y. Balasko and others, these results provide a fairly detailed understanding of the shape of the graph of the Walras correspondence.

In Chapter 14, we present a detailed comparison of our results with related results in the literature. As a general remark we repeat that our approach nowhere resorts to differentiability assumptions. All of our conceptualizations and results are based solely on continuity. In Section 14.1 we summarize the achievements of our results compared with the well-known global structural results on the graph of the Walras correspondence. Section 14.2 deals with the relationship of our approach and its results to the theory of regular economies and its extension to regular one-parametrizations by Mas-Colell (1985, Chapter 8). In a nutshell our conclusion is that the static regular theory produces stronger results than ours in the local sense, but if one leaves a connected component of the subspace of regular economies, these strong results break down. In this case our results have significant advantages.

There is certainly a close relationship between the theory of regular one-parametrizations and the approximation results in Section 11.1. However, there are advantages of our approach: Our method of achieving approximating evolutions by well-behaved equilibrium paths is constructive, whereas the theory of regular one-parametrizations merely provides an abstract existence result. Of course, equilibrium paths for an evolution are just selections from its equilibrium set. Thus they are non-unique in general, since the equilibrium set may well exhibit irregularities such as multifurcations, or 'thick' parts, i.e. continua. On the other hand, though they are isomorphic to linear segments, the 'regular equilibrium paths' found by the regular theory may exhibit geometrically wild features. For instance, they may have infinite Euclidean length as simple considerations show. Moreover, we can show here that the compartmentalization of the space of exchange economies as well as of the space of exchange one-parametrizations by the subspace of critical economies and critical one-parametrizations, respectively, is fairly complicated. We emphasize that this weakens the structure results of the regular theory considerably. Actually, slightly perturbing a critical economy, or one-parametrization, leads to a regular economy with probability one. But the complex structure of the subspace of critical economies makes it almost impossible to predict the properties of the obtained regular economy, or one-parametrization,respectively. A last issue concerns the labels 'critical' and 'regular'. In fact, speaking of 'non-regular', or 'critical', economies (one-parametrizations) means that they are exceptional, or negligible. To be precise, this implicitly presumes a uniform probability distribution on the space of economies (one-parametrizations). However, so far no consistent underpinning has been provided by economic theory which would justify the assumption of negligibility. Instead, experience with real social systems strongly suggests that 'critical' states are not at all negligible.

In **Part III** of the study the economic content of the preceding conceptualizations and results are explored. Following the common classification, a distinction is made between applications on the temporal and on the atemporal field.

On the atemporal field, i.e. in the mathematical economist's laboratory, two major strings of applications are presented (Chapter 17). The first one has to do with the computation of equilibria, and the second one with emancipating comparative statics from its paralysis through the indeterminateness phenomenon. More precisely, it is shown that our results, in a certain sense, achieve an extension of the well-known path following computational method of equilibria of regular exchange economies (see e.g. Mas-Colell, 1985, Section 5.6 for a survey on the topic). The method used here works in each of our nine basic set-ups and, particularly, is not confined to regular economies. This is, however, at the cost of loss of algorithmic comfort.

In our second atemporal application the notorious paralysis of comparative statics caused by multiplicity of equilibria is dealt with. In fact, it is our conviction that the multiplicity phenomenon is intrinsically linked to the present-day way of economic thinking. Our conclusion from this is that a way should be sought to give comparative statics a meaning, also in the multiplicity case. In Section 17.2 will be demonstrated that our preceding results indeed provide a way to reconstruct comparative statics when equilibria are multiple. Moreover, our main result from Chapter 13 implies that the proposed 'genetic comparative static method' in fact is *the only general way* to achieve this.

The main economic applications of our approach and of our results, however, are on the temporal field. In Chapter 18, the methodological viewpoint and the scope of the analysis are explained at some length. We will not strive for an analysis which is dynamic, or even evolutionary, in the strict sense, but confine ourselves to an analysis that is something like a continuous, or evolutionary comparative analysis (a '*genetic comparative analysis*'; for an evolutionary approach see e.g. Bosch 1990, Kirzner 1990, Loasby 1991, and Faber, Proops 1998, Witt 2003). From physics we have borrowed the term '*kinetic*' for our approach. As already mentioned the procedure used here is to conceptualize and formalize different types of classes of reasonable evolutions by means of continuous one-parametrizations, and then to analyze them for their general structural properties. *Kinetics* does not inquire into the causal explanation of the individual evolution of the real economic system in historical time, but searches for general regularity, or structural, properties of the dependent evolutions of the endogenous key variables. Thus, one can say, that while *dynamics* studies the *'laws of motion'* of the economic system, *kinetics* studies the *'laws of the effects of motion of the economic system'*. In this sense our temporal applications can be seen as being *complementary* to dynamics, and especially to evolutionary economics.

