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Xuexiang Huang · Xiaoqi Yang

Vector Optimization

Set-Valued and Variational Analysis

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To Our Parents

Preface

Vector optimization model has found many important applications in decision making problems such as those in economics theory, management science, and engineering design (since the introduction of the Pareto optimal solution in 1896). Typical examples of vector optimization model include maximization/minimization of the objective pairs (time, cost), (benefit, cost), and (mean, variance) etc.

Many practical equilibrium problems can be formulated as variational inequality problems, rather than optimization problems, unless further assumptions are imposed. The vector variational inequality was introduced by Giannessi (1980). Extensive research on its relations with vector optimization, the existence of a solution and duality theory has been pursued.

The fundamental idea of the Ekeland's variational principle is to assign an optimization problem a slightly perturbed one having a unique solution which is at the same time an approximate solution of the original problem. This principle has been an important tool for nonlinear analysis and optimization theory. Along with the development of vector optimization and set-valued optimization, the vector variational principle introduced by Nemeth (1980) has been an interesting topic in the last decade.

Fan Ky's minimax theorems and minimax inequalities for real-valued functions have played a key role in optimization theory, game theory and mathematical economics. An extension was proposed to vector payoffs was introduced by Blackwell (1955).

The Wardrop equilibrium principle was proposed for a transportation network. Until only recently, all these equilibrium models are based on a single cost. Vector network equilibria were introduced by Chen and Yen (1993) and are one of good examples of vector variational inequality applications.

This book studies vector optimization models, vector variational inequalities, vector variational principles, vector minimax inequalities and vector network equilibria and summarizes the recent theoretical development on these topics.

The outline of the book is as follows.

In Chapter 2, we examine vector optimization problems with a fixed domination structure, a variable domination structure and a set-valued function respectively. We will investigate optimality conditions, duality and topological properties of solutions for these problems.

In Chapter 3, we study existence, duality, gap function and characterization of a solution of vector variational inequalities. We will also explore set-valued vector variational inequalities and vector complementarity problems.

In Chapter 4, we present unified variational principles for vector-valued functions and set-valued functions respectively. We will also explore well-posedness properties of vector-valued/set-valued optimization problems.

In Chapter 5, we consider minimax inequalities for vector-valued and set-valued functions.

In Chapter 6, we consider weak vector equilibrium, vector equilibrium and continuous-time vector equilibrium principles.

One characteristic of the book is that special attention is paid to problems of set-valued and variable ordering nature. To deal with various nonconvex problems with vector objectives, the nonlinear scalarization method has been extensively used throughout the book. Most results of this book are original and should be interesting to researchers and graduates in applied mathematics and operations research. Readers can benefit from new methodologies developed in the book.

We are indebted to Franco Giannessi and Kok Lay Teo for their continuous encouragement and valuable advice and comments on the book. We are thankful to Xinmin Yang and Shengjie Li for their joint research collaboration on some parts of the book. The first draft of the book was typed by Hui Yu, whose assistance is appreciated. We acknowledge that the research of this book has been supported by the National Science Foundation of China and the Research Grants Council of Hong Kong, SAR, China.

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Contents

1

Introduction and Mathematical Preliminaries

In this chapter, we will present necessary mathematical concepts and results, which will be used in the later chapters. Most of the results can be found in the books: Aubin and Ekeland [5], Aubin and Frankowska [6], Rockafellar and Wets [168], Sawaragi, Nakayama and Tanino [176] and Yu [221]. Some new mathematical concepts and results on nonlinear scalarization functions will also be given.

1.1 Convex Cones and Minimal Points

Vector optimization problems (or multiobjective programming problems or multicriteria decision making problems) have close relations with orderings or preferences in objective spaces. It is known that orderings in a vector space can be defined by convex cones.

Let Y be a topological vector space, and $S \subset Y$ a nonempty subset. The topological interior, topological boundary and topological closure of S are denoted by $intS$, ∂S and clS, respectively.

A set $K \subset X$ is said to be convex if, for any $x_1, x_2 \in K, \lambda \in [0,1]$, we have $\lambda x_1 + (1-\lambda)x_2 \in K$.

A set C is called a cone if, for any $\lambda \geq 0, \lambda C \subset C$.

A set C is called a convex cone if $C + C \subset C$ and, for any $\lambda \geq 0, \lambda C \subset C$.

Let $B \subset C\backslash\{0\}$ be a subset. B is called a base of C if, for each $c \in C$, there exist $b \in B$ and $\lambda \geq 0$ such that $c = \lambda b$.

A convex cone C in Y is called pointed if

$$C \cap (-C) = \{0\}.$$

An ordering relation $\prec$ is said to be

(i) Reflexive if $x \prec x$;
(ii) Asymmetric if $x \prec y, y \prec x \Longrightarrow x = y$;

(iii) Transitive if $x \prec y, y \prec z \Longrightarrow x \prec z$.

An ordering relation is called a partial order if it satisfies reflexive, asymmetric and transitive conditions.

In principle, any nonempty subset C of Y can define an ordering relation by

$$y \leq_C z \iff z - y \in C, \quad \forall y, z \in Y.$$

However, only some particular subsets C of Y can define ordering relations with nice and useful properties. In this book, we restrict our attention to two cases: (i) C is a convex cone in Y and (ii) C is a convex subset of Y with $0 \in \partial C$. We emphasize that, throughout the book, we will discuss under case (i) unless explicitly stated otherwise.

If C is a convex cone in Y and C defines an ordering relation of Y, then C is called an ordering cone. If C is a pointed and convex cone, then the ordering relation $\leq_C$ is a partial order. If the interior $intC$ of C is nonempty, we can define a strict ordering relation "$\leq_{intC}$" in Y as follows: for any $y, z \in Y$,

$$y \leq_{intC} z \iff z - y \in intC.$$

Similarly, we can define an ordering relation "$\geq_C$" and a strict ordering relation "$\geq_{intC}$".

By (Y, C), we denote an ordered space with the ordering of Y defined by set C. Suppose that $intC \neq \emptyset$. We can define an ordering relation "$\not\leq_C$"and a strict ordering relation "$\not\leq_{intC}$" as follows: for any $y, z \in Y$

$$y \not\leq_C z \iff z - y \not\leq_C 0;$$

$$y \not\leq_{intC} z \iff z - y \notin intC.$$

Similarly, we can define an ordering relation "$\not\geq_C$" and a strict ordering relation "$\not\geq_{intC}$".

We also define the following ordering relations: for any $y, z \in Y$,

$$y \leq_{C\backslash\{0\}} z \iff z - y \in C\backslash\{0\},$$

$$y \not\leq_{C\backslash\{0\}} z \iff z - y \notin C\backslash\{0\}.$$

Given two subsets of Y, say A and B, the following ordering relationships on sets are defined:

$$\begin{aligned}
&A \leq_C B \iff \eta \leq_C \xi, \quad \forall \eta \in A,\ \xi \in B;\\
&A \leq_{intC} B \iff \eta \leq_{intC} \xi, \quad \forall \eta \in A,\ \xi \in B;\\
&A \leq_{C\backslash\{0\}} B \iff \eta \leq_{C\backslash\{0\}} \xi, \quad \forall \eta \in A,\ \xi \in B;\\
&A \not\leq_C B \iff \eta \not\leq_C \xi, \quad \forall \eta \in A,\ \xi \in B;\\
&A \not\leq_{intC} B \iff \eta \not\leq_{intC} \xi, \quad \forall \eta \in A,\ \xi \in B;\\
&A \not\leq_{C\backslash\{0\}} B \iff \eta \not\leq_{C\backslash\{0\}} \xi, \quad \forall \eta \in A, \xi \in B.
\end{aligned}$$

Let A and B be two sets. We denote by $A\backslash B$ the difference of A and B.

Lemma 1.1. *Let C be an ordering cone in Y. Then, for any $a, b, c \in Y$,*

(i) $a \geq_C b \implies a + c \geq_C b + c$;
(ii) $a \geq_{intC} b \implies a + c \geq_{intC} b + c$;
(iii) $a \geq_{C\setminus\{0\}} b \implies a + c \geq_{C\setminus\{0\}} b + c$;
(iv) $a \not\geq_C b \implies a + c \not\geq_C b + c$;
(v) $a \not\geq_{intC} b \implies a + c \not\geq_{intC} b + c$;
(vi) $a \not\geq_{C\setminus\{0\}} b \implies a + c \not\geq_{C\setminus\{0\}} b + c$.

