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International Centre for Mechanical Sciences

Poly-, Quasi- and Rank-One Convexity in Applied Mechanics

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INTERNATIONAL CENTRE FOR MECHANICAL SCIENCES

COURSES AND LECTURES - No. 516

POLY-, QUASI- AND RANK-ONE CONVEXITY IN APPLIED MECHANICS

EDITED BY

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PREFACE

Many mechanical applications are associated to "generalized" convexity conditions, e.g. the modeling of fracture and self contact, the status of elasticity with respect to atomistic models, the understanding of microstructure induced by phase transformations, the passage from three-dimensional elasticity to models of rods and shells, applications in the field of biomechanics, carbon nanotube modeling, and finite-element formulation of nematic liquid crystal elastomers.

Related to these problems are the conditions of polyconvexity (Ball 1977), quasiconvexity (Morrey 1952) and rank-one convexity (Legendre-Hadamard ellipticity). In contrast to isotropic models the construction of anisotropic polyconvex functions remains an open field of research and has been treated in the course. Some well-known material models do not fulfill the quasiconvexity inequality. In these cases the construction of quasiconvex hulls may be advisable. Applications have been discussed for the St. Venant-Kirchhoff model and for nematic liquid crystals. Furthermore, focussing on material models satisfying the Legendre-Hadamard condition, the construction of rank-one convex functions is another important strategy.

The CISM course on "Poly-, Quasi- and Rank-One Convexity in Applied Mechanics", held in Udine from September 24 to September 28, 2007, was adressed to master students, doctoral students, post docs and experienced researchers in engineering, applied mathematics and science who wished to broaden their knowledge in generalized convexity conditions and their impact in applied mechanics, particularly with regard to the constitutive modeling of complex material behavior as well as on the consequences of "validity" (existence) of solutions obtained within direct variational methods.

It is our pleasure to thank the lectures of the CISM course and contributors to this CISM lecture notes Sir John Ball (Oxford), Antonio DeSimone (Trieste), Annie Raoult (Paris), Miroslav Šilhavý (Prague), David J. Steigmannn (Berkeley), as well as the additional contributors Daniel Balzani (Essen) and Vera Ebbing (Essen). Finally, we thank the 59 participants from 13 countries who made the course a success. We extend our thanks to the Rectors, the Board, and the staff of CISM for the excellent support and kindful help.

Jörg Schröder and Patrizio Neff

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Progress and puzzles in nonlinear elasticity

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1 Introduction

These lectures are largely based on two previous survey articles Ball (2001), Ball (2002), and cover a selection of open problems with some new remarks and updates. But they also give an introduction to the convexity conditions that are the objects of study of this course.

We begin by considering the usual set-up for nonlinear elastostatics, in which an elastic body occupies in a reference configuration the bounded domain (i.e. open and connected set) $\Omega \subset \mathbf{R}^3$ having Lipschitz boundary $\partial\Omega$. We assume that the boundary can be decomposed as $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2 \cup N$, where $\partial\Omega_1, \partial\Omega_2$ are relatively open and disjoint, and where N has zero area (that is, its two-dimensional Hausdorff measure $\mathcal{H}^2(N) = 0$).

For a deformation $\mathbf{y} : \Omega \to \mathbf{R}^3$, the deformation gradient

$$D\mathbf{y}(\mathbf{x}) = \left(\frac{\partial y_i(x)}{\partial x_j}\right)$$

is required to belong to $M_+^{3\times 3}$, where $M^{m\times n} = \{\text{real } m \times n \text{ matrices}\}$, and $M_+^{n\times n} = \{\mathbf{A} \in M^{n\times n} : \det \mathbf{A} > 0\}$. We suppose that $\mathbf{y}$ satisfies mixed displacement zero-traction boundary conditions, so that

$$\mathbf{y}|_{\partial\Omega_1} = \bar{\mathbf{y}}(\cdot),$$

where $\bar{\mathbf{y}} : \partial\Omega_1 \to \mathbf{R}^3$ is given.

We further assume that the body is comprised of *homogeneous* material, that is the material response is the same at each point $\mathbf{x} \in \Omega$. (Note that this is not the same as having the same material at each point; think, for example, of two elastic bands stuck together, one stretched relative to the other, so that there is no stress-free reference configuration.) We also assume that the temperature is constant. The total elastic energy is then given by

$$I(\mathbf{y}) = \int_\Omega W(D\mathbf{y}(\mathbf{x}))\,d\mathbf{x}, \tag{1}$$

where the stored-energy function $W : M^{3\times 3}_+ \to \mathbf{R}$ is assumed to be C^1 and bounded below (in fact we may and do assume that $W \geq 0$). The Piola-Kirchhoff stress tensor is then given by $\mathbf{T}_R(\mathbf{A}) = D_\mathbf{A}W(\mathbf{A})$.

Let $\varphi : \Omega \to \mathbf{R}^3$ be smooth with $\varphi|_{\partial\Omega_1} = 0$. Formally computing

$$\frac{d}{d\tau}I(\mathbf{y}+\tau\varphi)|_{\tau=0} = 0$$

we obtain the *weak form of the Euler-Lagrange equation*

$$\int_\Omega D_\mathbf{A}W(D\mathbf{y})\cdot D\varphi\,d\mathbf{x} = 0 \quad \text{for all such } \varphi. \quad \text{(WEL)}$$

If $\mathbf{y}, \partial\Omega_1, \partial\Omega_2$ are sufficiently regular then (WEL) is equivalent to

$$\left.\begin{array}{ll} \mathrm{Div}\, D_\mathbf{A}W(D\mathbf{y}(\mathbf{x})) & = 0 \text{ for } \mathbf{x}\in\Omega, \\ D_\mathbf{A}W(D\mathbf{y}(\mathbf{x}))\mathbf{N}(\mathbf{x}) & = 0 \text{ for } \mathbf{x}\in\partial\Omega_2, \end{array}\right\}$$

where $\mathbf{N}(\mathbf{x})$ is the unit outward normal to $\partial\Omega_2$. (Thus the zero traction boundary condition on $\partial\Omega_2$ appears as a natural boundary condition.)

1.1 Function Spaces

To what function space should $\mathbf{y}$ belong? *This is part of the mathematical model*, since examples show that the minimum (or infimum) of I in different function spaces can be different. We will assume that $\mathbf{y}$ belongs to the (largest) Sobolev space $W^{1,1} = W^{1,1}(\Omega, \mathbf{R}^3)$, where for $1 \leq p \leq \infty$

$$\begin{aligned} W^{1,p}(\Omega;\mathbf{R}^3) &= \{\mathbf{z} : \Omega \to \mathbf{R}^3, \|\mathbf{z}\|_{1,p} < \infty\} \\ \|\mathbf{z}\|_{1,p} &= \left(\int_\Omega (|\mathbf{z}|^p + |D\mathbf{z}|^p)\,d\mathbf{x}\right)^{\frac{1}{p}} \text{ if } 1 \leq p < \infty \\ &= \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega}\,(|\mathbf{z}(\mathbf{x})| + |D\mathbf{z}(\mathbf{x})|) \text{ if } p = \infty. \end{aligned}$$

(For the formal definitions and basic facts see standard texts on Sobolev spaces e.g. Adams and Fournier (2003).)

If $\mathbf{y} \in W^{1,1}$ then $\mathbf{y}$ is absolutely continuous along a.e. line parallel to the coordinate axes (see Morrey (1966)). Hence deformations with planar cracks are excluded, though discontinuities on sets $S(\mathbf{y}) \subset \Omega$ with $\mathcal{H}^2(S(\mathbf{y})) = 0$ may be possible, as for example in cavitation (see Ball (1982) and Section 1.4 below). How should we decide on the 'correct' function space? We could hope to do this by means of a passage from an atomistic/molecular model to a continuum one. Such a 'derivation' of the continuum model would certainly lead to a larger function space than $W^{1,1}$ (allowing fracture, for example) and a modified energy functional. It would then be a task to understand the status of minimizers of I in $W^{1,1}$ with respect to the modified theory (e.g. as metastable states).

1.2 Properties of W

We make the following hypotheses on W. The first is *frame-indifference*:

$$W(\mathbf{RA}) = W(\mathbf{A}) \quad \text{for all } \mathbf{R} \in SO(3), \mathbf{A} \in M_+^{3\times 3},$$

where $SO(3) = \{\mathbf{R} \in M_+^{3\times 3} : \mathbf{R}^T\mathbf{R} = \mathbf{1}\}$, which holds if and only if

$$W(\mathbf{A}) = W(\mathbf{U}), \;\; \mathbf{U} = (\mathbf{A}^T\mathbf{A})^{\frac{1}{2}}.$$

The second is *material symmetry*:

$$W(\mathbf{AQ}) = W(\mathbf{A}) \text{ for all } \mathbf{Q} \in \mathcal{S}, \mathbf{A} \in M_+^{3\times 3},$$

where $\mathcal{S}$ is the *isotropy group* of the material. The case $\mathcal{S} \supset SO(3)$ corresponds to an *isotropic* material, for which we have the representation

$$W(\mathbf{A}) = \Phi(v_1, v_2, v_3),$$

where the v_i are the principal stretches (that is, the eigenvalues of $\mathbf{U}$) and Φ is symmetric with respect to permutations of the v_i.

The third condition says that infinite energy is required to compress the material to zero volume:

$$W(\mathbf{A}) \to \infty \text{ as } \det \mathbf{A} \to 0+. \tag{2}$$

We set $W(\mathbf{A}) = \infty$ if $\det \mathbf{A} \le 0$. Then $I(\mathbf{y}) \in [0, \infty]$ is well defined for $\mathbf{y} \in W^{1,1}$, and if $I(\mathbf{y}) < \infty$ then $\det D\mathbf{y}(\mathbf{x}) > 0$ a.e..

Are there any other conditions on W satisfied by 'all materials'? In the older literature there was a feeling that there should be such 'constitutive inequalities' that would correspond to 'stress increasing with strain' (for a comprehensive discussion see Truesdell and Noll (1965)). Two such candidates were the *strong ellipticity condition* and the *Coleman-Noll inequality*. The strong ellipticity condition is

$$D^2W(\mathbf{A})(\mathbf{a} \otimes \mathbf{N}, \mathbf{a} \otimes \mathbf{N}) > 0, \quad \text{for all nonzero } \mathbf{a}, \mathbf{N} \in \mathbf{R}^3, |\mathbf{N}| = 1.$$

where $(\mathbf{a} \otimes \mathbf{N})_{i\alpha} = a_i N_\alpha$, that is

$$\frac{d^2}{dt^2} W(\mathbf{A} + t\mathbf{a} \otimes \mathbf{N})|_{t=0} = \frac{\partial^2 W(\mathbf{A})}{\partial A_{i\alpha} \partial A_{j\beta}} a_i N_\alpha a_j N_\beta > 0.$$

In particular this condition implies the reality of wave speeds in elastodynamics.