Having clarified our method, we will start with applications in *discrete* historical time (Chapter 19.1). The first step is to formalize evolving economies in discrete historical time for the nine basic models. This is achieved in a natural way by employing the common 'period approach'. Essentially our applications in this context are based on the atemporal applications given in Chapter 17.

The main body of our temporal applications, however, are applications in *historical time* (Sections 19.2–19.4). In Section 19.2 we begin by designing two alternative models of evolving economies in continuous time. While the first one is based on the idea of continuous flows of commodities and services, the second one provides an entirely new approach. Its main idea is to describe the evolution of a market over time by varying time intervals between two successive demand, or supply, events. In our opinion, the resulting 'frequency model' achieves a realistic theoretical framework describing an evolving economy in continuous historical time. The main ingredient of the frequency model is the basic framework of an exchange economy that relaxes Walras' law and the homogeneity assumption on excess demand functions from Section 5.1.

What are the economic achievements of the application of the analytical work from Part II to these conceptualizations of evolving economies in historical continuous time? In a nutshell, it provides the opportunity to tune equilibria, at least piecewise, continuously to their changing values when the economy undergoes an evolution. In other words, we establish the existence of a 'homeostatic equilibrium' for evolving economies. It should be emphasized that this result merely ensures the opportunity for some policy making institution to achieve a (piecewise) fine tuning of equilibrium values, but does not endogenously model the policy making institution itself. In particular, our understanding of the notion of equilibrium is not that of a description of the real state of an economy. Indeed, this is made impossible by the multiplicity of equilibria. Instead, we understand the equilibria of a given momentary state of an evolving economy here solely as momentary, or transitory, coordination solutions to this state. Consequently, it is not our concern to explain the actual states of an evolving economy, but rather to support the provision of the opportunity for 'equilibrium engineering', i.e. for continually selecting equilibrating solution values with the least possible friction. Regarding the underlying evolution of the economic system, two model approaches will be applied: In the first one the open evolution of the economic system is not touched by the "equilibrium engineering" procedure, in the second one certain "backtracking" phases in the open evolution of the economy have to be employed. We will come back to this issue shortly.

At this point, however, we would like to mention a direct application of this result to the issue of time consuming equilibrium adjustment processes. It has been known for a long time that, in general, a time consuming equilibrium adjustment process faces a moving target (e.g. Kloek 1984, for a comprehensive survey see e.g. Fisher 1983). This has already been illustrated by V. Pareto in a different context by his famous 'courbes de pursuite'. The adjustment of a moving equilibrium is symbolized by him as a running hare being tracked by a hound. To our knowledge we show for the first time that for any evolution of any of our basic models there is something like 'the path of the hare' which can be actually tracked by an agent purposed to "catch the hound" (Section 19.3).

So far the results just show that a 'frictionless equilibrium engineering', or tuning, in general is only piecewise possible, i.e., up to finitely many discrete jumps. In the final Section 19.4 we will show, however, that this deficiency can also be removed. The key idea for this is to "re-manipulate" the evolution of economies continuously without bringing new momentary states into play such that no discrete jumps in the equilibrium values are necessary when tuning them. In fact, three of the nine basic models are, from their economic conceptualization, suitable for this. These are the two models from the framework with production, taxes, and subsidies (Chapter 6) and the multi-sectoral quantity constrained model (Chapter 7). All these models have in common that they contain explicit parameters that are, in principle, accessible to an external control by some economic policy institution. These are prices and wages in the case of the quantity constrained multi-sectoral model from Chapter 7, and tax and subsidy rates in the case of the two models from Chapter 6.

In order to ensure a perfect homeostatic equilibrium, i.e., a continually frictionless tuning of equilibrium values during an evolution, it is only necessary for a policy institution to intervene at finitely many dates. In concrete terms, an effective intervention means that the evolution of the control parameters governing the evolution of the economic system is partly backtracked, i.e., is in parts repeated in a continuous way. The reader should be well aware that we have a tuning on two different levels, namely on the level of economic state parameters and on the level of equilibrium values, whereas in Section

19.2 there is only a tuning of equilibrium values. We also want to emphasize again that our result only provides the general opportunity to realize a perfect homeostatic equilibrium during an evolution to an external policy institution, but does not endogenously model policy institutions, or their actions.