The same is true for $\leq_C, \leq_{intC}, \leq_{C\setminus\{0\}}, \not\leq_C, \not\leq_{intC}$ and $\not\leq_{C\setminus\{0\}}$ respectively.

Lemma 1.2. *Let C be a convex ordering cone in Y. Then, for any $a, b, c \in Y$,*

(i) $a \leq_C b \leq_C c \implies a \leq_C c$;
(ii) $a \leq_C b \leq_{C\setminus\{0\}} c \implies a \leq_{C\setminus\{0\}} c$;
(iii) $a \leq_C b \leq_{intC} c \implies a \leq_{intC} c$;
(iv) $a \not\leq_{intC} b \geq_{intC} c \implies a \not\geq_{intC} c$;
(v) $a \not\leq_{intC} b \geq_C c \implies a \not\geq_{intC} c$;
(vi) $a \not\geq_{intC} b \leq_{intC} c \implies a \not\leq_{intC} c$;
(vii) $a \not\geq_{intC} b \leq_C c \implies a \not\leq_{intC} c$.

Let Y^* be the topological dual space of Y and C a convex cone of Y. Set

$$C^* = \{f \in Y^* : \langle f, x \rangle \geq 0, \forall x \in C\},$$

where $\langle f, x \rangle$ denotes the value of f at x. C^* is called the dual cone (or positive polar cone) of C. Sometimes, we also use C^+ to denote the dual cone of C.

We set

$$C^{+i} = \{f \in Y^* : \langle f, x \rangle > 0, \forall x \in C\setminus\{0\}\}.$$

Proposition 1.3. *[96] Let (Y, C) be an ordered Banach space with $C \subset Y$ being a convex cone. Consider the following properties that a convex cone $C \subset Y$ may possess:*

(i) *C is a pointed and convex cone;*
(ii) *C has a base;*
(iii) *$intC^* \neq \varnothing$.*

Then (iii) $\implies$ (ii) $\implies$ (i); if Y is some Euclidean space, and C is closed, then all three properties are equivalent.

Definition 1.4. *Let Y be a topological vector space ordered by a convex cone C in Y or a convex subset C of Y with $0 \in \partial C$. Let $A \subset Y$ be a nonempty set. A point $y^* \in A$ is called a minimal point of A if*

$$(A - y^*) \cap (-C\setminus\{0\}) = \emptyset;$$

A point $y^* \in A$ is called a maximal point of A if

$$(A - y^*) \cap (C\setminus\{0\}) = \emptyset.$$

We denote the set of all minimal points of A and the set of all maximal points of A by $\mathrm{Min}_C A$ and $\mathrm{Max}_C A$, respectively.

Definition 1.5. *Let Y be a topological vector space ordered by a convex cone C in Y. Let A be a nonempty subset of Y. A is said to have the lower (upper) domination property if, for each y, there is a point $y^* \in Min_C A$ (or $Max_C A$) such that $y \in y^* + C$ (or $y \in y^* - C$).*

Proposition 1.6. *Let Y be a topological vector space ordered by a closed and convex cone C in Y. If $A \subset Y$ is a nonempty compact set, then A has the lower (upper) domination property, hence $Min_C A \neq \varnothing$ ($Max_C A \neq \varnothing$).*

Thus, we obtain immediately that $A \subset \mathrm{Min}_C A + C$ (or $A \subset \mathrm{Max}_C A - C$).

Definition 1.7. *Let $C \subset Y$ be a convex cone or a convex subset of Y with $0 \in \partial C$ and $intC \neq \emptyset$, $A \subset Y$ be a nonempty subset. A point $y^* \in A$ is called a weakly minimal point of A if*

$$A \cap (y^* - intC) = \varnothing.$$

A point $y^* \in A$ is called a weakly maximal point of A if

$$A \cap (y^* + intC) = \varnothing.$$

We denote the set of all weakly minimal points of A and the set of all weakly maximal points of A by $\mathrm{Min}_{intC} A$ and $\mathrm{Max}_{intC} A$, respectively.

Definition 1.8. *Let Y be a topological vector space ordered by a convex cone C or a convex subset C with $0 \in \partial C$. Let $K \subset X$ and $f : K \to Y$ be a vector-valued function. $x^* \in K$ is said to be a minimal solution of f on K if*

$$(f(K) - f(x^*)) \cap (-C \backslash \{0\}) = \emptyset.$$

Suppose that $intC \neq \emptyset$. $x^* \in K$ is said to be a weakly minimal solution of f on K if

$$(f(K) - f(x^*)) \cap (-intC) = \emptyset.$$

We denote the set of all minimal solutions of f on K and the set of all weakly minimal solutions of f on K by $\mathrm{Min}_C(f, K)$ and $\mathrm{Min}_{intC}(f, K)$ respectively.

Definition 1.9. *Let Y be a topological vector space ordered by a convex cone C or a convex subset C with $0 \in \partial C$. Let $K \subset X$ and $f : K \to Y$ be a vector-valued function. $y^* \in K$ is said to be a minimal point of f on K if there is a $x^* \in Min_C(f, K)$ such that $y^* = f(x^*)$. $y^* \in K$ is said to be a weakly minimal point of f on K if there is a $x^* \in Min_{intC}(f, K)$ such that $y^* = f(x^*)$.*

We denote the set of all minimal points of f on K and the set of all weakly minimal points of f on K by $\mathrm{Min}_C f(K)$ and $\mathrm{Min}_{intC} f(K)$ respectively.

Definition 1.10. *Let* Y *be a topological vector space ordered by a convex cone* C *or a convex subset* C *with* $0 \in \partial C$. *Let* $K \subset X$ *and* $f : K \longrightarrow Y$ *be a vector-valued function.* $x^* \in K$ *is said to be a local minimal solution of* f *on* K *if there exists a neighborhood* $U(x^*)$ *of* x^* *such that*

$$(f(K \cap U(x^*)) - f(x^*)) \cap (-C\backslash\{0\}) = \emptyset.$$

Suppose that $intC \neq \emptyset$. $x^* \in K$ *is said to be a local weakly minimal solution of* f *on* K *if there exists a neighborhood* $U(x^*)$ *of* x^* *such that*

$$(f(K \cap U(x^*)) - f(x^*)) \cap (-intC) = \emptyset.$$

Let $C : Y \rightrightarrows Y$ be a set-valued function (i.e., for every $y \in Y$, $C(y)$ is a subset of Y) such that for each $y \in Y$, $C(y)$ is a convex cone or a convex set with $0 \in \partial C(y)$, for all $y \in Y$.

The set-valued function C or the family of sets $\{C(y) : y \in Y\}$ is called a domination structure on Y. The domination structure describes a variable ordering structure or a variable preference structure when dealing with minimal points of a set.

We define relations $\leq_{C(y)}$, $\not\leq_{C(y)}$, $\leq_{C(y)\backslash\{0\}}$, $\not\leq_{C(y)\backslash\{0\}}$, $\leq_{intC(y)}$, and $\not\leq_{intC(y)}$ with respect to the convex cone $C(y)$ as follows: for any $y_1, y_2 \in Y$,

$$y_1 \leq_{C(y)} y_2 \iff y_2 - y_1 \in C(y);$$

$$y_1 \not\leq_{C(y)} y_2 \iff y_2 - y_1 \notin C(y);$$

$$y_1 \leq_{C(y)\backslash\{0\}} y_2 \iff y_2 - y_1 \in C(y) \setminus \{0\};$$

$$y_1 \not\leq_{C(y)\backslash\{0\}} y_2 \iff y_2 - y_1 \notin C(y) \setminus \{0\};$$

$$y_1 \leq_{intC(y)} y_2 \iff y_2 - y_1 \in intC(y);$$

$$y_1 \not\leq_{intC(y)} y_2 \iff y_2 - y_1 \notin intC(y).$$

Similarly, we can define $\geq_{C(y)}$, $\not\geq_{C(y)}$, $\geq_{C(y)\backslash\{0\}}$, $\not\geq_{C(y)\backslash\{0\}}$, $\geq_{intC(y)}$, and $\not\geq_{intC(y)}$.

Yu [221] proposed the following solution concepts for vector optimization problems with a variable domination structure.

Definition 1.11. *Let* $C : Y \rightrightarrows Y$ *be convex cone valued. Let* A *be a nonempty subset of* Y. *A point* $y^* \in A$ *is called a nondominated minimal point of* A *if*

$$A \cap (y^* - C(y)) = \{y^*\}, \quad \forall y \in A.$$

We denote the set of all nondominated minimal points of A by $\mathrm{Min}_{C(y)}A$. It is clear that a nondominated minimal point of A is a minimal point of A with respect to $C(y)$ for every $y \in A$.