We do not state the Coleman-Noll inequality here, but note that for an isotropic material it implies that $\Phi(v_1, v_2, v_3)$ is convex. It is easily seen that this is not satisfied for rubber because rubber is almost incompressible. For example, for moderately large v the convexity inequality

$$\Phi\left(\frac{1}{2}(v + v^{-1}), \frac{1}{2}(v + v^{-1}), 1\right) \le \frac{1}{2}\left(\Phi(v, v^{-1}, 1) + \Phi(v^{-1}, v, 1)\right)$$

is not satisfied because the volume change $\frac{1}{4}(v + v^{-1})^2$ is large and thus the left-hand side large compared to the right-hand side. Thus the Coleman-Noll inequality is not generally satisfied.

In fact 'stress increases with strain' should be regarded as a stability condition. For example, in one dimension consider the minimizers y of

$$I(y) = \int_0^1 W(y_x)\, dx, \text{ subject to } y(0) = 0, y(1) = \lambda > 0,$$

where $y_x = dy/dx$. Suppose $W \in C^1(0, \infty)$, $W(p) \to \infty$ as $p \to 0+$, $\lim_{p \to 0+} \frac{W(p)}{p} = \infty$. Let W^{**} be the convexification of W (that is the greatest convex function less than or equal to W). It is not difficult to show that W^{**} is C^1 (for a general result of this type see Kirchheim and Kristensen (2001)). A *Weierstrass point* p is a point at which $W(p) = W^{**}(p)$, so that the tangent at p to the graph of W does not lie above the graph. Let

$$I^{**}(y) = \int_0^1 W^{**}(y_x)\, dx.$$

We can think of W^{**} as being the macroscopic stored-energy function corresponding to the mesoscopic stored-energy function W. In fact, setting $\mathcal{A} = \{y \in W^{1,1}(0,1) : y(0) = 0, y(1) = \lambda, y_x > 0 \text{ a.e.}\}$ we have that

$$\inf_{\mathbf{y} \in \mathcal{A}} I(y) \ge \inf_{y \in \mathcal{A}} I^{**}(y) \ge \inf_{y \in \mathcal{A}} W^{**}\left(\int_0^1 y_x\, dx\right) = W^{**}(\lambda),$$

where the middle inequality follows from Jensen's inequality. But any $\lambda > 0$ can be written as $\lambda = \mu p + (1 - \mu) q$, where $W(p) = W^{**}(p), W(q) =$

$W^{**}(q), 0 \leq \mu \leq 1$, where $0 < p \leq \lambda \leq q < \infty$ and $W^{**}(r)$ is affine for $r \in [p,q]$. Thus

$$y^\lambda(x) = \begin{cases} px, & 0 \leq x \leq \mu \\ p\mu + q(x-\mu), & \mu \leq x \leq 1 \end{cases}$$

is such that

$$I(y^\lambda) = \mu W^{**}(p) + (1-\mu)W^{**}(q) = W^{**}(\lambda).$$

Hence y^λ is a minimizer, and $\inf_{\mathcal{A}} I = \inf_{\mathcal{A}} I^{**}$. For any minimizer y^* we have $W(y^*_x) = W^{**}(y^*_x)$ a.e., so that the only values of y^*_x that can be observed in minimizers (in fact even in strong local minimizers, i.e. local minimizers in the C^0 metric) are Weierstrass points. Also we have that $W_p(y^*_x) = W^{**}_p(\lambda)$ a.e., so that the stress is constant and a monotone function of the overall strain λ, even though no assumption has been made about monotonicity of $W_p(p) = dW(p)/p$ in p (i.e. of convexity of W).

In higher dimensions the role played in one dimension by convexity is played by *quasiconvexity* (in the sense of Morrey (1952)). Let $f : M^{m\times n} \to \mathbf{R} \cup \{+\infty\}$ be Borel measurable and bounded below. We say that f is *quasiconvex at* $\mathbf{A} \in M^{m\times n}$ if

$$\int_\Omega f(\mathbf{A} + D\varphi(\mathbf{x}))\,d\mathbf{x} \geq \int_\Omega f(\mathbf{A})\,d\mathbf{x}$$

for any $\varphi \in C_0^\infty(\Omega;\mathbf{R}^m)$, and is *quasiconvex* if it is quasiconvex at every $\mathbf{A} \in M^{m\times n}$. Here $\Omega \subset \mathbf{R}^n$ is any bounded open set whose boundary $\partial\Omega$ has zero n-dimensional Lebesgue measure. A standard scaling argument shows that these definitions do not depend on Ω.

1.3 Roles of quasiconvexity in the calculus of variations

Existence of global minimizers (Morrey (1952, 1966), Acerbi and Fusco (1984))

If $f : M^{m\times n} \to \mathbf{R}$ is quasiconvex and satisfies

$$C_1|\mathbf{A}|^p - C_0 \leq f(\mathbf{A}) \leq C_2(|\mathbf{A}|^p + 1) \text{ for all } \mathbf{A} \in M^{m\times n}, \tag{3}$$

where $p > 1$, C_0 and $C_1 > 0$, $C_2 > 0$ are constants, then

$$\mathcal{F}(\mathbf{y}) = \int_\Omega f(D\mathbf{y})\,d\mathbf{x}$$

attains a global minimum on

$$\mathcal{A} = \{\mathbf{y} \in W^{1,1}(\Omega;\mathbf{R}^m) : \mathbf{y}|_{\partial\Omega_1} = \bar{\mathbf{y}}\}.$$

Here we assume that Ω has Lipschitz boundary $\partial\Omega$, $\partial\Omega_1 \subset \partial\Omega$ is $\mathcal{H}^{n-1}$ measurable, and $\bar{\mathbf{y}} : \partial\Omega_1 \to \mathbf{R}^m$ is given such that $\mathcal{A}$ is nonempty. The proof is by the direct method of the calculus of variations, using the fact that under the growth conditions (3) quasiconvexity is necessary and sufficient for $\mathcal{F}$ to be sequentially weakly lower semicontinuous on $W^{1,p}(\Omega;\mathbf{R}^m)$. A result of Ball and Murat (1984) shows that if the minimum of $\mathcal{F}$ is attained whenever suitable lower order terms $g(\mathbf{y})$ are added to the integrand, then W is quasiconvex. This shows that the direct method is the right method for proving existence.

Relaxation (Dacorogna (1982))

Under similar hypotheses we have

$$\inf_{\mathcal{A}} \mathcal{F} = \inf_{\mathcal{A}} \mathcal{F}^{qc},$$

where

$$\mathcal{F}^{qc}(\mathbf{y}) = \int_\Omega f^{qc}(D\mathbf{y})\, dx$$

and f^{qc} is the *quasiconvex envelope* of f, i.e the supremum of all quasiconvex functions $g \le f$. In elasticity this has the interpretation that for problems (such as elastic crystals) for which the total elastic energy I does not attain a minimum, the macroscopic stored-energy function corresponding to the microscopic/mesoscopic stored-energy function W is W^{qc}. In Ball et al. (2000) it is shown that if $f \in C^1$ then f^{qc} is C^1.

Partial regularity of energy minimizers (Evans (1986), Kristensen and Taheri (2003))

Under similar hypotheses, with a slightly strengthened version of quasiconvexity, and assuming f smooth, any global (or $W^{1,p}$-local) minimizer is smooth on the complement of a closed set E of n-dimensional Lebesgue measure zero. For Lipschitz minimizers the Hausdorff dimension of the singular set is strictly less than n (see Kristensen and Mingione (2007)).

Necessary and sufficient conditions for local minimizers

Again consider

$$\mathcal{F}(\mathbf{y}) = \int_\Omega f(D\mathbf{y})\, d\mathbf{x},$$

where $f \in C^2$ is bounded below, and suppose that $\mathbf{y} \in \mathcal{A} \cap C^1(\bar{\Omega};\mathbf{R}^m)$ is a $W^{1,p}$ local minimizer of $\mathcal{F}$ in $\mathcal{A}$, i.e. for some $\varepsilon > 0$ we have that

$$\mathcal{F}(\mathbf{z}) \ge \mathcal{F}(\mathbf{y}) \text{ for all } z \in \mathcal{A} \text{ with } \|\mathbf{z} - \mathbf{y}\|_{1,p} < \varepsilon.$$

Then

(NC1) $\mathbf{y}$ satisfies (WEL):

$$\int_\Omega Df(D\mathbf{y})\cdot D\varphi\,d\mathbf{x}=0 \text{ for all smooth } \varphi \text{ with } \varphi|_{\partial\Omega_1}=0.$$

(NC2) (Positivity of the second variation)

For such φ

$$\frac{d^2}{d\tau^2}\mathcal{F}(\mathbf{y}+\tau\varphi)|_{\tau=0}\geq 0,$$

that is

$$\int_\Omega D^2_{\mathbf{A}}f(D\mathbf{y})(D\varphi,D\varphi)\,d\mathbf{x}\geq 0 \text{ for all smooth } \varphi \text{ with } \varphi|_{\partial\Omega_1}=0.$$

(NC3) (Interior quasiconvexity)

If $\mathbf{x}_0\in\Omega$ then f is quasiconvex at $D\mathbf{y}(\mathbf{x}_0)$.

(NC4) (Quasiconvexity at the free boundary) (Ball and Marsden (1984))

Let $\mathbf{x}_1\in\partial\Omega_2=\partial\Omega\backslash\partial\Omega_1$. We assume that $\partial\Omega$ is C^1 with unit outward normal $\mathbf{N}(\mathbf{x}_1)$ at $\mathbf{x}_1$. Let $B^-(\mathbf{x}_1)$ be the half-ball $\{\mathbf{x}\in\mathbf{R}^n:|\mathbf{x}|<1,\mathbf{x}\cdot\mathbf{N}(\mathbf{x}_1)<0\}$. Then

$$\int_{B^-(\mathbf{x}_1)}f(D\mathbf{y}(\mathbf{x}_1)+D\varphi(\mathbf{z}))\,d\mathbf{z}\geq\int_{B^-(\mathbf{x}_1)}f(D\mathbf{y}(\mathbf{x}_1))\,d\mathbf{z}$$

whenever $\varphi\in C^\infty(\overline{B^-(\mathbf{x}_1)};\mathbf{R}^m)$ with $\varphi|_{\partial B^-(\mathbf{x}_1)\cap\partial B(0,1)}=0$.

(NC3) and (NC4) are generalizations of the classical Weierstrass condition. It is natural to ask whether (NC1)-(NC4) can be slightly strengthened to form a set of sufficient conditions for $\mathbf{y}$ to be a strong local or $W^{1,p}$ local minimizer. For example, (NC2) can be strengthened to

$$(\mathrm{NC2})^+\qquad\int_\Omega D^2_{\mathbf{A}}f(D\mathbf{y})(D\varphi,D\varphi)\,d\mathbf{x}\geq\mu\int_\Omega|D\varphi|^2d\mathbf{x}$$

for all smooth φ with $\varphi|_{\partial\Omega_1}=0$, for some $\mu>0$. This has been achieved in very interesting recent work of Grabovsky and Mengesha (2009), in the more general context of integrands $f(\mathbf{x},\mathbf{y},D\mathbf{y})$ satisfying suitable p^{th} power growth conditions, thus generalizing the classical Weierstrass sufficiency theorem to this case. The idea is to split an arbitrary variation into a 'weak' and a 'strong' part.