The pros und cons of a continuous fine tuning of economic state parameters such as taxes, subsidies, or prices, according to the applied basic model set-up, have been, for the first time, extensively discussed in the literature during the debate on gradual versus bang-bang tax reform in the seventies (e.g. Hatta 1977, Hettich 1979). This controversy has later experienced a revival in a slightly different context, namely in the debate on macroeconomic policy design (e.g. Fellner et al. 1981, Zodrow 1985, Marangos 2002). To sum up, the following arguments are in favor of a continuous 'fine tuning' policy: Enactment of an "bang-bang", "cold turkey", or shock therapy policy entails greater administrative as well as greater social and political costs. This may largely be attributed to the agents' attitude of risk aversion and conservatism in economic affairs, which appears to be predominant in reality. Moreover, gradual control makes at least partial foresight possible for the economic agents. In economics this is generally considered as favorable for a stabilized evolution of the economy. This argument is beyond the scope of the model framework developed and employed in this book, but the reader should note that a discontinuous monitoring of equilibrium prices will not only cause sudden changes in consumed quantities, but also of individual wealth and thus of the agents' economic status. Last, but not least, tuning equilibria following a "well-behaved" path while the economy evolves is clearly much more comfortable for the agency than searching for new equilibria anywhere in the domain of all possible equilibria.

However, this does not mean that we take a one-sided position favouring a strict gradualism in economic policy making. For both positions of a gradualistic policy and a shock therapy there are striking metaphors: How would it be possible on one hand to change moving forward to moving backward other than gradually? The shock therapy position, on the other hand is favoured by the metaphor of changing from driving on the left to driving to the right in a state. We are well aware of the disadvantages of the gradualistic principle. The German reunification, for instance, may serve as an example of how political motives and uncertainties concerning the future evolution of boundary conditions may well favour a quasi bang-bang policy enactment of reforms. What we want to say is that it seems to be worthwhile investigating the conditions and opportunities for enacting a gradual, shock-free policy. A thorough assessment to decide whether a gradual adjustment, or a shock therapy policy adjustment is preferable can only be made on a case-by-case basis.

The monograph is rounded off in *Part IV* by general conclusions, an outlook on further possible research work and the Appendices A to C.

I have now reached the point where I would like to take the opportunity to thank all who have helped me with their comments and suggestions. In fact, there is a number of people who have contributed to the evolution of my personal ideas and views on my subject over the years and who have helped me to make them precise and comprehensible. Particular thanks are due to the Konrad Lorenz Institute for Evolution and Cognition Research in Altenberg near Vienna where I found the environment to do the last “finish” on this monograph, and to Barbara Feß from Springer Verlag for her help and encouragement as well as all people who gave technical support to the realization of this book.

2

Notations and Mathematical Preliminaries

Notations

$\mathbb{R}^n_+,\ \mathbb{R}^n_{++}$	closed (open) positive orthant of $\mathbb{R}^n$
$\mathbb{R}^n_-,\ \mathbb{R}^n_{--}$	closed (open) negative orthant of $\mathbb{R}^n$
$\partial\mathbb{R}^n_+$	boundary of $\mathbb{R}^n_+$, i.e. $\mathbb{R}^n_+\backslash\mathbb{R}^n_{++}$
Δ^{n-1}	closed $(n-1)$-dimensional unit simplex in $\mathbb{R}^n$, i.e. $\{x \in \mathbb{R}^n_+ \mid \sum x_i = 1\}$
$\partial\Delta^{n-1}$	boundary of the $(n-1)$unit simplex, i.e. $\{y \in \Delta^{n-1} \mid y_i = 0$ for at least one $i = 1, \ldots, n\}$
$\mathring{\Delta}^{n-1}$	boundaryless, or open, $(n-1)$-dimensional unit simplex, i.e. $\Delta^{n-1}\backslash\partial\Delta^{n-1}$
Δ^{n-1}_i	i-facet of Δ^{n-1}, i.e. the subspace $\{x \in \Delta^{n-1} \mid x_i = 0\}$ of the boundary $\partial\Delta^{n-1}(i \in \{1, \ldots, n\})$
Δ^{n-1}_ϵ	for $\epsilon > 0$ the inscribed "ϵ-unit simplex", i.e. $\{x \in \Delta^{n-1} \mid \forall_{i=1,\ldots,n}\ x_i \geq \epsilon\}$ (note that clearly the Euclidean distance from any i-facet of Δ^{n-1} to the i-facet of Δ^{n-1}_ϵ is greater than ϵ)
$\Delta^{n-1}_{\epsilon_i}$	the i-facet of Δ^{n-1}_ϵ, i.e. the subspace $\{x \in \Delta^{n-1}_\epsilon \mid x_i = \epsilon\}$
$\Delta^{n-1,\alpha}$	for any real number $\alpha > 0$ the $(n-1)$-dimensional simplex $\{y \in \mathbb{R}^n_+ \mid \sum_{i=1}^n y_i = \alpha\}$, also called the "$\alpha-(n-1)$-simplex"; thus $\Delta^{n-1,\alpha}$ is parallel to the unit simplex with intercepts α on the coordinate axes. For $\alpha \leq 1$ it is also called the "α-section of T^n" (see below); α is also called the "simplex-level" of $\Delta^{n-1,\alpha}$
$\mathring{\Delta}^{n-1,\alpha}$	open $\alpha-(n-1)$-simplex $\Delta^{n-1,\alpha} \cap \mathbb{R}^n_{++}$
$\langle v_1, \ldots, v_{k+1}\rangle$	for $k+1$ points $v_1, \ldots, v_{k+1}$ in $\mathbb{R}^n$ the m-dimensional simplex generated by them, i.e. their convex hull