Definition 1.12. *Let A be a nonempty subset of Y. Let $C : Y \rightrightarrows Y$ be convex cone valued with $intC(y) \neq \emptyset, \forall y \in Y$. A point $y^* \in A$ is called a weakly nondominated minimal point of A if*

$$A \cap (y^* - intC(y)) = \varnothing, \quad \forall y \in A. \tag{1.1}$$

We denote the set of all weakly nondominated minimal points of A by $\mathrm{Min}_{intC(y)}A$.

Using the ordering notation, (1.1) is equivalent to that, for any $y_1, y_2 \in A$, it follows that

$$y_1 \nleq_{intC(y_2)} y^*.$$

In fact, Definitions 1.11 and 1.12 deal with a similar "minimal" case as in Definitions 1.4 and 1.7. By the same way, we can define a nondominated maximal point and a weakly nondominated maximal point of A similar to "maximal" in Definitions 1.4 and 1.7.

We propose the following alternative concepts of nondominated minimal points for vector optimization problems with a variable domination structure.

Definition 1.13. *Let $C : Y \rightrightarrows Y$ be convex set valued or convex cone valued, and int $C(y) \neq \emptyset, \forall y \in Y$. Let A be a nonempty subset in Y. A point $y^* \in A$ is called a nondominated-like minimal point of A, if*

$$(A - y^*) \cap (-C(y^*)\backslash\{0\}) = \emptyset.$$

A point y^* is called to be a weakly nondominated-like minimal point of A, if

$$(A - y^*) \cap (-intC(y^*)) = \varnothing.$$

We denote the set of all nondominated-like minimal points of A and the set of all weakly nondominated-like minimal points of A by $\mathrm{LMin}_{C(y)}A$ and $\mathrm{LMin}_{intC(y)}A$, respectively.

The following example shows that the two definitions of weakly nondominated minimal points given in Definitions 1.12 and 1.13 may be different.

Example 1.14. Let $Y = \mathbb{R}^2$ be a 2-dimensional Euclidean space, and $A = \{(y_1, y_2)^\top \in \mathbb{R}^2 : 1 \leq y_1 \leq 2, y_2 = 1\}$. Let

$$C(y) = \{(d_1, d_2)^\top \in \mathbb{R}^2 : d_2 + kd_1 \geq 0, d_1 \geq 0\},$$

where $y = (2-k, 1)^\top,\ 0 \leq k \leq 1$. It is easy to verify that only $y^1 = (1,1)^\top$ is a weakly nondominated minimal point of A. But, by definition, both $y_1 = (1,1)^\top$ and $y_2 = (2,1)^\top$ are weakly nondominated-like minimal points of A.

Let $C : X \rightrightarrows Y$ be a set-valued function such that for each $x \in Y$, $C(x)$ is a nonempty convex cone or a nonempty convex set with $0 \in \partial C(x)$, for all $x \in X$.

The set-valued function C or the family of sets $\{C(x) : x \in X\}$ is also called a domination structure on Y. The domination structure describes a variable ordering structure or a variable preference structure in vector optimization problems with an objective function.

We define relations $\leq_{C(x)}$, $\not\leq_{C(x)}$, $\leq_{C(x)\setminus\{0\}}$, $\not\leq_{C(x)\setminus\{0\}}$, $\leq_{intC(x)}$, and $\not\leq_{intC(x)}$ with respect to the convex cone $C(x)$ as follows: for any $y_1, y_2 \in Y$,

$$y_1 \leq_{C(x)} y_2 \iff y_2 - y_1 \in C(x);$$

$$y_1 \not\leq_{C(x)} y_2 \iff y_2 - y_1 \notin C(x);$$

$$y_1 \leq_{C(x)\setminus\{0\}} y_2 \iff y_2 - y_1 \in C(x) \setminus \{0\};$$

$$y_1 \not\leq_{C(x)\setminus\{0\}} y_2 \iff y_2 - y_1 \notin C(x) \setminus \{0\};$$

$$y_1 \leq_{intC(x)} y_2 \iff y_2 - y_1 \in intC(x);$$

$$y_1 \not\leq_{intC(x)} y_2 \iff y_2 - y_1 \notin intC(x).$$

Similarly, we can define $\geq_{C(x)}$, $\not\geq_{C(x)}$, $\geq_{C(x)\setminus\{0\}}$, $\not\geq_{C(x)\setminus\{0\}}$, $\geq_{intC(x)}$, and $\not\geq_{intC(x)}$.

Definition 1.15. *Let $C : X \rightrightarrows Y$ be convex set valued with $0 \in \partial C(x), \forall x \in X$ or convex cone valued. Suppose that $K \subset X$ and $f : K \to Y$ is a vector-valued function. $x^* \in K$ is said to be a nondominated-like minimal solution of f with respect to $C(x)$ if*

$$(f(K) - f(x^*)) \cap (-C(x^*)\setminus\{0\}) = \emptyset.$$

The set of all nondominated-like minimal solutions of f with respect to $C(x)$ is denoted by $LMin_{C(x))}f(K)$.

Suppose that $intC(x) \neq \emptyset, \forall x \in X$. $x^* \in K$ is said to be a weakly nondominated-like minimal solution of f with respect to $C(x)$ if

$$(f(K) - f(x^*)) \cap (-intC(x^*)) = \emptyset.$$

The set of all weakly nondominated-like minimal solutions of f with respect to $C(x)$ is denoted by $LMin_{intC(x)}f(K)$.

Definition 1.16. *Let (Y, C) be an ordered Hausdorff topological vector space and $A \subset Y$. A point $z \in A$ is called an infimum point of A if,*

(i) $y \not\leq_{C\setminus\{0\}} z$, $\forall y \in A$ and

(ii) there exists a sequence $\{z_k\} \subset A$ such that $z_k \to z$ as $k \to \infty$.

We denote by $InfA$ the set of infimum points of A.

A point $z \in A$ is called a supremum point of A if,

(i) $y \not\geq_{C\setminus\{0\}} z$, $\forall y \in A$ and

(ii) there exists a sequence $\{z_k\} \subset A$ such that $z_k \to z$ as $k \to \infty$.

We denote by $SupA$ the set of supremum points of A.

Clearly, if z is a minimal point of A, then z is an infimum point of A.

Definition 1.17 ([177]). *Let (Y, C) be an ordered vector space, and $A \subset Y$ be nonempty. $a_0 \in A$ is called an upper bound of A if $a_0 \geq_C a, \forall a \in A$. If a_0 is an upper bound of A and $a_0 \leq_C b$ for any upper bound b of A, then a_0 is unique and called the absolute supremum (least upper bound) of A. We denote $a_0 = ASup_C A$. Similarly, we can define the absolute infimum (largest lower bound) of A and denote it by $AInf_C A$.*

Definition 1.18 (Luc [142]).

(**i**) *The cone C is called Daniell if any decreasing sequence having a lower bound converges to its infimum;*
(**ii**) *A subset A of Y is said to be minorized, if there is a $y \in Y$ such that $A \subset \{y\} + C$.*

Consider the scalar optimization problem:

$$\text{(P)} \quad \min_{x \in K} \varphi(x),$$

where $K \subset X$ is a nonempty set and $\varphi : X \to \mathbb{R}$ is a real-valued function.

(i) $x^* \in K$ is called an optimal solution of (P) if

$$\varphi(x^*) \leq \varphi(x), \quad \forall x \in K.$$

(ii) $x^* \in K$ is called a local optimal solution of (P) if there exists a neighborhood $U(x^*)$ of x^* such that

$$\varphi(x^*) \leq \varphi(x), \quad \forall x \in K \cap U(x^*).$$

1.2 Elements of Set-Valued Analysis

In this section, we present necessary concepts and results in set-valued analysis. More detailed investigation of set-valued analysis can be found in Aubin and Frankowska [6] and Aubin and Ekeland [5]. Some particular concepts and results of set-valued analysis are presented in the following context.

Let X, Y be two Hausdorff topological spaces and $F : X \rightrightarrows Y$ a set-valued function.

Definition 1.19. *F is said to be closed if its graph*

$$Gr(F) = \{(x, y) : x \in X, y \in F(x)\}$$

is closed.