Unfortunately none of these results applies directly to elasticity, since the growth conditions assumed are inconsistent with the condition (2). This is related to the lack of a tractable characterization of quasiconvexity, which might lead to different proof techniques. It is known (Kristensen (1999)) that there is no *local* characterization. In general we have that for f taking finite values

$$f \text{ polyconvex } \Rightarrow f \text{ quasiconvex } \Rightarrow f \text{ rank-one convex.}$$

Here f *polyconvex* means that $f(\mathbf{A}) = g(\mathbf{J}(\mathbf{A}))$ for some convex function g of the list $\mathbf{J}(\mathbf{A})$ of all minors of $\mathbf{A}$, while f *rank-one convex* means that $t \mapsto f(\mathbf{A} + t\mathbf{a} \otimes \mathbf{N})$ is convex for all $\mathbf{a} \in \mathbf{R}^m, \mathbf{N} \in \mathbf{R}^n$. The converse implications are false for $m > 1, n > 1$ except that when $m = 2$ it is not known whether f rank-one convex implies f quasiconvex (for $m > 2$ this is the famous counterexample of Šverák (1991)). Although there are examples of quasiconvex f that are not polyconvex, no useful class of examples is known. Existence theorems based on polyconvexity remain of interest both because of this lack of examples and because they can handle the blow-up of $W(\mathbf{A})$ as $\det \mathbf{A} \to 0+$. The following such result is due to Müller et al. (1994), following Ball (1977).

Theorem 1.1. *Suppose that* W *satisfies*
(H1) W *is polyconvex, i.e.* $W(\mathbf{A}) = g(\mathbf{A}, \operatorname{cof} \mathbf{A}, \det \mathbf{A})$ *for all* $\mathbf{A} \in M^{3\times3}_+$ *and some convex* g,
(H2) $W(\mathbf{A}) \geq c_0(|\mathbf{A}|^2 + |\operatorname{cof} \mathbf{A}|^{\frac{3}{2}}) - c_1$, *for all* $\mathbf{A} \in M^{3\times3}_+$, *where* $c_0 > 0$.
Then if $\mathcal{A}$ *is nonempty, there exists a global minimizer* $\mathbf{y}^*$ *of* I *in* $\mathcal{A}$.

1.4 Open problems in elastostatics

When is the minimizer $\mathbf{y}^*$ smooth? No such result is known even in the simplest special cases, such as

$$W(\mathbf{A}) = |\mathbf{A}|^2 + |\mathbf{A}|^4 + h(\det \mathbf{A}),$$

where h is smooth, convex, with $h(\delta) \to \infty$ as $\delta \to 0+$, $\frac{h(\delta)}{\delta} \to \infty$ as $\delta \to \infty$. Although there are counterexamples to regularity for minimizers of

$$\mathcal{F}(\mathbf{y}) = \int_\Omega f(D\mathbf{y})\, d\mathbf{x},$$

where f is strictly convex (see Nečas (1977), Šverák and Yan (2000)), none are known for the dimensions $m = n = 2$ or 3.

Does $\mathbf{y}^*$ satisfy (WEL)? The difficulty is that (WEL) requires that $D_A W(D\mathbf{y}^*)$ be at least locally integrable, but $I(\mathbf{y}^*) < \infty$ only tells us that

$W(D\mathbf{y}^*) \in L^1$, and $|D_{\mathbf{A}}W(\mathbf{A})|$ may be much bigger than $W(\mathbf{A})$ when $|\mathbf{A}|$ is large or $\det \mathbf{A}$ is small, so that there is no obvious way to pass to the limit $t \to 0$ in the difference quotient

$$\int_\Omega \frac{W(D\mathbf{y}^* + tD\varphi) - W(D\mathbf{y}^*)}{t}\, d\mathbf{x}.$$

In fact it need not be the case that $\det(D\mathbf{y}^*(x) + D\varphi(\mathbf{x})) > 0$.

There is no general such theorem even in the one-dimensional calculus of variations. An example (see Ball and Mizel (1985)) is the problem of minimizing

$$I(u) = \int_{-1}^{1} [(x^4 - u^6)^2 u_x^{28} + \varepsilon u_x^2]\, dx \tag{4}$$

subject to $u(-1) = -1, u(1) = 1$, where $0 < \varepsilon < \varepsilon_0 << 1$, which has a global minimizer u^* with

$$u^*(x) \sim |x|^{-\frac{1}{3}} x$$

as $x \sim 0$. In one dimension (WEL) is equivalent to the integrated form

$$f_{u_x} = \int_0^x f_u\, ds + \text{const.},$$

but here $f_{u_x}(x, u^*, u_x^*)$ is unbounded.

It is possible to derive two different forms of the Euler-Lagrange equation for (1) by taking variations that are compositions, thus preserving the sign of the determinant. For example, by considering the variation

$$\mathbf{y}_\tau(\mathbf{x}) = \mathbf{y}^*(\mathbf{x}) + \tau\varphi(\mathbf{y}^*(\mathbf{x}))$$

we can prove that Cauchy's equilibrium equations hold in the weak form

$$\int_\Omega \left[D_{\mathbf{A}}W(D\mathbf{y}^*)D\mathbf{y}^{*T} \cdot D\varphi(\mathbf{y}^*)\right]\, d\mathbf{x} = 0$$

for all $\varphi \in C^1(\mathbf{R}^3; \mathbf{R}^3)$ with $\varphi, D\varphi$ uniformly bounded and such that $\varphi(\mathbf{y}^*)|_{\partial\Omega_1} = 0$, provided that W satisfies

$$|D_{\mathbf{A}}W(\mathbf{A})A^T| \le K(W(\mathbf{A}) + 1) \quad \text{for all } \mathbf{A} \in M_+^{3\times 3},$$

a condition that holds for many models of natural rubber (for the details see Ball (2002)).

Prove or disprove that under suitable growth conditions on W, $\det D\mathbf{y}^*(\mathbf{x}) \geq \varepsilon > 0$. For this we seem to need some variation that increases $\det D\mathbf{y}^*$ where it is small. Perhaps related to this is the open problem

If $\mathbf{y} \in W^{1,p}(\Omega, \mathbf{R}^3)$ is invertible, can $\mathbf{y}$ be approximated in $W^{1,p}$ by piecewise affine invertible maps? The difficulty can be seen even in two dimensions, where a Lipschitz $\mathbf{y}$ can map three points A, B, C to points A', B', C' in such a way that the orientation of the triangle ABC is opposite to that of $A'B'C'$. For some recent partial results see Bellido and Mora-Corral (2008), Mora-Corral (2009).

If Ω is homeomorphic to a ball, $\partial\Omega = \partial\Omega_1$, W strictly polyconvex, are minimizers (or sufficiently smooth equilibrium solutions) unique? There are well-known counterexamples to uniqueness when $\partial\Omega_1 \neq \partial\Omega$ or if Ω has holes (see Ball (2002) Section 2.6). The answer to the problem as stated is probably no, as explained in Ball (2002). A recent paper by Spadaro (2009) gives some counterexamples with $\mathbf{y} : \Omega \to \mathbf{R}^3, \Omega \subset \mathbf{R}^2$, with injective boundary values, using ideas from minimal surfaces. However it is not clear how to extend these examples to $\mathbf{y} : \Omega \to \mathbf{R}^n$, $n = 2$ or $n = 3$, where $\Omega \subset \mathbf{R}^n$ and the boundary values are injective.

Now consider the example

$$W(\mathbf{A}) = |\mathbf{A}|^2 + h(\det \mathbf{A}),$$

where h is convex, $h(\delta) \to \infty$ as $\delta \to 0$, $\frac{h(\delta)}{\delta} \to \infty$ as $\delta \to \infty$. This W is polyconvex, but does not satisfy the growth condition (H2). It is an example of a function that is $W^{1,p}$ quasiconvex, i.e.

$$\int_\Omega W(\mathbf{A} + D\varphi(\mathbf{x}))\, d\mathbf{x} \geq \int_\Omega W(\mathbf{A})\, d\mathbf{x} \quad \text{for all } \varphi \in W_0^{1,p}(\Omega; \mathbf{R}^3)$$

if $p \geq 3$, but not if $p < 3$. In fact if $\mathbf{A} = \lambda\mathbf{1}$, with $\lambda > 0$ sufficiently large, we can find a radial deformation of the form

$$\mathbf{y}(\mathbf{x}) = \frac{r(R)}{R} x$$

with $r(0) > 0, r(1) = \lambda$, and $I(\mathbf{y}) < I(\lambda\mathbf{1})$. This is the phenomenon of *cavitation.*

For such a W, is the minimum of I attained? Here is a strange argument, perhaps suggesting that the answer is no. Let us suppose that the minimum is attained for the cube $Q = (-1,1)^3$ and linear boundary data $\mathbf{y}|_{\partial Q} = A\mathbf{x}$, and that the minimizer $\mathbf{y}^*$ is C^1 up to the flat parts of the boundary ∂Q. We can deduce from this that $D\mathbf{y}^*$ is constant on each face of the cube. To see this pick some point $\mathbf{a}$ in the interior of one face of the cube, having normal $\mathbf{e}_1$ say, and another point $\mathbf{b}$ in the interior of the opposite face. Now for some small $\varepsilon > 0$ consider the two cubes $Q_1 = \varepsilon Q$ and $Q_2 = \varepsilon(Q + \mathbf{a} - \mathbf{b})$. These cubes are disjoint, interior to Q, and their closures meet on part of the surface $\{\mathbf{x} : \mathbf{x}\cdot\mathbf{e}_1 = \varepsilon\mathbf{a}\cdot\mathbf{e}_1\}$ which has $\varepsilon\mathbf{a}$ as an interior point. Now let $\mathbf{c}_1 = 0, \mathbf{c}_2 = \varepsilon(\mathbf{a}-\mathbf{b})$, $\varepsilon_1 = \varepsilon_2 = \varepsilon$ and choose cubes $Q_j = \varepsilon_j Q + \mathbf{c}_j$, $j \geq 3$ such that the $\{Q_i\}_{i=1}^\infty$ are disjoint with

$$\text{meas}\left(Q \backslash \bigcup_{i=1}^\infty Q_i\right) = 0,$$

which is possible by Vitali's covering theorem. Define for $\mathbf{x} \in \bar{Q}$

$$\mathbf{y}(\mathbf{x}) = \begin{cases} A\mathbf{c}_i + \varepsilon\mathbf{y}^*\left(\frac{\mathbf{x}-\mathbf{c}_i}{\varepsilon}\right) & \text{if } \mathbf{x} \in Q_i, \\ A\mathbf{x} & \text{otherwise.} \end{cases}$$

Then $\mathbf{y} \in \mathbf{Ax} + W_0^{1,p}(Q;\mathbf{R}^3)$ and

$$I(\mathbf{y}) = \sum_{i=1}^\infty \int_{Q_i} W\left(D\mathbf{y}^*\left(\frac{\mathbf{x}-\mathbf{c}_i}{\varepsilon}\right)\right) d\mathbf{x} = I(\mathbf{y}^*).$$

Hence $\mathbf{y}$ is also a minimizer, and since $\mathbf{y}$ is piecewise C^1 in the neighbourhood of $\varepsilon\mathbf{a}$ it follows that in this neighbourhood it satisfies (WEL). Consequently the stress at $\varepsilon\mathbf{a}$ across the surface $\{\mathbf{x} : \mathbf{x}\cdot\mathbf{e}_1 = \varepsilon\mathbf{a}\cdot\mathbf{e}_1\}$ is continuous, i.e.

$$DW(D\mathbf{y}^*(\mathbf{a}))\mathbf{e}_1 = DW(D\mathbf{y}^*(\mathbf{b}))\mathbf{e}_1.$$

But since W is strictly polyconvex it is strictly rank-one convex, and hence by a result in Ball (1980) (see also Knowles and Sternberg (1978)) we have $D\mathbf{y}^*(\mathbf{a}) = D\mathbf{y}^*(\mathbf{b})$. Since $\mathbf{a}$ and $\mathbf{b}$ are arbitrary points on opposite faces of Q the claim follows.