	$(m \le k \le n)$
S^{n-1}	$(n-1)$-dimensional unit sphere, i.e. $\{x \in \mathbb{R}^n \mid \|\|x\|\| = 1\}$
S_+^{n-1}	closed positive part of S^{n-1}, i.e. $\{x \in \mathbb{R}_+^n \mid \|\|x\|\| = 1\}$
S_{++}^{n-1}	strictly positive part of S^{n-1}, i.e. $\{x \in \mathbb{R}_{++}^n \mid \|\|x\|\| = 1\}$
S_ϵ^{n-1}	for $\epsilon > 0$ the contained closed ϵ-unit sphere, i.e. $\{x \in S_+^{n-1} \mid \forall_{i=1,\ldots,n}\ x_i \ge \epsilon\}$
$T^{n,\alpha}$	for any real number $\alpha > 0$ the orthogonal projection of $\Delta^{n,\alpha} \subset \mathbb{R}_+^{n+1} := \mathbb{R}_+^n \times \mathbb{R}_+$ into the coordinate hyperplane $\mathbb{R}_+^n$, i.e. $T^{n,\alpha} := \{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i \le \alpha\}$, also called "embedded α-n-simplex"
$\overset{\circ}{T}{}^{n,\alpha}$	the open embedded α-n-simplex $\{x \in \mathbb{R}_{++}^n \mid \sum_{i=1}^n x_i < \alpha\}$
T^n	abbreviation of $T^{n,1}$, also called "embedded n-dimensional unit simplex"
0^n	the null vector of $\mathbb{R}^n$
T_0^n	the pointed embedded n-dimensional unit simplex $T^n \backslash \{0^n\}$
$T_\gamma^{n,\alpha}$	for arbitrarily large real positive α and arbitrarily small real positive γ the "inscribed embedded α-γ-n-simplex" $$\{x \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i \le \alpha \text{ and } x_i \ge \gamma \ \text{ for all } \ i = 1, \ldots, n\}$$
T_γ^n	$T_\gamma^{n,1}$
$\{pt\}$	the single point space (singleton set)
e_n	the vector $(1, 1, \ldots, 1)$ of $\mathbb{R}^n$
e^i	the i-th unit vector $(0, \ldots, 0, 1, 0, \ldots, 0)$ of $\mathbb{R}^n$
$(x_1, \ldots, \hat{x}_i, \ldots, x_n)$	the $n-1$-vector $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$
$x \ge y$, $x > y$	for n-vectors x and y means that the weak (strong) inequality holds for every component
$\overline{xy}$	for n-vectors x and y the (straight line) segment with endpoints x and y
x'	for a column vector $x \in \mathbb{R}^n$ the transposed row vector
$Y \backslash X$	for spaces $X \subset Y$ the difference set $\{y \in Y \mid y \notin X\}$
X^c	for spaces $X \subset Y$ the complement of X in Y, i.e. $Y \backslash X$
coX	for a subspace $X \subset \mathbb{R}^n$ denotes the convex hull of X in $\mathbb{R}^n$