Definition 1.20. (**i**) *F is said to be upper semicontinuous (u.s.c. in short) at $x_0 \in X$ if, for any neighborhood $V(F(x_0))$ of the set $F(x_0)$, there exists a neighborhood $U(x_0)$ of the point x_0 such that*

$$F(x) \subset V(F(x_0)), \quad \forall x \in U(x_0).$$

(ii) *F is said to be lower semicontinuous (l.s.c., in short) at $x_0 \in X$ if, for any $y \in F(x_0)$ and any neighborhood $V(y_0)$ of y_0, there exists a neighborhood $U(x_0)$ of the point x_0 such that*

$$F(x) \cap V(y_0) \neq \varnothing, \quad \forall x \in U(x_0).$$

(iii) *F is said to be continuous at x_0 if F is both u.s.c. and l.s.c. at x_0.*
(iv) *F is said to be continuous on X if it is continuous at every $x \in X$.*

Proposition 1.21. *[5] Let X be a topological space and Y a locally convex topological vector space. Suppose that $F : X \rightrightarrows Y$ is a set-valued function which is u.s.c., nonempty and closed-valued. Then F is closed.*

Definition 1.22. *A set-valued function $F : X \rightrightarrows Y$ is said to have open lower sections if the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is open in X for every $y \in Y$.*

Proposition 1.23 (Tian [192]). *Let X be a topological space and Y a convex set of a topological vector space. Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ be set-valued functions with open lower sections. Then*

(i) *the set-valued function $M : X \rightrightarrows Y$, defined by $M(x) = co(G(x))$ for all $x \in X$, has open lower sections;*
(ii) *the set-valued function $Q : X \rightrightarrows Y$, defined by $Q(x) = G(x) \cap F(x)$ for all $x \in X$, has open lower sections.*

Definition 1.24. *Let $F : Y \rightrightarrows Y$ be a set-valued function.*

(i) The vector-valued function $e : Y \to Y$ is said to be a selection of F if $e(y) \in F(y)$, for every $y \in Y$.

(ii) $e : Y \to Y$ is said to be a continuous selection of F if e is a selection of F and e is continuous on Y.

Theorem 1.25 (Generalized Browder Selection Theorem). *Let K be a nonempty compact subset of a Hausdorff topological vector space, and let V be a subset of a topological vector space. Suppose that $H : K \rightrightarrows V$ is a set-valued function with nonempty convex values and has open lower sections. Then there exists a continuous selection $h : K \to V$ of H. Moreover, $h(K)$ is contained in the convex hull of a finite subset $M \subset V$.*

Proof. For each $v \in V$, $H^{-1}(v)$ is open, and each point $x \in K$ lies in at least one of these open subsets. Since K is compact, there exists a finite set $M = \{v_1, \cdots, v_k\} \subset V$ such that $K = \cup_{i=1}^{k} H^{-1}(v_i)$. Let $\{\beta_1, \cdots, \beta_k\}$ be a partition of unit subordinated to this covering, i.e., each β_i is a continuous function from K to $[0, 1]$, which vanishes outside of $H^{-1}(v_i)$, while $\sum_{i=1}^{k} \beta_i(x) = 1$ for all x in K.

Now, we define the continuous function $h : K \to co(M)$ by $h(x) := \sum_{i=1}^{k} \beta_i(x) v_i$. Clearly, $\beta_i(x) > 0$ implies that $x \in H^{-1}(v_i)$ and therefore $v_i \in H(x)$. Thus $h(x)$ is a convex linear combination of points of $H(x)$. Since $H(x)$ is assumed to be convex for each $x \in K$, it follows that $h(x) \in H(x)$. The theorem is proved. ■

Theorem 1.26 (Browder Fixed Point Theorem). *Let K be a nonempty, compact and convex subset of a Hausdorff topological vector space. Suppose $F : K \rightrightarrows K$ is a set-valued function with nonempty convex values and open lower sections. Then F has a fixed point in K.*

Theorem 1.27 (Fan-Glicksber-Kakutani). *Let K be a nonempty compact subset of a real locally convex Hausdorff vector topological space. If $F : K \rightrightarrows K$ is upper semi-continuous and, for any $x \in K$, $F(x)$ is a nonempty, convex and closed subset, then F has a fixed point in K.*

Definition 1.28. *A nonempty topological space is said to be acyclic if all of its reduced Čech homology groups over the rational vanish.*

In particular, any contractible space is acyclic, and hence any convex or star-shaped set in a topological vector space is acyclic.

Theorem 1.29. *[159] Let K be a compact convex subset of a locally convex Hausdorff topological vector space, and $F : K \rightrightarrows K$ be an upper semicontinuous set-valued function with nonempty, closed and acyclic values. Then F has a fixed point in K.*

Now, we introduce the concepts of the contingent tangent cone of a set and the contingent derivative of a set-valued function.

Let X and Y be two topological vector spaces, and K a nonempty subset of X.

Definition 1.30. *Let $\bar{x} \in K$. The set $T(K, \bar{x}) \subset X$ is called a contingent tangent cone to K at $\bar{x}$ if*

$$T(K, \bar{x}) = \{x \in X : \exists \{x_k\} \subset X \text{ and } \{h_k\} \subset \mathbb{R}_+ \backslash \{0\}, \\ \text{s.t. } x_k \to x, h_k \to 0 \text{ and } \bar{x} + h_k x_k \in K, \forall k\}.$$

We know that (i) if X is a normed space, then $T(K, \bar{x})$ is closed and (ii) if K is a convex set, then $T(K, \bar{x})$ is also convex.

Obviously, if $(X, ||\cdot||)$ is a normed space, then

$$T(K, \bar{x}) = \cap_{\epsilon>0} \cap_{\alpha>0} \cap_{0<h<\alpha} (((K - \bar{x})/h) + \epsilon B),$$

where $B = \{x \in X : ||x|| = 1\}$.

Definition 1.31. *[6] Let $G : X \rightrightarrows Y$ be a set-valued function, and let $(\bar{x}, \bar{y})$ be a point of $Gr(G)$. We denote by $DG(\bar{x}, \bar{y})$ the set-valued function from X to Y whose graph is the contingent tangent cone $T(Gr(G), (\bar{x}, \bar{y})) \subset X \times Y$. $DG(\bar{x}, \bar{y})$ is called the contingent derivative of G at $(\bar{x}, \bar{y})$.*

It is useful to note that $y \in DG(\bar{x}, \bar{y})(x)$ if and only if there exist $\{h_k\} \subset \mathbb{R}_+ \backslash \{0\}$ and $\{(x_k, y_k)\} \subset X \times Y$, such that $h_k \to 0$, $(x_k, y_k) \to (x, y)$ and $\bar{y} + h_k y_k \in G(\bar{x} + h_k x_k)$ for all n.

Let X and Y be two Banach spaces. We denote by $L(X, Y)$ the set of all linear continuous operators from X to Y. The value of a linear operator $f : X \to Y$ at a point x is denoted by $\langle f, x \rangle$. For any $A \in L(X, Y)$, we introduce a norm

$$||A||_L = \sup\{||A(x)|| : ||x|| \leq 1\}.$$

Since Y is a Banach space, $L(X, Y)$ is also a Banach space with the norm $|| \cdot ||_L$ (or $\| \cdot \|$ in short).

Definition 1.32. *Let $f : K \subset X \to L(X, Y)$ be a vector-valued function. f is said to be Fréchet differentiable at $x_0 \in K$ if there exists a linear continuous operator $\Phi : X \to L(X, Y)$, such that*

$$\lim_{x \to x_0} \frac{||f(x) - f(x_0) - \Phi(x - x_0)||}{||x - x_0||} = 0.$$

Φ is called the Fréchet derivative of f at x_0. If f is Fréchet differentiable at every x of K, f is said to be Fréchet differentiable on K.

Definition 1.33. *Let $f : K \subset X \to Y$ be a vector-valued function. f is said to be Gâteaux differentiable at $x_0 \in K$ if there exists a linear function $Df(x_0) : X \to Y$ such that, for any $v \in X$,*

$$\langle Df(x_0), v \rangle = \lim_{t \searrow 0} \frac{f(x_0 + tv) - f(x_0)}{t}.$$

$Df(x_0)$ is called the Gâteaux derivative of f at x_0. If f is Gâteaux differentiable at every x of K, f is said to be Gâteaux differentiable on K.