Can we incorporate cavitation into a more general theory of fracture? The 'free-discontinuity' variational models of fracture (see e.g. Francfort and Marigo (1998)) are based on minimization of an energy such as

$$I(\mathbf{y}) = \int_\Omega W(D\mathbf{y})\,d\mathbf{x} + \int_{S_\mathbf{y}} g(\mathbf{y}^+ - \mathbf{y}^-, \nu_\mathbf{y})\,d\mathcal{H}^2,$$

where $\mathbf{y}$ belongs to the space $SBV(\Omega)$ of mappings of special bounded variation, i.e. those whose gradient is a bounded measure having no Cantor part. $S_{\mathbf{y}}$ denotes the set of jump points of $\mathbf{y}$, $\nu_{\mathbf{y}}$ is the normal to $S_{\mathbf{y}}$ at $\mathbf{y}$, and $\mathbf{y}^+, \mathbf{y}^-$ are the traces of $\mathbf{y}$ from the positive and negative sides of $S_{\mathbf{y}}$ respectively. It is tempting to think of a progression from zero-dimensional (cavitation) to one-dimensional (line singularities) to two-dimensional (cracks) fracture singularities, and there is some evidence that, for example, cavities can coalesce to form cracks. Thus a framework in which all these kinds of singularities can energetically compete is desirable. Recent progress in this direction, leading to a theory in which both cavitation and cracks are possible, has been made by Henao and Mora-Corral (2009a,b).

1.5 Dynamics

We end with some brief remarks on dynamics. The balance laws of linear momentum and energy lead to the pointwise forms of the governing equations:

$$\left.\begin{array}{l}\rho_R\mathbf{y}_{tt} - \operatorname{Div}\mathbf{T}_R - \mathbf{b} = 0,\\ (\frac{1}{2}\rho_R|\mathbf{y}_t|^2 + U)_t - \mathbf{b}\cdot\mathbf{y}_t - \operatorname{Div}(\mathbf{y}_t\mathbf{T}_R) + \operatorname{Div}\mathbf{q}_R - r = 0,\end{array}\right\} \tag{5}$$

where $\rho_R > 0$ is the density in the reference configuration, $\mathbf{b}$ is the body force, U is the internal energy density, and $\mathbf{q}_R$ the reference heat flux vector.

The balance of angular momentum holds if and only if the Cauchy stress tensor

$$\mathbf{T} = (\det D\mathbf{y})^{-1}\mathbf{T}_R(D\mathbf{y})^T$$

is symmetric.

For a thermoelastic material, we assume that $\mathbf{T}_R$, the entropy density η, the Helmholtz free energy $\psi = U - \theta\eta$ and $\mathbf{q}_R$ depend on $D\mathbf{y}, \theta$, and $\operatorname{Grad}\theta$. Use of the Clausius-Duhem inequality then leads to

$$\psi = \psi(D\mathbf{y}, \theta), \quad \mathbf{T}_R = D_{\mathbf{A}}\psi, \quad \eta = -D_\theta\psi$$

and

$$-\mathbf{q}_R \cdot \operatorname{Grad}\theta \geq 0.$$

Frame-indifference is equivalent to

$$\psi(\mathbf{RA}, \theta) = \psi(\mathbf{A}, \theta) \quad \text{for all } \mathbf{R} \in SO(3),$$

and this implies that $\mathbf{T}$ is symmetric. We need to solve (5) for the unknowns $\mathbf{y}$ and θ.

If we assume that $\theta(\mathbf{x},t)=\theta_0=$ constant then we obtain the equation of motion

$$\rho_R\mathbf{y}_{tt}-\mathrm{Div}\,D_{\mathbf{A}}W(D\mathbf{y})-\mathbf{b}=0,$$

where $W(\mathbf{A})=\psi(\mathbf{A},\theta_0)$. This is a multi-dimensional system of conservation laws about which very little is known. One might ask, however, if polyconvexity or quasiconvexity play any role. Whereas nothing seems to be known about their implications for existence of solutions, there are two such results as regards uniqueness:

1. (Dafermos (2005)) If W is quasiconvex then Lipschitz solutions of uniformly small oscillation are unique within the class of weak solutions.

2. (Qin (1998)) If W is polyconvex, the hypothesis of uniform small oscillation in the Dafermos result can be removed.

What if we add dissipation? The simplest situation is that of viscoelasticity of rate type, for which the equation of motion is

$$\rho_R\mathbf{y}_{tt}-\mathbf{T}_R(D\mathbf{y},D\mathbf{y}_t)=0.$$

Frame-indifference of $\mathbf{T}_R$ holds if and only if

$$\mathbf{T}_R(D\mathbf{y},D\mathbf{y}_t)=D\mathbf{y}\,\mathbf{G}(\mathbf{U},\mathbf{U}_t),$$

where $\mathbf{G}$ is symmetric. No large data existence theorem is known for this case (though one would expect to have one that would cover even non-quasiconvex elastic energies). With a good existence and uniqueness theory we could study the questions of approach to equilibrium and address qualitative features of the dynamic evolution.

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Quasiconvex envelopes in nonlinear elasticity

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Abstract We give several examples of modeling in nonlinear elasticity where a quasiconvexification procedure is needed. We first recall that the three-dimensional Saint Venant-Kirchhoff energy fails to be quasiconvex and that its quasiconvex envelope can be obtained by means of careful computations. Second, we turn to the mathematical derivation of slender structure models: an asymptotic procedure using Γ-convergence tools leads to models whose energy is quasiconvex by construction. Third, we construct an homogenized quasiconvex energy for square lattices.

1 The Saint Venant-Kirchhoff stored energy function

1.1 Non quasiconvexity of the Saint Venant-Kirchhoff stored energy function

This section is based on Raoult (1986) from which it is immediately derived that the Saint Venant-Kirchhoff stored energy function is not rank-one convex, and as a consequence not polyconvex, nor quasiconvex.

The internal energy of an elastic material reads $J(\phi) = \int_\Omega W(\nabla\phi(x))dx$ where $\Omega \subset \mathbb{R}^3$ is a reference configuration (here assumed to be homogeneous), $W : \mathbb{M}_{3\times 3} \mapsto \mathbb{R}$ is the stored energy function that is most of the time assumed to be continuous and the deformation $\phi : \Omega \mapsto \mathbb{R}^3$ is sufficiently regular. This is the energy due to the deformation ϕ. Actually, the domain of W should be restricted to the set $\mathbb{M}^+_{3\times 3}$ of matrices with positive determinant and ϕ should satisfy in some sense $\det\nabla\phi(x) > 0$ in order to express that orientation is preserved by realistic deformations and to prevent matter interpenetration. This restriction leads to mathematical difficulties and is quite often left aside. The total energy is the sum of the internal energy and of the external energy which takes into account the action of external loads (body forces such as gravity, surface forces such as pressure ...). The equilibrium problem in an energy form reads:

$$\text{Find } \phi \in \Phi \text{ such that } I(\phi) = \min_{\Phi} I, \tag{1}$$

where I is the total energy and Φ is a functional space that takes into account placement conditions. A basic hypothesis for proving the existence of a minimizer of a given functional is that this functional be lower semicontinuous. We are dealing here with functionals defined on infinite dimensional spaces, namely $W^{1,p}(\Omega;\mathbb{R}^3)$ spaces (for simple growth conditions on W, J is well defined on $W^{1,p}(\Omega;\mathbb{R}^3)$). Therefore, the appropriate lower semicontinuity is lower semicontinuity with respect to the weak topology. Then, it is classical (see Morrey (1995) and Dacorogna (2007) for a survey) that J is weakly lower semicontinuous if and only if W is quasiconvex. Recall that the quasiconvexity condition reads

$$\forall F \in \mathbb{M}_{3\times 3},\ \forall \phi \in W_0^{1,\infty}(\Omega;\mathbb{R}^3),\ W(F) \leq \frac{1}{|\Omega|}\int_\Omega W(F+\nabla\phi(x))\,dx. \tag{2}$$

Quasiconvexity is a nonlocal notion which makes it difficult to check or to contradict. Two pointwise notions make a lower-upper frame for quasiconvexity. Indeed, the implications

$$\text{polyconvexity} \Rightarrow \text{quasiconvexity} \Rightarrow \text{rank-one convexity} \tag{3}$$

are valid. This chain of implications can be easily remembered by noticing that the three notions are listed in alphabetical order. The convexity notion can even be added at the left extremity of the chain. But for applications in nonlinear elasticity, this is not useful since convexity of the energy density has to be ruled out for modeling reasons. Rank-one convexity is simply convexity along the straight line generated by two matrices whose difference is of rank 1. It reads

$$\begin{aligned} &\forall F, G \in \mathbb{M}_{3,3} \text{ such that } \operatorname{rank}(F-G) \leq 1,\\ &\forall \lambda \in [0,1],\ W(\lambda F + (1-\lambda)G) \leq \lambda W(F) + (1-\lambda)W(G). \end{aligned} \tag{4}$$

Let us mention that rank-one convexity does not imply quasiconvexity as was proved by Sverak (1992). Polyconvexity is a more complex notion that was introduced in Ball (1977): an energy is polyconvex if one can find a convex function w such that

$$\forall F \in \mathbb{M}_{3\times 3},\ W(F) = w(F, \operatorname{adj}F, \det F). \tag{5}$$

The function w in the above equation is defined on $\mathbb{M}_{3\times 3} \times \mathbb{M}_{3\times 3} \times \mathbb{R}$. Note that this definition may be restricted to matrices with positive determinant in which case w is defined on $\mathbb{M}_{3\times 3} \times \mathbb{M}_{3\times 3} \times \mathbb{R}^{+*}$ which is the convex hull of $\{(F, \mathrm{adj}F, \det F), \det F > 0\}$. From implications (3), it is obvious that a way of proving that an energy is quasiconvex – the important notion – is to prove that it is polyconvex. This is why the possible polyconvexity of the Saint Venant-Kirchhoff energy was examined two decades ago. In fact, the proof provided the non rank-one convexity.