$dist(F,G)$	means for any two nonempty compact subsets F, G of a metric space X the Hausdorff distance, i.e. $$\min\{\epsilon \geq 0 \vert F \subset B_\epsilon(G) \text{ and } G \subset B_\epsilon(F)\}$$ where $B_\epsilon(Y) = \{x \in X \vert d(x,y) < \epsilon \text{ for some } y \in Y\}$ for any $Y \subset X$
$B^n_r(x)$	the closed n-ball with center x and radius $r \geq 0$, i.e. $\{y \in \mathbb{R}^n \vert\ \vert\vert y - x\vert\vert \leq r\}$
$im\, f$	for a mapping $f : X \to Y$ the image $f(X) \subset Y$
$Fix\, f$	for a self-mapping $f : X \to X$ the fixed point set, i.e. $\{x \in X \vert f(x) = x\}$
$Fix\, F$	for a homotopy $F : X \times [0,1] \longrightarrow X$ the set $\{(x,s) \in X \times [0,1] \vert F(x,s) = x\}$
$X \times [0,1]$	homotopy space, i.e. the domain of a homotopy $F : X \times [0,1] \to Y$; due to their geometrical shape the special homotopy spaces $\Delta^{n-1} \times [0,1]$ and $T^{n-1} \times [0,1]$ are called "homotopy prisms"
$X \times \{s\}$	for $0 \leq s \leq 1$ the "s-slice" of the homotopy space $X \times [0,1]$
$\mathbb{N}$	natural numbers including 0
$M(n \times m; \mathbb{R})$	the set of $n \times m$-matrices with real entries
$\vert A\vert$	for an $m \times n$-matrix $A = (a_{ij})$ with real entries the $m \times n$-matrix $(\vert a_{ij}\vert)$ of absolute values of the entries

Mathematical Preliminaries

Now we are going to provide the reader with the formal standard notions from general and algebraic topology and algebraic geometry as well which will play an important role in our analysis[1]. Our exposition will be self-contained as regards our subsequent analysis. The reader who still misses further background informations is referred to the relevant textbook literature.

At the heart of our formalizations stands the notion of a continuous one-parametrization. Generally a *continuous one-parametrization,* or *evolution, homotopy, deformation*, *family* or *perturbation,* is a continuous mapping $F : X \times [0,1] \longrightarrow Y$ where X and Y are topological spaces. To be sure, the notion of a continuous one-parametrization has intuitive appeal since it can be viewed as a continuous one-parameter family of "ordinary" continuous mappings $(F_s)_{s \in [0,1]} : X \longrightarrow Y$ where $F_s(x) := F(x,s)$. For instance, any continuous movement process is an example of a continuous one-parametrization (cf. Figure 2.1). The subspace $X \times \{s\},\ s \in [0,1]$, is called the *s-slice*

[1] As this Section consists of a collection of definitions we will omit the term "definition" throughout.

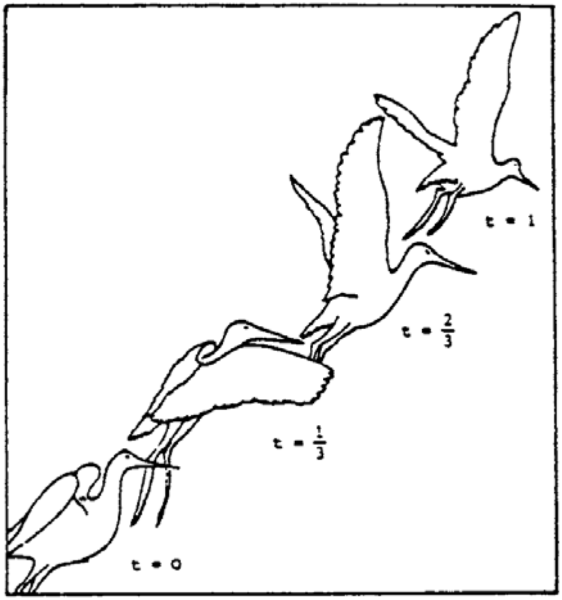

Fig. 2.1: Formal Representation of a Movie as a Homotopy

of the *homotopy space* $X \times [0, 1]$ (corresponds to the snap–shot photo at time $t = s$ in Figure 2.1). We furthermore call s the *homotopy*, *deformation*, or *evolution*, *parameter* and the mapping $F_s(-) = F(-, s)$ the *s-state mapping of the one-parametrization F*. Thus one can visualize a one-parametrization $F : X \times [0, 1] \longrightarrow Y$ for Euclidean subspaces $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ by the continuous evolution of the graphs of the n component functions $F_i(x, s)$. *Evolutions of economic systems* which we will employ in our study will always be formally representable by one-parametrizations. Recall from the General Introduction that in our study we neither restrict the term 'evolution' to economic systems which are characterized by 'evolutionary' (technical) progress, nor do we even stick to the narrow understanding of evolutions as necessarily being evolutions over (historical) time. Rather, we will introduce evolutions of economic systems in the general notion of *any* continuous changes governed by a scalar parameter s. In Part III of our study we will study evolutions of economies in both interpretations of the evolution parameter s: in the technical atemporal interpretation, and in the interpretation as elapsing historical time.