Theorem 1.34 (Knaster, Kuratowski and Mazurkiewicz (KKM, in short) Theorem). *Let E be a subset of a topological vector space V. For each $x \in E$, let a closed and convex set $F(x)$ in V be given such that $F(x)$ is compact for at least one $x \in E$. If the convex hull of every finite subset $\{x_1, x_2, \cdots, x_k\}$ of E is contained in the corresponding union $\cup_{i=1}^n F(x_i)$, then $\cap_{x \in E} F(x) \neq \emptyset$.*

A set-valued $F : E \rightrightarrows E$ function is called a KKM map if we have $co\{x_1, \cdots, x_k\} \subset \cup_{i=1}^k F(x_i)$ for every finite subset $\{x_1, \cdots, x_k\}$ of E.

Definition 1.35. *Let T be a mapping from X into $L(X, Y)$. T is called v-hemicontinuous if, for every $x, y \in X$, the mapping $t \to \langle T(x + ty), y \rangle$ is continuous at 0^+.*

Definition 1.36. *Let X and Y be topological vector spaces, $C \subset Y$ be a nonempty convex cone with $intC \neq \varnothing$ and $C \neq \{0\}$ or Y. Let $T : X \to L(X,Y)$ be a mapping.*

(i) *T is called C-monotone, if, for every $x, y \in X$,*

$$\langle T(x) - T(y), x - y \rangle \geq_C 0;$$

(ii) *T is called strictly C-monotone, if, for every x, $y \in X$ and $x \neq y$,*

$$\langle T(x) - T(y), x - y \rangle \geq_{intC} 0.$$

Definition 1.37. *Let X and Y be topological vector spaces and $C \subset Y$ be a convex cone. The set-valued function $T : X \rightrightarrows L(X,Y)$ is said to be C-monotone if and only if*

$$\langle u_2 - u_1, y - x \rangle \geq_C 0, \quad \forall x, y \in X, u_1 \in F(x), u_2 \in F(y).$$

It is clear that any selection of a C-monotone set-valued function is also C-monotone.

Definition 1.38. *Let X and Y be Banach spaces, $C \subset Y$ be a convex cone with nonempty interior $intC$ and $intC^* \neq \varnothing$. Let K be a convex and unbounded subset of X. We say that a mapping $T : K \to L(X,Y)$ is weakly coercive on K if there exist $x_0 \in K$ and $c \in intC^*$ such that*

$$\langle c \circ T(x) - c \circ T(x_0), x - x_0 \rangle / ||x - x_0|| \to +\infty,$$

whenever $x \in K$ and $||x|| \to +\infty$.

It is easy to see that if $Y = \mathbb{R}$, then $L(X,Y) = X^*$, $intC^* = \mathbb{R}_+ = \{\alpha \in \mathbb{R} : \alpha > 0\}$, and the weakly coercive condition reduces a standard coercive condition in "scalar" variational inequality.

1.3 Nonlinear Scalarization Functions

A useful approach for analyzing a vector optimization problem is to reduce it to a scalar optimization problem. Nonlinear scalarization functions play an important role in this reduction in the context of nonconvex vector optimization problems.

Let Y be a Hausdorff topological vector space, $C \subset Y$ a closed and convex cone of Y with nonempty interior $intC$.

Definition 1.39. *A function $\psi : Y \to I\!R$ is monotone if, for any $y_1, y_2 \in Y$,*

$$y_1 \geq_C y_2 \Longrightarrow \psi(y_1) \geq \psi(y_2).$$

A function $\psi : Y \to I\!R$ is strictly monotone if, for any $y_1, y_2 \in Y$,

$$y_1 \geq_{intC} y_2 \Longrightarrow \psi(y_1) > \psi(y_2).$$

A function $\psi : Y \to I\!R$ is strongly monotone if, for any $y_1, y_2 \in Y$,

$$y_1 \geq_{C \setminus \{0\}} y_2 \Longrightarrow \psi(y_1) > \psi(y_2).$$

The following nonlinear scalarization function is of fundamental importance to our analysis. The original version is due to Gerstewitz [77]. Its first appearance in English seems to be due to Luc [142].

Definition 1.40. *Given a fixed $e \in intC$ and $a \in Y$, the nonlinear scalarization function is defined by:*

$$\xi_{ea}(y) = \min\{t \in I\!R : y \in a + te - C\}, \quad y \in Y. \tag{1.2}$$

Proposition 1.41. *The function ξ_{ea} is well-defined, that is, the minimum in (1.2) is attained.*

Proof. For any $y \in Y$, define

$$L = \{\ \lambda \in \mathbb{R} :\ y \in \lambda e - C\}.$$

It is sufficient to show that L is bounded from below and a closed subset in $\mathbb{R}$.

Suppose that

$$\{\lambda_k\} \subset L \text{ and } \lambda_k \to \lambda^*, \qquad \text{as } k \to +\infty.$$

We have

$$\lambda_k e - y \in C, \quad \forall k.$$

By the closedness of C, we have

$$\lambda^* e - y \in C.$$

It implies that $\lambda^* \in L$. Thus, L is closed.

Assume that, for each $r \in \mathbb{R}$, there exist $\lambda_r \in \mathbb{R}$ such that $\lambda_r < r$ and $y \in \lambda_r e - C$. By Lemma 1.51 (ii), there exists $\alpha \in \mathbb{R}$ such that $y \notin \alpha e - C$. By Lemma 1.51(iii),

$$y \notin \mu e - C, \quad \forall \mu < \alpha,$$

a contradiction. Thus, L is bounded from below. ■

If Y is the ℓ-dimensional Euclidean space $\mathbb{R}^\ell$, and $C = \mathbb{R}^\ell_+$, $e = (e_1, e_2, \cdots, e_\ell)^\top$, $a = (a_1, a_2, \cdots, a_\ell)^\top$, then the function ξ_{ea} may be rewritten as

$$\xi_{ea}(y) = \max\{(y_i - a_i)/e_i : 1 \le i \le \ell\}, \text{ for } y = (y_1, y_2, \cdots, y_\ell)^\top.$$

It can be verified that ξ_{ea} is a continuous and convex function on Y, and it is monotone and strictly monotone.

Remark 1.42. The function ξ_{ea} plays an important role in many areas of multicriteria, or vector optimization problems. Note, however, that the function ξ_{ea} is not strongly monotone. It is for this reason that the function ξ_{ea} is more useful in dealing with weakly minimal points.

Proposition 1.43. *For any fixed $e \in intC$, $y \in Y$ and $r \in \mathbb{R}$, we have*

(i) $\xi_{e0}(y) < r \iff y \in re - intC$;
(ii) $\xi_{e0}(y) \le r \iff y \in re - C$;
(iii) $\xi_{e0}(y) = r \iff y \in re - \partial C$;
(iv) $\xi_{e0}(re) = r$.

Proof. Follows directly from Definition 1.40 of ξ_{ea}. ∎

Sometimes, we denote ξ_{e0} by ξ_e.

Proposition 1.44. *Let $C = \{y \in Y : f(y) \le 0, f \in \Gamma\}$, where $\Gamma \subset Y^* \backslash \{0\}$. Assume that $intC \ne \varnothing$. Let $e \in intC$ and $a \in Y$. Then, for $y \in Y$,*

$$\xi_{ea}(y) = \sup_{f \in \Gamma} \left\{ \frac{f(y) - f(a)}{f(e)} \right\}.$$

Proof. Firstly, we prove that, for all $f \in \Gamma$, $f(e) < 0$. Assume to the contrary, i.e., there exists $f_0 \in \Gamma$ such that $f_0(e) \ge 0$. Since $f_0 \ne 0$ and f_0 is a linear functional, there exists an $y_0 \in Y$ such that $f_0(y_0) < 0$. Observe that $e \in intC$. Thus, if $\alpha > 0$ is small enough, we have $e - \alpha y_0 \in C$. It follows from the definition of C that

$$0 \ge f_0(e - \alpha y_0) = f_0(e) - \alpha f_0(y_0) > 0,$$

a contradiction.