Theorem 1.1. *The Saint Venant-Kirchhoff energy is not rank-one convex.*

The following statements immediately follow.

Corollary 1.2. *The Saint Venant-Kirchhoff energy is not polyconvex, nor quasiconvex.*

Proof of Theorem 1.1. - The Saint Venant Kirchhoff energy reads

$$W(F) = \frac{\mu}{4}\|F^T F - I\|^2 + \frac{\lambda}{8}(\|F\|^2 - 3)^2, \tag{6}$$

where $\|F\|^2 = \mathrm{tr}\, F^T F$. Letting $C(F) = F^T F$, we have equivalently

$$W(F) = \frac{\mu}{4}\mathrm{tr}(C(F))^2 + \frac{\lambda}{8}\|F\|^4 - \frac{3\lambda + 2\mu}{4}\|F\|^2 + \frac{9\lambda + 6\mu}{8}. \tag{7}$$

The first term reads $\frac{\mu}{4}(v_1^4 + v_2^4 + v_3^4)$ where v_1, v_2, v_3 are the singular values of F. Since g defined by $g(v_1, v_2, v_3) = (v_1^4 + v_2^4 + v_3^4)$ is a convex, symmetric function that is not decreasing with respect to each of its variable, it is known that this first term is convex in F. Proofs of such results can be found in Ball (1977), Ciarlet (1987), Le Dret (1990), Thompson and Freede (1971). The second term is obviously convex in F. But, the minus sign in the third term prevents the whole of the expression of being rank-one convex as shown by the following counter-example which uses the fact that for $\|F\|$ small, this term is greater than the two previous ones. Let $F = \varepsilon\, Id$ and $G = \varepsilon\, \mathrm{diag}(1, 1, 3)$, so that $\dfrac{F+G}{2} = \varepsilon\, \mathrm{diag}(1, 1, 2)$. Notice that $F - G$ is of rank 1. Matrices F and G are such that

$$\mathrm{adj}\,(\frac{F+G}{2}) = \frac{\mathrm{adj}\, F + \mathrm{adj}\, G}{2},\ \det(\frac{F+G}{2}) = \frac{\det F + \det G}{2}.$$

If W were rank-one convex, we would get $W(\dfrac{F+G}{2}) \leq \dfrac{1}{2}(W(F) + W(G))$, *i.e.*,

$$\begin{aligned}
&\mu\, \mathrm{tr}(C(\frac{F+G}{2}))^2 + \frac{\lambda}{2}\|\frac{F+G}{2}\|^4 - (3\lambda + 2\mu)\|\frac{F+G}{2}\|^2 \\
&\leq \frac{1}{2}[\mu\, \mathrm{tr}(C(F))^2 + \frac{\lambda}{2}\|F\|^4 - (3\lambda + 2\mu)\|F\|^2] \\
&+\frac{1}{2}[\mu\, \mathrm{tr}(C(G))^2 + \frac{\lambda}{2}\|G\|^4 - (3\lambda + 2\mu)\|G\|^2].
\end{aligned}$$

Since terms $\mathrm{tr}(C(F))^2$, $\mathrm{tr}(C(G))^2$, $\mathrm{tr}(C(\frac{F+G}{2}))^2$ and terms $\|F\|^4$, $\|G\|^4$, $\|\frac{F+G}{2}\|^4$ are of order 4 with respect to ε while the remaining ones are of order 2, this amounts to

$$-\frac{1}{\varepsilon^2}\|\frac{F+G}{2}\|^2 \leq -\frac{1}{\varepsilon^2}(\frac{1}{2}\|F\|^2 + \frac{1}{2}\|G\|^2).$$

The contradiction follows from the equalities $\|F\|^2 = 3\varepsilon^2$, $\|G\|^2 = 11\varepsilon^2$, $\|\frac{F+G}{2}\|^2 = 6\varepsilon^2$.

1.2 The quasiconvex envelope of the Saint Venant-Kirchhoff stored energy function

The quasiconvex envelope of the three-dimensional Saint Venant-Kirchhoff stored energy function was first computed in Le Dret and Raoult (1995b). Computations were made in a systematic but somewhat tedious way, and inspired by a preliminary work by the authors in a 2d×3d setting, see Le Dret and Raoult (1995a). Extending some results by Pipkin allows to simplify the proofs, see Pipkin (1994), Le Dret and Raoult (1995c).

We denote by $\mathbb{S}_m^+$ the set of symmetric, positive semidefinite matrices.

Lemma 1.3. *Let $m \leq n$ and $Y : \mathbb{M}_{n\times m} \mapsto \mathbb{R}$ be a left $O(n)$-invariant, rank-one convex mapping. Then the mapping $\tilde{Y} : \mathbb{S}_m^+ \mapsto \mathbb{R}$ such that $Y(F) = \tilde{Y}(F^TF)$ for all F in $\mathbb{M}_{n\times m}$ satisfies*

$$\tilde{Y}(C) \leq \tilde{Y}(C+S) \text{ for all } C,\, S \in \mathbb{S}_m^+. \tag{8}$$

Remark 1.4. In the case when $m < n$, this result is due to Pipkin (1994). However, the argument does not apply to square matrices. In Pipkin's terminology, $\tilde{Y}$ is said to be increasing.

Proof - Following Pipkin (1993), we first remark that proving (8) amounts to proving that

$$\tilde{Y}(C) \leq \tilde{Y}(C+\mu v\otimes v) \text{ for all } C \in \mathbb{S}_m^+ \text{ and for all } \mu \geq 0,\ v \in \mathbb{R}^m \setminus \{0\}. \tag{9}$$

Indeed, (8) clearly implies (9). Conversely, any S in $\mathbb{S}_m^+$ admits a spectral decomposition $S = \sum_{i=1,m} \mu_i v_i \otimes v_i$ where $\mu_i \geq 0$, and v_i, $i = 1, \cdots, m$ are orthonormal eigenvectors of S. Applying inequality (9) m times, we obtain (8).

Let us now prove (9). Let $C \in \mathbb{S}_m^+$ and $v \in \mathbb{R}^m \setminus \{0\}$ be given. Without loss of generality, we assume $\|v\|^2 := v^Tv = 1$.

We first consider the case when either $m < n$ or $m = n$ and C is not invertible. In both cases, C can be written as $C = F^TF$ where F^T is a noninjective $m \times n$ matrix. Therefore, there exists u in $\ker F^T$ with $\|u\| = 1$. From the rank-one convexity of Y, we know that the function $y : t \in \mathbb{R} \mapsto y(t) = Y(F + t\, u \otimes v) \in \mathbb{R}$ is convex. Moreover, since $F^Tu = 0$, $y(t) = \tilde{Y}(C + t^2 v \otimes v)$. Therefore, y is even. It follows that y is monotone increasing on $\mathbb{R}^+$; in particular, $y(0) \leq y(t)$ for all $t \geq 0$. Choosing $t = \sqrt{\mu}$, we obtain (9).

We now turn to the case when $m = n$ and C is invertible. For all $\mu \geq 0$, the matrix $C_\mu = C + \mu\, v \otimes v$ is symmetric, positive definite. Hence, $F_\mu = C_\mu^{1/2}$ is invertible. We define a function h on $\mathbb{R}$ by

$$h : t \in \mathbb{R} \mapsto h(t) = Y(F_\mu + t\, F_\mu^{-1} v \otimes v).$$

It follows from the rank-one convexity of Y that h is convex. Moreover,

$$h(t) = \tilde{Y}(C + (\mu + 2\,t)\, v \otimes v + t^2 v \otimes v(C + \mu v \otimes v)^{-1} v \otimes v).$$

An easy computation shows that the function

$$t \in \mathbb{R} \mapsto 2\,t\, v \otimes v + t^2 v \otimes v(C + \mu\, v \otimes v)^{-1} v \otimes v$$

is symmetric with respect to $\bar{t} = -(v^T\,(C+\mu\, v\otimes v)^{-1}\, v)^{-1} < 0$. The function h, in turn, is symmetric with respect to $\bar{t}$. Therefore, h attains its minimum at $\bar{t}$ and is monotone increasing on $[\bar{t}, +\infty[$. Obviously, $h(0) = \tilde{Y}(C+\mu\, v\otimes v)$. If we can find t such that $\bar{t} \leq t \leq 0$ and

$$h(t) = \tilde{Y}(C), \tag{10}$$

then inequality (9) is proved. A sufficient condition for a real number t to solve (10) is

$$t^2 v^T (C + \mu v \otimes v)^{-1} v + 2\,t + \mu = 0. \tag{11}$$

The discriminant of equation (11) is positive if and only if

$$\mu v^T (C + \mu v \otimes v)^{-1} v \leq 1. \tag{12}$$

Let us check that this is indeed the case. Let $z = \mu^{1/2} C^{-1/2} v$. Then we have

$$\mu v^T (C + \mu v \otimes v)^{-1} v = z^T (I + z \otimes z)^{-1} z = \frac{\|z\|^2}{1 + \|z\|^2} \leq 1,$$

hence the roots of equation (11) are real. Moreover, they are nonpositive and symmetric with respect to $\bar{t}$. The largest root thus satisfies $\bar{t} \leq t \leq 0$ and (10), which proves our claim.

With this lemma at hand, we can now state our main result.

Theorem 1.5. *Let $m \le n$ and let $W : F \in \mathbb{M}_{n\times m} \mapsto \mathbb{R}$ be a left $O(n)$-invariant, bounded from below, stored energy function such that the associated function $\tilde{W} : C \mapsto \tilde{W}(C)$ is convex on $\mathbb{S}_m^+$. Then,*

$$QW(F) = \inf_{S\in\mathbb{S}_m^+} \tilde{W}(F^T F + S). \tag{13}$$

Proof - Since $\tilde{W}$ is bounded from below, we can define, with Pipkin's notation, a function $\tilde{W}_r$ on $\mathbb{S}_m^+$ by $\tilde{W}_r(C) = \inf_{S\in\mathbb{S}_m^+} \tilde{W}(C+S)$. It is easy to check that $\tilde{W}_r(C)$ is convex. Indeed, given C_1 and $C_2 \in \mathbb{S}_m^+$ and an arbitrary $\varepsilon > 0$, there exists S_1 and S_2 in $\mathbb{S}_m^+$ such that $\tilde{W}_r(C_i) \le \tilde{W}(C_i + S_i)$ and $\tilde{W}(C_i+S_i) \le \tilde{W}_r(C_i)+\varepsilon$ for $i = 1, 2$. Let $t \in [0, 1]$ and $S = t\,S_1 + (1-t)S_2$. Then,

$$\begin{aligned} \tilde{W}_r(t\,C_1 + (1-t)C_2) &\le \tilde{W}(t\,C_1 + (1-t)\,C_2 + S) \\ &\le t\,\tilde{W}(C_1 + S_1) + (1-t)\,\tilde{W}(C_2 + S_2) \\ &\le t\,\tilde{W}_r(C_1) + (1-t)\tilde{W}_r(C_2) + \varepsilon. \end{aligned}$$

The convexity of $\tilde{W}_r$ follows at once.