Clearly, one can *combine,* or say *compose,* two homotopies $F^1, F^2 : X \times [0, 1] \longrightarrow Y$ when $F^1_1 = F^2_0$, i.e. $F^1(x, 1) = F^2(x, 0)$ for all $x \in X$. We also say that the obtained homotopy $F : X \times [0, 1] \longrightarrow Y$,

$$F(x,s) = \begin{cases} F^1(x,2s) & \text{if } s \in [0,1/2], \\ F^2(x,2s-1) & \text{if } s \in [1/2,1], \end{cases}$$

is the *composition* of the two homotopies F^1 and F^2, or the *composite homotopy* of F^1 and F^2. Composing k homotopies in this way accordingly leads to a $(k-1)$*-fold composite homotopy.*

A *contractible* topological space X is *homotopic* to the single point space, i.e. there is a homotopy $F : X \times [0,1] \longrightarrow X$ with $F(-,0) = id_X$, and $F(-,1)$ is a constant mapping into some point $x \in X$. Contractible spaces are special examples of *acyclic* spaces. The *Lefschetz number* of a space is an algebraic topological characteristic. For an acyclical space it is $+1$ (the interested reader is referred to Brown, 1971, II). A subspace X of $\mathbb{R}^n$ which is not convex can still be *star-shaped,* that means there is a point $x_0 \in X$ such that any two points $x, y \in X$ can be connected by the two segments $\overline{xx_0}$ and $\overline{x_0y}$. Clearly, a star-shaped space is contractible.

If the homotopy space equals the unit interval a special type of a homotopy called "*path*" obtains. Indeed, the concept of a *Euclidean path* $w : [0,1] \longrightarrow \mathbb{R}^n$ will be crucial for our study, and it especially gives rise to the following concept of a connected component of a space.

A *connected component* Z of some topological space X cannot be separated into two disjoint open subsets, i.e. there are no disjoint open subsets A, B of X with $(A \cup B) \cap Z = Z$. For any two points x, y of a *path (connected) component* Z' of X there is a continuous path $w : [0,1] \longrightarrow X$ with $w(0) = x,\ w(1) = y$. *w is a path in X connecting x with y.* A path connected component is maximal with this property.

One has to distinguish carefully between the notion of a path w and of its *arc,* i.e. its image $w[0,1]$ in $X \subset \mathbb{R}^n$. Identifying $[0,1]$ with $\{y\} \times [0,1]$ a path $w : [0,1] \longrightarrow X$ can also be viewed as a continuous one-parametrization $w : \{y\} \times [0,1] \longrightarrow X$ of its arc $w[0,1]$. Note particularly that in a graphic representation the parameter $t \in [0,1]$ in general is *not identifiable on the coordinate axes.* The *Euclidean length of a path* $w : [0,1] \longrightarrow X \subset \mathbb{R}^n$ is defined as $\sup_{W_k} L(w, W_k)$ where W_k denotes a subdivision of $[0,1]$ by $k+1$ points $0 = t_0 < t_1 < \ldots < t_k = 1$ and $L(w, W_k) := \sum_{j=1}^k d(w(t_{j-1}), w(t_j)) = \sum_{j=1}^k ||w(t_j) - w(t_{j-1})||$. If $\sup_{W_k} L(w, W_k)$ is finite then one says that w is of finite length, or w is *rectifiable.* It is well-known that a path w is rectifiable if and only if each of its component functions $w_i,\ i = 1, \ldots, n$, is of *bounded variation over* $[0,1]$, that means

$$\sup_{W_k} \sum_{j=1}^{k} ||w_i(t_j) - w_i(t_{j-1})|| < \infty.$$

The everyday connotation of the term 'path' clearly is 'to be viable, or passable' in the intuitive geometrical sense. This is also our intuition in this study. Unfortunately, arcs of *continuous* paths can still have wild shapes as the following examples show: the graph of the continuous function $x \cdot \sin 1/x$ on

$[-1,1]$ has *infinite length* (one estimates from below by the divergent harmonic series). But even if the arc of a continuous path is of finite length, it still may *oscillate,* or *tremble,* infinitely often, as the function

$$x \mapsto \begin{cases} x^3 \sin 1/x, & x \in [-1,0[\ \cup \ ]0,+1] \\ 0, & x = 0 \end{cases}$$

shows ('a damped oscillation', see Fig. 5).