Furthermore, since $y \in a + \xi_{ea}(y)e - C$,

$$f(y - \xi_{ea}(y)e - a) \ge 0, \quad \forall f \in \Gamma.$$

Since f is linear,

$$f(y) - \xi_{ea}(y)f(e) - f(a) \ge 0.$$

As $f(e) < 0$, we have

$$\xi_{ea}(y) \geq \frac{f(y) - f(a)}{f(e)}, \quad \forall f \in \Gamma.$$

Consequently,

$$\xi_{ea}(y) \geq \sup_{f \in \Gamma} \left\{ \frac{f(y) - f(a)}{f(e)} \right\}.$$

Conversely, let

$$t_0 = \sup_{f \in \Gamma} \left\{ \frac{f(y) - f(a)}{f(e)} \right\}.$$

Then

$$\frac{f(y) - f(a)}{f(e)} \leq t_0, \quad \forall f \in \Gamma.$$

Observing that $f(e) < 0$ and that f is linear, we have

$$f(y - a - t_0 e) \geq 0, \quad \forall f \in \Gamma,$$

which implies that $y - a - t_0 e \leq_C 0$ by the definition of C. By the definition of ξ_{ea}, we have

$$t_0 \geq \xi_{ea}(y) = \min\{t \in \mathbb{R} : y \in a + te - C\}. \qquad \blacksquare$$

Corollary 1.45. *Let* $C := \{y \in Y : f_i(y) \leq 0, f_i \in Y^*, i = 1, 2, \cdots, m\}$. *Then*

$$\xi_{ea}(y) = \max_{1 \leq i \leq m} \left\{ \frac{f_i(y) - f_i(a)}{f_i(e)} \right\}, \quad \forall y \in Y,$$

$$\xi_{e0}(y) = \max_{1 \leq i \leq m} \left\{ \frac{f_i(y)}{f_i(e)} \right\}, \quad \forall y \in Y.$$

Corollary 1.46. *Let* $Y = \mathbb{R}^\ell$ *and* $C = \mathbb{R}^\ell_+$, $e = (1, 1, \cdots, 1)^\top \in \mathbb{R}^\ell$. *Then, for any* $a \in \mathbb{R}^\ell$, $y \in \mathbb{R}^\ell$,

$$\xi_{ea}(y) = \max_{1 \leq i \leq \ell} [y_i - a_i],$$

$$\xi_{e0}(y) = \max_{1 \leq i \leq \ell} [y_i],$$

Proof. In Corollary 1.45, let $m = \ell$ and $f_i(y) = -y_i, i = 1, 2, \cdots, \ell$. Thus $C = \{y \in Y : f_i(y) \leq 0, f_i \in Y^* \backslash \{0\}, i = 1, 2, \cdots, \ell\}$. Then the conclusion follows directly from Corollary 1.45. $\blacksquare$

Proposition 1.47. *For* $e \in intC$, $a \in Y$ *and* $b \in -C$,

$$\xi_{ea}(y - b) \geq \xi_{ea}(y),$$

and the equality holds for $b \in C \cap (-C)$.

Proof. The conclusions follow directly from the monotonicity of ξ_{ea}. ■

Now we introduce a nonlinear scalarization function for a variable domination structure.

Let Y be a locally convex Hausdorff topological vector space. Let $C : Y \rightrightarrows Y$ be a set-valued function and, for any $y \in Y$, $C(y)$ be a proper, closed and convex cone with $intC(y) \neq \varnothing$ and $e : Y \to Y$ be a vector-valued function and for any $y \in X$, $e(y) \in intC(y)$. Let Y^* be the dual space of Y, equipped with weak star topology. Let $C^* : Y \rightrightarrows Y^*$ be defined by

$$C^*(y) = \{\phi \in X^* : \langle \phi, z\rangle \geq 0, \quad \forall z \in C(y)\}, \quad \forall y \in Y.$$

Thus, the set

$$B^*(y) = \{\phi \in C^*(y) : \langle \phi, e(y)\rangle = 1\}$$

is a weak star compact base of the cone $C^*(y)$.

Definition 1.48. *The nonlinear scalarization function $\xi : Y \times Y \to I\!R$ is defined by*

$$\xi(y,z) = \min\ \{\lambda \in I\!R : \ z \in \ \lambda e(y) - C(y)\}, \quad (y,z) \in Y \times Y.$$

Remark 1.49. **(i)** Let C be a proper, closed and convex cone in Y with $intC \neq \varnothing$, and let $e \in intC$. Recall that in Definition 1.40

$$\xi_{e0}(z) = \min\{t \in \mathbb{R} : z \in te - C\}, \quad z \in Y.$$

If, for any $y \in Y$, $C(y) = C$ and $e(y) = e$ in Definition 1.48, then $\xi(y,z)$ reduces to $\xi_{e0}(z)$.

(ii) Let $e \in int \bigcap_{y \in Y} C(y) \neq \varnothing$. A nonlinear scalarization function in [42] is defined as

$$\xi_e(y,z) = \inf\{t \in \mathbb{R} : z \in te - C(y)\}. \tag{1.3}$$

We note that if for any $y \in Y$, $e(y) = e$, the function $\xi(y,z)$ reduces to $\xi_e(y,z)$. In the new definition of $\xi(y,z)$ (Definition 1.48), the assumption $int \bigcap_{y \in Y} C(y) \neq \varnothing$ is removed.

Lemma 1.50. *[78] For each $y \in Y$,*

$$Y = \cup\{\lambda e(y) - intC(y) : \ \lambda \in I\!R^+ \backslash \{0\}\}.$$

Lemma 1.51. *For $\lambda \in I\!R$ and $y \in Y$, we set $C_\lambda(y) = \lambda e(y) - \ C(y)$.*

(i) *If $z \in C_\lambda(y)$ holds for some $\lambda \in I\!R$, and $y \in Y$, then*

$$z \in \mu e(y) - intC(y), \quad \textit{for each } \mu > \lambda;$$

moreover,

$$z \in \mu e(y) - C(y), \quad \textit{for each } \mu > \lambda.$$

(ii) *For each $y, z \in Y$, there exists a real number $\lambda \in I\!R$ such that $z \notin C_\lambda(y)$.*
(iii) *Let $z \in Y$. If $z \notin C_\lambda(y)$ for some $\lambda \in I\!R$, and $y \in Y$, then*

$$z \notin C_\mu(y), \text{ for each } \mu < \lambda.$$

Proof. (i) Let $\mu > \lambda$ and let $z \in C_\lambda(y)$ hold for some $y \in Y$. We have

$$\mu e(y) - z = (\mu - \lambda)e(y) + \lambda e(y) - z \in intC(y) + C(y) \subset intC(y).$$

Thus,

$$z \in \mu e(y) - intC(y) \subset \mu e(y) - C(y).$$

(ii) Let us assume that there exist $y_0, z_0 \in Y$ such that, for all $\lambda \in \mathbb{R}$, $z_0 \in C_\lambda(y_0)$. From (i), we have

$$z_0 \in \lambda e(y_0) - intC(y_0), \qquad \text{for all } \lambda \in \mathbb{R}.$$

Thus,

$$\{\lambda e(y_0) - z_0 : \lambda \in \mathbb{R}\} \subset intC(y_0);$$

equivalently,

$$\{-\lambda e(y_0) - z_0 : \lambda \in \mathbb{R}\} \subset intC(y_0).$$

From Lemma 1.50, we have

$$Y = \{\lambda e(y_0) - intC(y_0) : \lambda \in \mathbb{R}^+ \backslash \{0\}\}.$$

Therefore, for each $y \in Y$, there exist $c \in intC(y_0)$ and $\alpha \in \mathbb{R}^+ \backslash \{0\}$ such that

$$-y = \alpha e(y_0) - c;$$

then,

$$\begin{aligned} y &= -\alpha e(y_0) + c \\ &= (-\alpha e(y_0) - z_0) + c + z_0 \\ &\in intC(y_0) + intC(y_0) + z_0 \\ &= z_0 + intC(y_0). \end{aligned}$$

Thus

$$Y \subset z_0 + intC(y_0).$$

This contradicts $C(y_0) \neq Y$.

(iii) Let

$$z \notin C_\lambda(y), \qquad \text{for some } \lambda \in \mathbb{R} \text{ and } y \in Y.$$

Suppose that, for some $\mu < \lambda$, $z \in C_\mu(y)$. From (ii), we have that $z \in C_\lambda(y)$. This contradicts the assumption. ■

Proposition 1.52. *The function* $\xi : Y \times Y \to I\!R$ *is well defined.*

Proof. For any $y, z \in Y$, define

$$L = \{\ \lambda \in \mathbb{R} : \ z \in \lambda e(y) - C(y)\}.$$

It is sufficient to show that L is bounded from below and a closed subset in $\mathbb{R}$.

Suppose that

$$\{\lambda_k\} \subset L \text{ and } \lambda_k \to \lambda^*, \quad \text{as } k \to +\infty.$$

We have

$$\lambda_k e(y_0) - z \in C(y), \quad \forall n.$$

By the closedness of $C(y)$, we have

$$\lambda^* e(y) - z \in C(y).$$

It implies that $\lambda^* \in L$. Thus, L is closed.