Let us now remark that $\tilde{W}_r$ obviously satisfies $\tilde{W}_r(C) \le \tilde{W}_r(C+S)$ for all C and S in $\mathbb{S}_m^+$. This implies that the function $Z := F \in \mathbb{M}_{n\times m} \mapsto \tilde{W}_r(F^T F)$ is convex. Indeed, for all F and G in $\mathbb{M}_{n\times m}$ and for all $t \in [0, 1]$,

$$\begin{aligned} (tF + (1-t)G)^T(tF + (1-t)G) &= t\,F^T F + (1-t)G^T G \\ &- t(1-t)(F-G)^T(F-G). \end{aligned}$$

Therefore, since $t(1-t)(F-G)^T(F-G)$ is positive semidefinite,

$$Z(t\,F + (1-t)G) \le \tilde{W}_r(t\,F^T F + (1-t)G^T G) \le t\,Z(F) + (1-t)Z(G),$$

by the convexity of $\tilde{W}_r$. Consequently, since Z is convex and below W, we see that $Z \le QW$.

The reverse inequality is obtained as follows. From Le Dret and Raoult (1995a), we know that QW is also left $O(n)$-invariant. Applying Lemma 1 to $Y = QW$, which is rank-one convex, we obtain

$$\begin{aligned} QW(F) = \widetilde{QW}(F^T F) &\le \widetilde{QW}(F^T F + S) = QW\left((F^T F + S)^{1/2}\right) \\ &\le \tilde{W}(F^T F + S) \end{aligned}$$

for all $S \in \mathbb{S}_m^+$. Therefore, $QW(F) \le Z(F)$ and the proof is complete.

Remark 1.6. i) It follows clearly from the proof that the quasiconvex envelope is also in this case the convex and rank-one convex envelope of the stored energy function.

ii) We now proceed to show by means of a simple counterexample that Pipkin's formula fails for $m > n$. Let $m = 2$, $n = 1$. Consider the function $W : \mathbb{M}_{1\times 2} \mapsto \mathbb{R}$, $W(F) = \|F^T F - I\|^2$. We thus let $\tilde{W}(C) = \|C - I\|^2$. This function is clearly convex with respect to C. If we denote $F = (z_1, z_2)$ with $z_i \in \mathbb{R}$, then we have

$$W(F) = (z_1^2 - 1)^2 + (z_2^2 - 1)^2 + 2z_1^2 z_2^2 = \left(\|F\|^2 - 1\right)^2 + 1.$$

Therefore, $QW(F) = CW(F)$ and $QW(F) = W(F)$ if $\|F\| \geq 1$, 1 if $\|F\| \leq 1$, see Dacorogna (2007). Let us now take $F = (1, -1)$ so that $C = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $\tilde{W}(C) = 2$. With the choice $S = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, we obtain $\tilde{W}_r(C) \leq \tilde{W}(C + S) = 1 < 2 = \tilde{W}(C) = W(F) = QW(F)$.

Application to an explicit computation: The Saint Venant-Kirchhoff stored energy function defined in (6) can equivalently be written under the form $W(F) = \tilde{W}(F^T F)$ where

$$\tilde{W}(C) = \frac{\mu}{4}\,\|C - I\|^2 + \frac{\lambda}{8}\,(\operatorname{tr} C - 3)^2 \tag{14}$$

for all C in $\mathbb{S}_3^+$. The mapping $\tilde{W}$ is clearly convex with respect to C.

Therefore, Theorem 1.5 applies. Let us briefly show how computations can be organized. For any C in $\mathbb{S}_3^+$, let $J_C : S \in \mathbb{S}_3^+ \mapsto \tilde{W}(C + S) \in \mathbb{R}$. This is a strictly convex, coercive mapping. Consequently, J_C admits one and only one minimizer on $\mathbb{S}_3^+$. By (13), we have to evaluate $\inf_{S\in\mathbb{S}_3^+} J_C(S) = \min_{S\in\mathbb{S}_3^+} J_C(S)$. Assume first that C is diagonal. We deduce from (14) that $J_C(S) \geq J_C(\operatorname{diag}(s_{11}, s_{22}, s_{33}))$. Minimizing $J_C(S)$ among semidefinite positive matrices thus amounts to minimizing $J_C(S)$ among diagonal positive matrices. Equivalently, we have to minimize on $(\mathbb{R}^+)^3$ the mapping j_C such that

$$j_C(s_1, s_2, s_3) = \frac{\mu}{4}\sum_{i=1}^{3}(c_{ii} - 1 + s_i)^2 + \frac{\lambda}{8}\,(\sum_{i=1}^{3}(c_{ii} - 1 + s_i))^2.$$

Without loss of generality, we assume that $c_{11} \leq c_{22} \leq c_{33}$. The optimality conditions for j_C on $(\mathbb{R}^+)^3$ read

$$Dj_C(s_1, s_2, s_3)(t_1, t_2, t_3) \geq 0 \;\text{ for all } (t_1, t_2, t_3) \in (\mathbb{R}^+)^3,$$
$$Dj_C(s_1, s_2, s_3)(s_1, s_2, s_3) = 0,\; (s_1, s_2, s_3) \in (\mathbb{R}^+)^3.$$

They are equivalent to

$$\partial_i j_C(s_1, s_2, s_3) \geq 0 \ \text{ for } \ i = 1,2,3,$$
$$\partial_i j_C(s_1, s_2, s_3)\, s_i = 0, \ s_i \geq 0 \ \text{ for } \ i = 1,2,3,$$

that is to say

$$(2\,\mu + \lambda)(c_{ii} - 1 + s_i) + \lambda \textstyle\sum_{k\neq i}(c_{kk} - 1 + s_k) \geq 0,$$
$$((2\,\mu + \lambda)(c_{ii} - 1 + s_i) + \lambda \textstyle\sum_{k\neq i}(c_{kk} - 1 + s_k))\, s_i = 0\ , \ s_i \geq 0.$$

We distinguish four different cases:

1) If $c_{33} \leq 1$, then we can choose $s_i = 1 - c_{ii}$ and $\min\limits_{(\mathbb{R}^+)^3} j_C = 0$.

2) If $c_{33} \geq 1$ and $2\,(\lambda + \mu)\, c_{22} + \lambda\, c_{33} \leq 3\,\lambda + 2\,\mu$, then we can choose $s_3 = 0$, $s_j = -c_{jj} - \dfrac{\lambda}{2(\lambda+\mu)} c_{33} + \dfrac{3\lambda + 2\mu}{2(\lambda+\mu)} \geq 0, \ j = 1,2$ and

$$\min_{(\mathbb{R}^+)^3} j_C = \frac{\mu\,(3\lambda + 2\mu)}{8(\lambda+\mu)}(c_{33} - 1)^2.$$

3) If $2(\lambda{+}\mu)\, c_{22}{+}\lambda\, c_{33} \geq 3\,\lambda{+}2\,\mu$ and $(\lambda{+}2\mu)\, c_{11}{+}\lambda\,(c_{22}{+}c_{33}) \leq 3\lambda{+}2\mu$, then $s_2 = s_3 = 0$, $s_1 = -c_{11} - \dfrac{\lambda}{(\lambda + 2\,\mu)}(c_{22} + c_{33}) + \dfrac{3\lambda + 2\mu}{(\lambda + 2\,\mu)} \geq 0$ and

$$\min_{(\mathbb{R}^+)^3} j_C = \frac{\mu}{4}((c_{22} - 1)^2 + (c_{33} - 1)^2) + \frac{\lambda\mu}{4(\lambda + 2\,\mu)}(c_{22} + c_{33} - 2)^2.$$

4) If $(\lambda + 2\mu)\, c_{11} + \lambda\,(c_{22} + c_{33}) \geq 3\lambda + 2\mu$, then we can choose $s_1 = s_2 = s_3 = 0$ and $\min\limits_{(\mathbb{R}^+)^3} j_C = \tilde{W}(C)$.

So far, we have determined $QW(F)$ when $C = F^T F$ is diagonal. To extend the result to arbitrary matrices C in $\mathbb{S}_3^+$, we make use of the right $O(3)$-invariance of the Saint Venant-Kirchhoff density W which is inherited by QW. Therefore, $QW(F)$ only depends on the singular values of F. We denote by $v_1(F) \leq v_2(F) \leq v_3(F)$ the singular values arranged in increasing order. It suffices to replace c_{ii} by $v_i(F)^2$ in the above formulas to obtain an explicit expression for $QW(F)$.

The expressions thus obtained are the same as those obtained in Le Dret and Raoult (1995a) that we recall below. First, for making comparisons between the energy and its quasiconvex envelop easier, we express the Saint Venant-Kirchhoff energy in terms of the singular values and we obtain

$$W(F) = \frac{E}{8(1+\nu)} \sum_{i=1}^{3} (v_i(F)^2 - 1)^2 + \frac{E\nu}{8(1+\nu)(1-2\nu)} \Big(\sum_{i=1}^{3} v_i(F)^2 - 3 \Big)^2, \tag{15}$$

where the Young modulus and the Poisson ratio are given by

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \text{ and } \nu = \frac{\lambda}{2(\lambda + \mu)}.$$

Let $\mathcal{T} = \{v \in \mathbb{R}^3;\, 0 \leq v_1 \leq v_2 \leq v_3\}$ be the convex tetrahedral cone of $\mathbb{R}^3_+$ delimited by the planes $v_1 = 0$, $v_1 = v_2$ and $v_2 = v_3$. We define a mapping q on $\mathcal{T}$ by

$$\begin{aligned} q(v) = \frac{E}{8}[v_3^2 - 1]_+^2 + \frac{E}{8(1-\nu^2)}[v_2^2 + \nu v_3^2 - (1+\nu)]_+^2 \\ + \frac{E}{8(1-\nu^2)(1-2\nu)}[(1-\nu)v_1^2 + \nu(v_2^2 + v_3^2) - (1+\nu)]_+^2, \end{aligned} \tag{16}$$

where $[x]_+^2 = x^2$ if $x \geq 0$, $[x]_+^2 = 0$ if $x < 0$. Previous computations allow to write the following theorem.