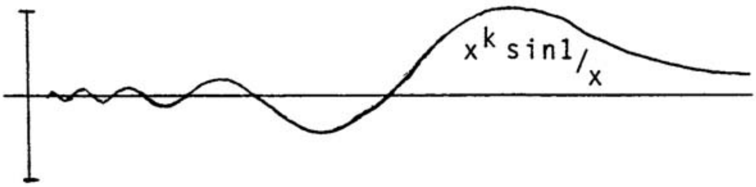

Fig. 2.2: Damped Oscillation

Fortunately, there is a way to analytically design a broad class of paths whose arcs are really "nice" in the intuitive geometrical sense. In other words they do not display any features of impassableness. We will come back to this later in our study (Section 11.1).

A *gluing (hat) function* $\alpha : \Delta^{n-1} \longrightarrow [0,1]$ which continuously glues some continuous function $f : \Delta^{n-1} \longrightarrow \mathbb{R}^m$ with some other continuous function $g : \Delta^{n-1} \longrightarrow \mathbb{R}^m$ over the area $\Delta_\epsilon^{n-1} \backslash \mathring{\Delta}_{2\epsilon}^{n-1}$ such that f prevails on the inner part $\Delta_{2\epsilon}^{n-1}$ and g on the boundary area $\Delta^{n-1} \backslash \Delta_\epsilon^{n-1}$ is a continuous function with the properties

$$\alpha\big|_{\Delta^{n-1} \backslash \Delta_\epsilon^{n-1}} \equiv 0$$
$$\alpha\big|_{\Delta_{2\epsilon}^{n-1}} \equiv 1.$$

The *glued function* is given by the *convex,* or *linear, combination*

$$\alpha(x)f(x) + (1 - \alpha(x))g(x).$$

An "*(affine) simplex* of dimension $k \geq 0$ embedded into $\mathbb{R}^n$" is the convex hull of $k+1$ different points $v_0, v_1, \ldots, v_k$ in $\mathbb{R}^n$ *which moreover are in general position.* The latter means that the affine linear subspace of $\mathbb{R}^n$

$$\left\{ y \in \mathbb{R}^n \;\middle|\; y = v_0 + \sum_{i=1}^{k} \lambda_i (v_i - v_0), \ \lambda_i \in \mathbb{R} \right\}$$

spanned by $v_0, v_1, \ldots, v_k$ is *not* spanned by any subset of $\{v_0, v_1, \ldots, v_k\}$. Thus, the *affine simplex* $\langle v_0, \ldots, v_k \rangle \subset \mathbb{R}^n$ *generated by* $v_0, v_1, \ldots, v_k$ is the subspace

$$\left\{ \sum_{i=0}^{k} \lambda_i v_i \,\middle|\, \sum_{i=0}^{k} \lambda_i = 1, \quad \text{every} \quad \lambda_i \geq 0 \right\}.$$

$v_0, \ldots, v_k$ are also called the *vertices* of the simplex $\langle v_0, \ldots, v_k \rangle$. Each subset $\{v_{i_0}, \ldots, v_{i_l}\}$ of the set of vertices spans an affine subsimplex $\langle v_{i_0}, \ldots, v_{i_l} \rangle$ of $\langle v_0, \ldots, v_k \rangle$. $\langle v_{i_0}, \ldots, v_{i_l} \rangle$ is also called an l-dimensional *face* of the simplex $\langle v_0, \ldots, v_k \rangle$. In this terminology, the vertices are precisely the 0-faces, and the 1-faces are called the *edges* of the simplex. The maximal dimension of an affine simplex in $\mathbb{R}^n$ is clearly n.

A subspace X of $\mathbb{R}^n$ is a *finite simplicial complex* $\sum_X$ *of dimension* k if it is the union of finitely many affine simplices of dimension $\leq k$ which satisfy the following rules of adjacency: for any simplex from $\sum_X$ each of its faces also belongs to $\sum_X$. The intersection of any two simplices from $\sum_X$ is either empty or is a common face.

One also calls $X = \bigcup_{\sigma \in \sum_X} \sigma$ the *support* of the simplicial complex $\sum_X$, and says that X is *finitely simplicially decomposed,* or *finitely triangulated,* by the simplicial complex $\sum_X$. In this study we will only deal with finite simplicial decompositions of simple Euclidean subspaces like Δ^{n-1} or Δ^{n-1}_ϵ, for instance.

There is obviously no difficulty to *extend* a given simplicial triangulation $\sum_X$ of Δ^{n-1}_ϵ to Δ^{n-1}, i.e. to provide a simplicial decomposition $\sum'_X$ of Δ^{n-1} whose restriction to Δ^{n-1}_ϵ equals $\sum_X$. Furthermore it is straightforward for these simple spaces to obtain a finite triangulation $\sum_X$ for any two given triangulations $\sum'_X$ and $\sum''_X$ which is a *common refinement* of $\sum'_X$ and $\sum''_X$, i.e. which contains both complexes $\sum'_X$ and $\sum''_X$ as subcomplexes.