Assume that, for each $r \in \mathbb{R}$, there exist $\lambda_r \in \mathbb{R}$ such that $\lambda_r < r$ and $z \in \lambda_r e(y) - C(y)$. By Lemma 1.51 (ii), there exists $\alpha \in \mathbb{R}$ such that $z \notin \alpha e(y_0) - C(y)$. By Lemma 1.51(iii),

$$z \notin \mu e(y) - C(y), \quad \forall \mu < \alpha,$$

a contradiction. Thus, L is bounded from below. ■

Proposition 1.53. *For any* $(y, z) \in Y \times Y$,

$$\xi(y, z) = \max_{\phi \in B^*(y)} \frac{\langle \phi,\ z\rangle}{\langle \phi, e(y)\rangle},$$

where $B^*(y)$ *is a base of* $C^*(y)$.

Proof. We show firstly,

$$\xi(y, z) = \sup\nolimits_{\phi \in C^*(y)\setminus\{0\}} \frac{\langle \phi,\ z\rangle}{\langle \phi, e(y)\rangle}.$$

Since $\xi(y, z) = \min\{\lambda \in \mathbb{R} : z \in \lambda e(y) - C(y)\}$, $z \in \xi(y, z)e(y) - C(y)$, equivalently,

$$\xi(y, z)e(y) - z \in C(y).$$

For any $\phi \in C^*(y)\setminus\{0\} \subset C^*(y)$, we have $\langle \phi, \xi(y, z)e(y) - z\ \rangle \geq 0$, equivalently,

$$\xi(y, z)\langle \phi,\ e(y)\rangle - \langle \phi,\ z\rangle \geq 0.$$

Because $e(y) \in intC(y)$ and $\phi \in C^*(y)\backslash\{0\}$, then, we have $\langle \phi, e(y) \rangle > 0$. So $\xi(y,z) \geq \frac{\langle \phi, y \rangle}{\langle \phi, e(y) \rangle}$. That is to say,

$$\xi(y,z) \geq \sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi,\ z \rangle}{\langle \phi, e(y) \rangle}.$$

On the other hand, let

$$\lambda_0 = \sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi,\ z \rangle}{\langle \phi, e(y) \rangle}.$$

So, for any $\phi \in C^*\backslash\{0\}$, $\lambda_0 \geq \frac{\langle \phi,\ z \rangle}{\langle \phi, e(y) \rangle}$. Since $\langle \phi, e(y) \rangle > 0$, $\lambda_0 \langle \phi, e(y) - z \rangle \geq 0$. Then, $\lambda_0 e(y) - z \in C(y)$, i.e. $z \in \lambda_0 e(y) - C(y)$. From the definition of ξ, $\lambda_0 \geq \xi(y,z) = \min\{\lambda \in \mathbb{R} : z \in \lambda e(y) - C(y)\}$, i.e.,

$$\xi(y,z) \leq \sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi,\ z \rangle}{\langle \phi, e(y) \rangle}.$$

So we have

$$\xi(y,z) = \sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi,\ z \rangle}{\langle \phi, e(y) \rangle}.$$

Since $B^*(y)$ is the base of $C^*(y)$ for any $y \in Y$, $\phi \in C^*(y)\backslash\{0\}$, there are $\lambda > 0$, and $\varphi \in B^*(y)$ such that $\phi = \lambda\varphi$. So for any $y \in Y$,

$$\frac{\langle \phi, y \rangle}{\langle \phi, e(y) \rangle} = \frac{\langle \lambda\varphi, y \rangle}{\langle \lambda\varphi, e(y) \rangle} = \frac{\langle \varphi, y \rangle}{\langle \varphi, e(y) \rangle}.$$

So we have

$$\sup_{\phi \in C^*(y)\backslash\{0\}} \frac{\langle \phi, z \rangle}{\langle \phi, e(y) \rangle} = \sup_{\phi \in B^*(y)} \frac{\langle \phi, z \rangle}{\langle \phi, e(y) \rangle}.$$

i.e.

$$\xi(y,z) = \sup_{\phi \in B^*(y)} \frac{\langle \phi,\ z \rangle}{\langle \phi, e(y) \rangle}.$$

Since $B^*(y)$ is weak star compact, $\xi(y,z) = \max_{\phi \in B^*(y)} \frac{\langle \phi,\ z \rangle}{\langle \phi, e(y) \rangle}$. ■

Proposition 1.54. *For each $r \in \mathbb{R}$ and $y, z \in Y$, the following statements are true.*

(i) $\xi(y,z) < r \Longleftrightarrow z \in re(y) - intC(y)$.
(ii) $\xi(y,z) \leq r \Longleftrightarrow z \in re(y) - C(y)$.
(iii) $\xi(y,z) \geq r \Longleftrightarrow z \notin re(y) - intC(y)$.
(iv) $\xi(y,z) > r \Longleftrightarrow z \notin re(y) - C(y)$.

(v) $\xi(y,z) = r \Longleftrightarrow z \in re(y) - \partial C(y)$.

Proof. We only prove (i). The proofs for other assertions are similar and omitted. Indeed,

$$\begin{aligned}\xi(y,z) < r &\Longleftrightarrow \max_{\phi \in B^*(y)} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle} < r \\ &\Longleftrightarrow \langle \phi, z\rangle < r\langle \phi, e(y)\rangle, \forall \phi \in B^*(y) \\ &\Longleftrightarrow \langle \phi, re(y) - z\rangle > 0, \forall \phi \in B^*(y) \\ &\Longleftrightarrow \langle \phi, re(y) - z\rangle > 0, \forall \phi \in C^*(y)\backslash\{0\} \\ &\Longleftrightarrow re(y) - z \in intC(y) \\ &\Longleftrightarrow z \in re(y) - intC(y).\end{aligned}$$

■

Proposition 1.55. *Let Y be a locally convex Hausdorff topological vector space. Then, for any given $y \in Y$,*

(i) $\xi(y,\cdot)$ *is positively homogenous;*
(ii) $\xi(y,\cdot)$ *is strictly monotone, that is, if $z_1 \geq_{intC(y)} z_2$, then*

$$\xi(y,z_2) < \xi(y,z_1).$$

Proof. (i) Let $\mu > 0$. For $z \in Y$, we have

$$\begin{aligned}\xi(y,\mu z) &= \max_{\phi \in B^*(y)} \frac{\langle \phi, \mu\, z\rangle}{\langle \phi, e(y)\rangle} \\ &= \mu \max_{\phi \in B^*(y)} \frac{\langle \phi, z\rangle}{\langle \phi, e(y)\rangle} \\ &= \mu\, \xi(y,z).\end{aligned}$$

(ii) Let $z_1 \geq_{intC(y)} z_2$. Set $r = \xi(y,z_1)$. By the definition of $\xi(y,z_1)$, we have

$$z_2 \in z_1 - intC(y) \subset re(y) - C(y) - intC(y) \subset re(y) - intC(y).$$

By Proposition 1.54 (i), we have

$$\xi(y,z_2) < r = \xi(y,z_1).$$

■

Proposition 1.56. *For any fixed $y \in Y$, and any $z_1, z_2 \in Y$,*

(i) $\xi(y, z_1 + z_2) \leq \xi(y,z_1) + \xi(y,z_2)$;

(ii) $\xi(y, z_1 - z_2) \geq \xi(y,z_1) - \xi(y,z_2)$.