Theorem 1.7. *The quasiconvex envelope of the Saint Venant-Kirchhoff stored energy function W is given by*

$$\forall F \in \mathbb{M}_3, \qquad QW(F) = q\left(v_1(F),\, v_2(F),\, v_3(F)\right). \tag{17}$$

Let us examine more deeply the values taken by QW. We introduce three subsets of $\mathcal{T}$:

$$\begin{aligned} \mathcal{H} &= \{v \in \mathcal{T};\, v_3 \leq 1\}, \\ \mathcal{C} &= \{v \in \mathcal{T};\, v_2^2 + \nu v_3^2 - (1+\nu) \leq 0\}, \\ \mathcal{E} &= \{v \in \mathcal{T};\, (1-\nu)v_1^2 + \nu(v_2^2 + v_3^2) - (1+\nu) \leq 0\}, \end{aligned}$$

which correspond to the various positive parts that appear in formula (16). It is easily checked that $\mathcal{H} \subset \mathcal{C} \subset \mathcal{E}$ and that

i) if $v \in \mathcal{H}$, then $q(v) = 0$,
ii) if $v \in \mathcal{C} \setminus \mathcal{H}$, then $q(v) = \frac{E}{8}(v_3^2 - 1)^2$,
iii) if $v \in \mathcal{E} \setminus \mathcal{C}$, then

$$q(v) = \frac{E}{8(1+\nu)} \sum_{i=2}^{3} (v_i^2 - 1)^2 + \frac{E\nu}{8(1-\nu^2)} \Big(\sum_{i=2}^{3} v_i^2 - 2 \Big)^2,$$

iv) if $v \notin \mathcal{E}$, then

$$q(v) = \frac{E}{8(1+\nu)} \sum_{i=1}^{3} (v_i^2 - 1)^2 + \frac{E\nu}{8(1+\nu)(1-2\nu)} \Big(\sum_{i=1}^{3} v_i^2 - 3 \Big)^2.$$

For singular values outside of $\mathcal{E}$, the energy and its quasiconvex envelope coincide. At the other end of the scale, for sufficiently small singular values, the quasiconvex energy is equal to 0. It is indeed a general fact that the quasiconvex envelope of a material indifferent and isotropic material vanishes on the set of singular values less than 1, see Le Dret and Raoult (1994).

2 Quasiconvexity in the derivation of slender structure models

In this section, we turn to the rigorous derivation of models for bidimensional structures from three-dimensional models. The idea of deriving simplified models from complete models goes back at least to the 50s with the works by R.D. Mindlin and by E.Reissner among others. In the linear case, correct bidimensional models can be obtained by quick, but mathematically frightening ways: assuming for instance that the 33 *component of the linearized strain tensor is equal to* 0 *in some equations of the three-dimensional model, but not in other ones. Later on, came the idea of considering a sequence of structures with thickness* 2ε*, of writing elasticity models for each of these structures and of studying the asymptotic behavior of the solutions of the models. Many researchers in applied mathematics worked on this subject and most of them consider that this procedure was first formalized by Ciarlet and Destuynder (1979).*

The first result obtained by this method was of no surprise: the usual linear plate model is recovered. In the linear case, this line of work was pursued by identifying more precise models (i.e. not only identifying the limit of the three-dimensional solutions, but identifying a higher-order term as well), studying dynamical cases, or considering more general materials such that piezo-electric materials, visco-elastic materials. Things get trickier when dealing with nonlinear models: existence results are not always available and convergence proofs are much harder when products of terms have to be considered. Identifying limit models was first obtained by formal asymptotic methods on the system of partial differential equations describing finite elasticity written under variational form. Then came the realm of rigorous Γ*-convergence arguments and their escort of quasiconvexification tools. The work we present here is taken from Le Dret and Raoult (1995a).*

We mention that a previous work by Acerbi et al. (1991) existed and, although in a one-dimesional setting that only requires convexity arguments, gave a path to follow.

2.1 The three-dimensional and rescaled problems

For all $\varepsilon > 0$, let $\Omega_\varepsilon = \{x \in \mathbb{R}^3; (x_1, x_2) \in \omega, |x_3| < \varepsilon\}$, where ω is an open, bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. For all $z_i \in \mathbb{R}^3$, $i = 1, 2, 3$, we note $(z_1|z_2|z_3)$ the 3×3 matrix whose i-th column is z_i. Let $W : \mathbb{M}_{3\times 3} \mapsto \mathbb{R}$ be a continuous function that satisfies the following growth and coercivity hypotheses:

$$\exists C > 0, \exists p \in]1, +\infty[, \forall F \in \mathbb{M}_{3\times 3}, |W(F)| \le C(1 + \|F\|^p), \tag{18}$$

$$\exists \alpha > 0, \exists \beta \ge 0, \forall F \in \mathbb{M}_{3\times 3}, W(F) \ge \alpha \|F\|^p - \beta. \tag{19}$$

We assume that Ω_ε is the reference configuration of a hyperelastic homogeneous three-dimensional body whose stored energy function is W. We assume for simplicity that the bodies are submitted to the action of dead loading body force densities $f^\varepsilon \in L^q(\Omega_\varepsilon; \mathbb{R}^3)$ and surface traction densities $g^\varepsilon \in L^r(S_\varepsilon; \mathbb{R}^3)$ on $S_\varepsilon = \omega \times \{\pm\varepsilon\}$, the top and bottom surfaces of Ω_ε. For the sake of definiteness, we assume that $q = r$ and $1/p + 1/q = 1$, but other choices are indeed possible at no extra cost. Let $\Gamma_\varepsilon = \partial\omega \times]-\varepsilon, \varepsilon[$ be the lateral surface of Ω_ε. We assume that the deformations of the bodies satisfy a boundary condition of place on Γ_ε. The equilibrium problem may be formulated as a minimization problem:

$$\text{Find } \phi^\varepsilon \in \Phi_\varepsilon \text{ such that } I_\varepsilon(\phi^\varepsilon) = \inf_{\psi \in \Phi_\varepsilon} I_\varepsilon(\psi), \tag{20}$$

where the total energy I_ε is

$$I_\varepsilon(\psi) = \int_{\Omega_\varepsilon} W(\nabla\psi)\, dx - \int_{\Omega_\varepsilon} f^\varepsilon \cdot \psi\, dx - \int_{S_\varepsilon} g^\varepsilon \cdot \psi\, d\sigma, \tag{21}$$

and the set of admissible deformations is

$$\Phi_\varepsilon = \{\psi \in W^{1,p}(\Omega_\varepsilon; \mathbb{R}^3); \psi(x) = x \text{ on } \Gamma_\varepsilon\}. \tag{22}$$

We do not assume that W is quasiconvex and problem (20) may well not possess any solutions. Naturally, if it does have solutions which are thus actual equilibrium deformations of the bodies, our results apply to these deformations.

Let us thus be given a diagonal minimizing sequence ϕ^ε for the sequence of energies I_ε over the sets Φ_ε. More specifically, we assume that

$$\phi^\varepsilon \in \Phi_\varepsilon, \quad I_\varepsilon(\phi^\varepsilon) \le \inf_{\psi \in \Phi_\varepsilon} I_\varepsilon(\psi) + \varepsilon s(\varepsilon), \tag{23}$$

where s is a positive function such that $s(\varepsilon) \to 0$ when $\varepsilon \to 0$. Such a sequence always exists and if the minimization problems have solutions, ϕ^ε may be chosen to be such a solution.

In order to obtain a membrane model in the limit, it is of crucial importance to specify the order of magnitude of the applied loads. In effect, it is always possible to stretch all thin cylinders Ω_ε into the same block, say Ω_1, by applying sufficiently large forces. For such forces, the limit behavior is obviously not that of a membrane.

It turns out that the right order of magnitude is given by $\|f^\varepsilon\|_{L^q(\Omega_\varepsilon;\mathbb{R}^3)} \leq C\varepsilon^{1/q}$ and $\|g^\varepsilon\|_{L^q(S_\varepsilon;\mathbb{R}^3)} \leq C\varepsilon$ where the constant C does not depend on ε. For example, the weight of the material, $f^\varepsilon(x) = (0, 0, -\rho g)^T$, is allowed.

In order to rescale the problem, we let $\Omega = \Omega_1$, $\Gamma = \Gamma_1$ and $S = S_1$ and define a rescaling operator Θ_ε by $(\Theta_\varepsilon \psi)(x_1, x_2, x_3) = \psi(x_1, x_2, \varepsilon x_3)$. Let $\phi(\varepsilon) = \Theta_\varepsilon \phi^\varepsilon$ and $\phi_0(\varepsilon)(x) = (x_1, x_2, \varepsilon x_3)^T$. Note that all components are treated in the same way: we only transport ϕ^ε on the fixed domain Ω. This is the same rescaling as that used in Fox et al. (1993). The rescaled displacement $u(\varepsilon) = \phi(\varepsilon) - \phi_0(\varepsilon)$ belongs to $V = W^{1,p}_\Gamma(\Omega;\mathbb{R}^3)$. We accordingly rescale the energies by setting $I(\varepsilon)(\psi) = \varepsilon^{-1} I_\varepsilon(\Theta_\varepsilon^{-1}\psi)$, *i*.e.,

$$I(\varepsilon)(\psi) = \int_\Omega W\Big(\Big(\partial_1\psi\Big|\partial_2\psi\Big|\frac{\partial_3\psi}{\varepsilon}\Big)\Big)\,dx - \int_\Omega f(\varepsilon)\cdot\psi\,dx - \int_S \varepsilon^{-1} g(\varepsilon)\cdot\psi\,d\sigma, \tag{24}$$

or in terms of the rescaled displacements

$$J(\varepsilon)(v) = \int_\Omega W\Big(\Big(e_1 + \partial_1 v\Big|e_2 + \partial_2 v\Big|e_3 + \frac{\partial_3 v}{\varepsilon}\Big)\Big)\,dx$$
$$- \int_\Omega f(\varepsilon)\cdot(\phi_0(\varepsilon) + v)\,dx - \int_S \varepsilon^{-1} g(\varepsilon)\cdot(\phi_0(\varepsilon) + v)\,d\sigma,$$

where $f(\varepsilon) = \Theta_\varepsilon f^\varepsilon$ and $g(\varepsilon) = \Theta_\varepsilon g^\varepsilon$. It is immediate that

$$J(\varepsilon)(u(\varepsilon)) \leq \inf_{v\in V} J(\varepsilon)(v) + s(\varepsilon). \tag{25}$$

For simplicity, we assume that $f(\varepsilon) = f$ and $\varepsilon^{-1} g(\varepsilon) = g$ are independent of ε.