The spaces Δ^{n-1} and S^{n-1}_+ are standard examples of *neighborhood retracts in a Euclidean space.* Generally, a *Euclidean neighborhood retract* A in $\mathbb{R}^n$ is a subspace which is a *retract* of some of its neighborhoods, i.e. there is a neighborhood $\mathcal{U}(A)$ of A in $\mathbb{R}^n$ and a continuous mapping

$$r : \mathcal{U}(A) \longrightarrow A \text{ with } r|_A = id_A.$$

For the purpose of our present study, i.e. for the equilibrium analysis of evolutions of economic systems, the notion of a homotopy is still not quite satisfactory. The reason for this is that the continuity of a homotopy is a fairly weak property still allowing for some pathologies of the one-parametrized family of state mappings if the domain is not a compact space. More formally, the continuity of a homotopy F is equivalent to C^0-uniform convergence of the state mappings F_s on compacta, i.e. to convergence of the state mappings F_s on any compact subset $A \subset X$ with respect to the maximum norm. However, if X is an *open* subspace of $\mathbb{R}^n$ this admits for instance the following pathology for a continuously one-parametrized family of excess demand functions $(\zeta_s)_{s\in[0,1]} : \mathring{\Delta}^{n-1} \longrightarrow \mathbb{R}$ (see Figure 2.3): The sequence of excess demand functions $(\zeta^k)_{k=1,2,\ldots}$ obviously converges to the excess demand function ζ^0

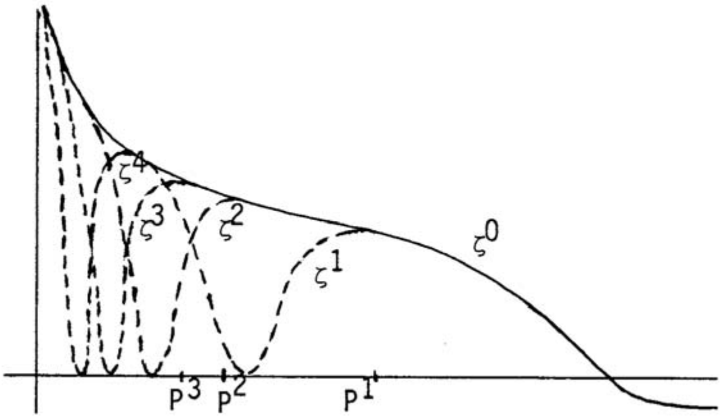

Fig. 2.3: Convergence Pathology

on compacta, though there is an increasing deviation of the functional values for the critical arguments $P^1, P^2, \ldots$. This is possible since the critical arguments run to the boundary of the non-closed domain. Clearly such a behavior strongly contradicts the intuition underlying the notion of a continuous evolution of economic behavior functions. That means that neighboring state functions of an evolution should have similar values *on their whole domain* – not only on compacta. Thus, throughout our whole study we will employ the stronger concept of *overall C^0-uniform convergence* for one-parametrizations instead of mere continuity, i.e. the state functions must converge on their whole domain with respect to the usual supremum norm.

Given subspaces $X \subseteq Y \subseteq \mathbb{R}^n$, a real $\epsilon > 0$, and a function $g : X \longrightarrow \mathbb{R}^m$, we say that a function $f : Y \longrightarrow \mathbb{R}^m$ *ϵ-approximates g uniformly on X* when the restriction $f^3{}_X$ is in the ϵ-neighbourhood of g, i.e.

$$||f(x) - g(x)|| = \sqrt{\sum_{i=1}^{m} (f_i(x) - g_i(x))^2} \;<\; \epsilon \;\text{ for all }\; x \in X.$$

Now we are going to introduce the concept of *semi-algebraic subsets* of $\mathbb{R}^n$. We will employ these sets since they have very nice geometrical properties and help us to formalize the notion of "nice paths". Let us first recall some elementary definitions from algebraic geometry: a *polynomial in n-variables over* $\mathbb{R}$ is a continuous mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ of the form

$$f(x_1, \ldots, x_n) = \sum a_{i_1 \ldots i_n} \cdot x^{i_1} \cdot \ldots \cdot x^{i_n}$$

where the coefficients $a_{i_1 \ldots i_n}$ are fixed real numbers and the sum is taken over a *finite* set of n-tuples $(i_1, \ldots, i_n)$ of positive integers. $\mathbb{R}[x_1, \ldots, x_n]$ denotes the