Proof. (i)

$$\begin{aligned}\xi(y,z_1+z_2) &= \max_{\phi\in B^*(y)} \frac{\langle \phi, z_1+z_2\rangle}{\langle \phi, e(y)\rangle}\\ &\le \max_{\phi\in B^*(y)} \frac{\langle \phi, z_1\rangle}{\langle \phi, e(y)\rangle} + \max_{\phi\in B^*(y)} \frac{\langle \phi, z_2\rangle}{\langle \phi, e(y)\rangle}\\ &= \xi(y,z_1)+\xi(y,z_2).\end{aligned}$$

(ii) It follows from (i) that

$$\xi(y,z_1) = \xi(y,z_1-z_2+z_2) \le \xi(y,z_1-z_2)+\xi(y,z_2).$$

Then, $\xi(y,z_1)-\xi(y,z_2) \le \xi(y,z_1-z_2)$. This implies that (ii) holds. ∎

Theorem 1.57. *Let Y be a locally convex Hausdorff topological vector space, and let $C : Y \rightrightarrows Y$ be a set-valued function such that for each $y \in Y$, $C(y)$ is a proper, closed, convex cone in Y with $intC(y) \neq \varnothing$. And let $e : Y \to Y$ be a continuous selection of the set-valued function $intC(\cdot)$. Define a set-valued function $W : Y \rightrightarrows Y$ by $W(y) = Y\backslash intC(y)$, for $y \in Y$. We have*

(i) *If W is upper semi-continuous, then $\xi(\cdot,\cdot)$ is upper semi-continuous on $Y \times Y$;*
(ii) *If C is upper semi-continuous, then $\xi(\cdot,\cdot)$ is lower semi-continuous on $Y \times Y$.*

Proof. (i) In order to show that $\xi(\cdot,\cdot)$ is upper semi-continuous, we must check, for any $\lambda \in \mathbb{R}$, the set

$$A := \{(y,z) \in Y\times Y : \xi(y,z) \ge r\}$$

is closed. Let $(y_\alpha, z_\alpha) \in A$ and $(y_\alpha, z_\alpha) \to (y_0, z_0)$. We have $\xi(y_\alpha, z_\alpha) \ge r$, that is to say, by Proposition 1.54 (iii), that

$$z_\alpha \notin re(y_\alpha) - intC(y_\alpha).$$

Namely, $re(y_\alpha) - z_\alpha \in Y\backslash intC(y_\alpha) = W(y_\alpha)$. Since $e(\cdot)$ is continuous on Y, $(re(y_\alpha) - z_\alpha, y_\alpha) \to (re(y_0) - z_0, y_0)$. Since W is upper semi-continuous and closed-valued, by Proposition 1.21, W is closed. So $re(y_0) - z_0 \in W(y_0)$. Namely, $z_0 \notin re(y_0) - intC(y_0)$. By Proposition 1.54 (iii), it is equivalent to $\xi(y_0, z_0) \ge r$. So, A is closed, i.e., $\xi(\cdot,\cdot)$ is upper semi-continuous on $Y \times Y$.

(ii) In order to show $\xi(\cdot,\cdot)$ is lower semi-continuous, we must check, for any $\lambda \in \mathbb{R}$, the set

$$B := \{(y,z) \in Y\times Y : \xi(y,z) \le r\}$$

is closed. Let $(y_\alpha, z_\alpha) \in B$ and $(y_\alpha, z_\alpha) \to (y_0, z_0)$. We have $\xi(y_\alpha, z_\alpha) \le r$, it is to say, by Proposition 1.54 (ii),

$$z_\alpha \in re(y_\alpha) - C(y_\alpha).$$

Since $e(\cdot)$ is continuous on Y, $(re(y_\alpha) - z_\alpha, y_\alpha) \to (re(y_0) - z_0, y_0)$. Since $C(\cdot)$ is upper semi-continuous and closed-valued, by Proposition 1.21, C is closed. So $re(y_0) - z_0 \in C(y_0)$. Namely, $z_0 \in re(y_0) - C(y_0)$. By Proposition 1.54 (ii), it is equivalent to $\xi(y_0, z_0) \leq r$. So, B is closed, i.e., $\xi(\cdot,\cdot)$ is lower semi-continuous on $Y \times X$. ■

Remark 1.58. (**i**) If Y is a paracompact space, and $intC^{-1}(x) = \{y \in Y : x \in intC(y)\}$ is an open set and for each $y \in Y$, $intC(y) \neq \varnothing$ and $C(y)$ is convex, by the Browder continuous selection theorem, $intC(\cdot)$ has a continuous selection $e(\cdot)$.
(**ii**) If $e \in int \bigcap_{y \in Y} C(y)$, we could let, for any $y \in Y$, $e(y) = e$. The function e is also continuous.

The following examples are to show that if C (W, respectively) is not upper semi-continuous, then $\xi(\cdot,\cdot)$ is not lower semi-continuous (upper semi-continuous, respectively) even if all the other conditions of Theorem 1.57 are satisfied.

Example 1.59. Let $Y = \mathbb{R}^2$, the 2-dimensional Euclidean space. Let

$$A = cone(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 2, \frac{1}{2} \leq y_1 \leq \frac{3}{2}\}),$$
$$B = cone(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 2, 0 \leq y_1 \leq \frac{3}{2}\}),$$
$$C = cone(\{(y_1, y_2)^\top \in \mathbb{R}^2 : y_1 + y_2 = 2, \frac{1}{2} \leq y_1 \leq 2\}).$$

The set-valued map $C : Y \rightrightarrows Y$ is defined by

$$C((y_1, y_2)^\top) = \begin{cases} A, & \text{if } y_1 = 0; \\ B, & \text{if } y_1 > 0; \\ C, & \text{if } y_1 < 0. \end{cases}$$

Thus,

$$W((y_1, y_2)^\top) = \begin{cases} Y \backslash intA, & \text{if } y_1 = 0; \\ Y \backslash intB, & \text{if } y_1 > 0; \\ Y \backslash intC, & \text{if } y_1 < 0. \end{cases}$$

Let $e = (1, 1)^\top$ and for any $y = (y_1, y_2)^\top \in Y$, $e(y) = e$.

Note that for any $y \in Y$, $intC(y) \neq \varnothing$ and $e \in intC(y)$. We also note that $W(\cdot)$ is upper semi-continuous, so $\xi(\cdot,\cdot)$ is upper semi-continuous on $Y \times Y$. But $C(\cdot)$ is not upper semi-continuous. Note that the level set of the function ξ at 0,

$$\begin{aligned}L(\xi,0) &= \{((y_1,y_2)^\top,(z_1,z_2)^\top)\in \mathbb{R}^2\times\mathbb{R}^2 : \xi((y_1,y_2)^\top,(z_1,z_2)^\top)\le 0\}\\ &= (\{(y_1,y_2)^\top\in\mathbb{R}^2 : y_1=0\}\times(-A))\cup\\ &\quad (\{(y_1,y_2)^\top\in\mathbb{R}^2 : y_1>0\}\times(-B))\\ &\quad \cup(\{(y_1,y_2)^\top\in\mathbb{R}^2 : y_1<0\}\times(-C)),\end{aligned}$$

is not a closed set. That is to say, $\xi(\cdot,\cdot)$ is not lower semi-continuous.

Example 1.60. Let $Y=\mathbb{R}^2$, the 2-dimensional Euclidean space. Let

$$A = cone(\{(y_1,y_2)^\top\in\mathbb{R}^2 : y_1+y_2=2, \frac{1}{2}\le y_1\le\frac{3}{2}\}),$$
$$B = cone(\{(y_1,y_2)^\top\in\mathbb{R}^2 : y_1+y_2=2, 0\le y_1\le 2\}),$$

The set-valued map $C: Y\rightrightarrows Y$ is defined by

$$C((y_1,y_2)^\top)=\begin{cases}B, & \text{if } y_1=0;\\ A, & \text{if } y_1\ne 0.\end{cases}$$

Then,

$$W((y_1,y_2)^\top)=\begin{cases}Y\backslash intB, & \text{if } y_1=0;\\ Y\backslash intA, & \text{if } y_1\ne 0.\end{cases}$$

Let $e=(1,1)^\top$ and for any $y=(y_1,y_2)^\top\in Y$, $e(y)=e$.

Note that for any $y\in Y$, $intC(y)\ne\varnothing$ and $e\in intC(y)$. We also note that $C(\cdot)$ is upper semi-continuous, so $\xi(\cdot,\cdot)$ is lower semi-continuous on $Y\times Y$. But $W(\cdot)$ is not upper semi-continuous. Note that the strict level set of the function ξ at 0,

$$\begin{aligned}L_s(\xi,0) &= \{((y_1,y_2)^\top,(z_1,z_2)^\top)\in \mathbb{R}^2\times\mathbb{R}^2 : \xi((y_1,y_2)^\top,(z_1,z_2)^\top)< 0\}\\ &= (\{(y_1,y_2)^\top\in\mathbb{R}^2 : y_1=0\}\times(-intB))\cup\\ &\quad (\{(y_1,y_2)^\top\in\mathbb{R}^2 : y_1\ne 0\}\times(-intA))\end{aligned}$$

is not an open set. That is to say, $\xi(\cdot,\cdot)$ is not upper semi-continuous.

1.4 Convex and Generalized Convex Functions

In this section, we introduce some concepts of (generalized) convexity for vector-valued and set-valued functions.

Let X, Y be two topological vector spaces, $C\subset Y$ a convex cone with nonempty interior $intC$.

Definition 1.61. *(i) A set $A\subset Y$ is said to be C-bounded below if there exists b such that $A\subset b+C$.*