2.2 Computation of the Γ-limit of the rescaled energies

We use Γ-convergence theory to determine the asymptotic behavior of the rescaled displacements $u(\varepsilon)$ when $\varepsilon \to 0$. In the sequel, the thickness parameter ε will take its values in a sequence $\varepsilon_n \to 0$. Since the results do not depend on the sequence in question, and for notational brevity, we will simply use the notation ε. Let us recall that a sequence of functions G_ε from a metric space X into $\bar{\mathbb{R}}$ is said to Γ-converge toward G_0 for the topology of X if the following two conditions are satisfied for all $x \in X$:

$$\forall x_\varepsilon \to x, \liminf G_\varepsilon(x_\varepsilon) \geq G_0(x),$$
$$\exists y_\varepsilon \to x, G_\varepsilon(y_\varepsilon) \to G_0(x).$$

If the sequence G_ε Γ-converges, its Γ-limit is lower semicontinuous and is alternatively given by

$$G_0(x) = \min\{\liminf G_\varepsilon(x_\varepsilon); x_\varepsilon \to x\}.$$

In addition, the set of functions from X into $\bar{\mathbb{R}}$ has a sequential compactness property with respect to Γ-convergence in the sense that any sequence $G_\varepsilon : X \to \bar{\mathbb{R}}$ admits a Γ-convergent subsequence. The main interest of Γ-convergence is that if the minimizers of G_ε stay in a compact set of X for all ε, then their limit points are minimizers of G_0, see De Giorgi and Franzoni (1975), Attouch (1984), Dal Maso (1993).

We do not use $J(\varepsilon)$ directly, since this would imply working with the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$, which is non metrizable. Instead, we extend the energies to $L^p(\Omega;\mathbb{R}^3)$ by setting

$$\forall v \in L^p(\Omega;\mathbb{R}^3),\ \tilde{J}(\varepsilon)(v) = J(\varepsilon)(v) \text{ if } v \in V, \quad +\infty \text{ otherwise.} \tag{26}$$

This is a classical trick used in the applications of Γ-convergence: obviously, this does not change the minimization problem. It has the additional virtue of incorporating the boundary conditions in the energy functional.

Let us now proceed to compute the Γ-limit of the sequence $\tilde{J}(\varepsilon)$ for the strong topology of $L^p(\Omega;\mathbb{R}^3)$. Let $\mathbb{M}_{3\times 2}$ be the space of 3×2 real matrices endowed with the usual Euclidean norm $\|\bar{F}\| = \sqrt{\mathrm{tr}\,(\bar{F}^T\bar{F})}$. We note $(z_1|z_2)$ the matrix of $\mathbb{M}_{3\times 2}$ whose α-th column is $z_\alpha \in \mathbb{R}^3$. For all $\bar{F} = (z_1|z_2) \in \mathbb{M}_{3\times 2}$ and $z \in \mathbb{R}^3$, we also note $(\bar{F}|z)$ the matrix whose first two columns are z_1 and z_2 and whose third column is z.

As in Acerbi et al. (1991) for elastic strings, we define $W_0 \colon \mathbb{M}_{3\times 2} \to \mathbb{R}$ by

$$W_0(\bar{F}) = \inf_{z\in\mathbb{R}^3} W((\bar{F}|z)). \tag{27}$$

Due to the coercivity assumption on W, it is clear that this function is well defined. Besides, since W is continuous, the infimum is attained.

Proposition 2.1. *The function* W_0 *is continuous and satisfies the growth and coercivity estimates:*

$$\exists C' > 0, \forall \bar{F} \in \mathbb{M}_{3\times 2}, |W_0(\bar{F})| \leq C'(1 + \|\bar{F}\|^p), \tag{28}$$
$$\forall \bar{F} \in \mathbb{M}_{3\times 2}, W_0(\bar{F}) \geq \alpha\|\bar{F}\|^p - \beta. \tag{29}$$

Proof - Since W_0 is an infimum of continuous functions, it is upper semicontinuous. Let $\bar{F} \in \mathbb{M}_{3\times 2}$ and consider a sequence $\bar{F}^n \in \mathbb{M}_{3\times 2}$ such that $\bar{F}^n \to \bar{F}$ as $n \to +\infty$. Because of the coercivity assumption on W, there exists a compact set K such that for all $\bar{F}^n$ the infimum in definition (27) is attained at a point $z^n \in K$. Consider a subsequence, still denoted n, such that $W_0(\bar{F}^n)$ converges when $n \to +\infty$. We extract a further subsequence such that $z^n \to z \in K$. By continuity of W, $W_0(\bar{F}^n) = W((\bar{F}^n|z^n)) \to W((\bar{F}|z)) \geq W_0(\bar{F})$. As this is true for all subsequences such that $W_0(\bar{F}^n)$ converges, it follows that $\liminf W_0(\bar{F}^n) \geq W_0(\bar{F})$, hence W_0 is lower semicontinuous.

For all $\bar{F} \in \mathbb{M}_{3\times 2}$, let z_0 be a point where the infimum in definition (27) is attained. Thus, $W_0(\bar{F}) = W((\bar{F}|z_0)) \geq \alpha\|(\bar{F}|z_0)\|^p - \beta \geq \alpha\|\bar{F}\|^p - \beta$. Hence W_0 is coercive. Therefore, W_0 is nonnegative outside of a compact set K'. Since $|W_0|$ is continuous, it is bounded on K' and for $\bar{F} \notin K'$, $|W_0(\bar{F})| = W_0(\bar{F}) \leq W((\bar{F}|0)) \leq C(1 + \|(\bar{F}|0)\|^p) = C(1 + \|\bar{F}\|^p)$, which proves the growth estimate.

Let $QW_0 = \sup\{Z\colon \mathbb{M}_{3\times 2} \to \mathbb{R}, Z \text{ quasiconvex}, Z \leq W_0\}$ be the quasiconvex envelope of W_0. Let us introduce the space

$$V_M = \{v \in V; \partial_3 v = 0\}, \tag{30}$$

which we call the space of membrane displacements. It is canonically isomorphic to $W_0^{1,p}(\omega;\mathbb{R}^3)$ and we let $\bar{v}$ denote the element of $W_0^{1,p}(\omega;\mathbb{R}^3)$ that is associated with $v \in V_M$ through this isomorphism. The expression of the Γ-limit of the sequence $\tilde{J}(\varepsilon)$ is given in the following theorem.

Theorem 2.2. *The sequence* $\tilde{J}(\varepsilon)$ Γ*-converges for the strong topology of* $L^p(\Omega;\mathbb{R}^3)$ *when* $\varepsilon \to 0$. *Let* $\tilde{J}(0)$ *be its* Γ*-limit. For all* $v \in L^p(\Omega;\mathbb{R}^3) \cap V_M$,

$$\tilde{J}(0)(v) = 2\int_\omega QW_0((e_1+\partial_1\bar{v}|e_2+\partial_2\bar{v}))\,dx_1dx_2 - \int_\omega \mathcal{F}\cdot(\phi_0(0)+\bar{v})\,dx_1dx_2 \tag{31}$$

where $\mathcal{F}(x_1,x_2) = \int_{-1}^1 f(x_1,x_2,x_3)\,dx_3 + g(x_1,x_2,1) + g(x_1,x_2,-1)$, *and* $\tilde{J}(0)(v) = +\infty$ *if* $v \in L^p(\Omega;\mathbb{R}^3) \setminus V_M$.

For clarity, we break the proof of Theorem 2.2 into a series of lemmas and propositions.

We begin by extracting a Γ-convergent subsequence and call $\tilde{J}(0)$ its Γ-limit. The uniqueness of $\tilde{J}(0)$ will make the extraction of this subsequence superfluous *a posteriori*.

Lemma 2.3. *Let $v(\varepsilon) \in L^p(\Omega;\mathbb{R}^3)$ be a sequence such that $\tilde{J}(\varepsilon)(v(\varepsilon)) \leq C < +\infty$ where C does not depend on ε. Then $v(\varepsilon)$ is uniformly bounded in V and its limit points for the weak topology of V belong to V_M.*

Proof- Let $v(\varepsilon) \in L^p(\Omega;\mathbb{R}^3)$ be such that $\tilde{J}(\varepsilon)(v(\varepsilon)) \leq C < +\infty$. Then, the definition (26) of the function $\tilde{J}(\varepsilon)$ implies first of all that $v(\varepsilon) \in V$ for all $\varepsilon > 0$. Let us call $\psi(\varepsilon) = v(\varepsilon) + \phi_0(\varepsilon)$ the deformation that is associated with the displacement $v(\varepsilon)$. The coercivity of the function W and the assumed uniform bound for the energies imply that

$$\alpha \int_\Omega \big\|(\partial_1\psi(\varepsilon)|\partial_2\psi(\varepsilon)|\varepsilon^{-1}\partial_3\psi(\varepsilon))\big\|^p\,dx \leq C'\big(1 + \|\psi(\varepsilon)\|_{W^{1,p}(\Omega;\mathbb{R}^3)}\big) \quad (32)$$

where C' does not depend on ε. It is clear that for all $\varepsilon \leq 1$, $\|(z_1|z_2|\varepsilon^{-1}z_3)\| \geq \|(z_1|z_2|z_3)\|$. Therefore, (32) implies that

$$\alpha\|\nabla\psi(\varepsilon)\|^p_{L^p(\Omega;\mathbb{R}^3)} \leq C'\big(1 + \|\psi(\varepsilon)\|_{W^{1,p}(\Omega;\mathbb{R}^3)}\big), \quad (33)$$

which, together with the boundary condition of place $\psi(\varepsilon) = \phi_0(\varepsilon)$ on Γ, yields the desired uniform bound for $\psi(\varepsilon)$ in $W^{1,p}(\Omega;\mathbb{R}^3)$ by Poincaré's inequality. Since $\phi_0(\varepsilon)$ is obviously uniformly bounded in $W^{1,p}(\Omega;\mathbb{R}^3)$, the same holds true for $v(\varepsilon)$.

On the other hand, since $\|(z_1|z_2|\varepsilon^{-1}z_3)\| \geq \varepsilon^{-1}|z_3|$, where $|\cdot|$ denotes the Euclidean norm into $\mathbb{R}^3$, upon using the bound just established above in inequality (33) we obtain that $\|\partial_3\psi(\varepsilon)\|_{L^p(\Omega;\mathbb{R}^3)} \leq C''\varepsilon$, so that $\partial_3\psi(\varepsilon) \to 0$ strongly in $L^p(\Omega;\mathbb{R}^3)$. If we let ψ denote any limit point of the sequence $\psi(\varepsilon)$ for the weak topology of $W^{1,p}(\Omega;\mathbb{R}^3)$, it follows at once that $\partial_3\psi = 0$. If v denotes the corresponding limit point of the sequence $v(\varepsilon)$, since $v = \psi - \phi_0(0)$ and $\partial_3\phi_0(0) = 0$, we obtain that v belongs to V_M.

Corollary 2.4. *If $v \in L^p(\Omega;\mathbb{R}^3)$ but $v \notin V_M$, then $\tilde{J}(0)(v) = +\infty$.*

Proof - Indeed, if $\tilde{J}(0)(v) < +\infty$, there exists a sequence $v(\varepsilon)$ that converges strongly to v in $L^p(\Omega;\mathbb{R}^3)$ and such that $\tilde{J}(\varepsilon)(v(\varepsilon)) \to \tilde{J}(0)(v)$. Therefore, by Lemma 2.3, $v \in V_M$.

We thus only have to compute the value of the Γ-limit for displacements in V_M.