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Holm Altenbach
Victor A. Eremeyev *Editors*

Generalized Continua – from the Theory to Engineering Applications

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Generalized Continua from the Theory to Engineering Applications

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Editors

Holm Altenbach
Otto Von Guericke University
Magdeburg, Germany

Victor A. Eremeyev
South Federal University and South
Scientific Center of Rasci
Rostov on Don, Russian Federation

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PREFACE

The course Generalized continua - from the theory to engineering applications *brought together doctoral students, young researcher, senior researchers, and practicing engineers. The need of generalized continua models is coming from the practice. Complex material behavior sometimes cannot be presented by the classical Cauchy continua.*

Generalized Continua are in the focus of scientists from the end of the 19th century. A first summary was given in 1909 by the Cosserat brothers. After World War II a true renaissance in this field occurred with a publication of Ericksen & Truesdell in 1958. Further developments were connected with the fundamental contributions of, among others, Kröner (Germany), Aero and Palmov (Soviet Union), Nowacki (Poland), Eringen (USA), and Maugin (France).

The Mechanics of Generalised Continua is an established research topic since the end of the 50s - early 60s of the last century. The starting point was the monograph of the Cosserat brothers from 1909 Théorie des corps déformables *and some previous works of such famous scientists like Lord Kelvin. All these contributions were focussed on the fact that in a continuum one has to define translations and rotations independently (or in other words, one has to establish force and moment actions as it was done by Euler).*

The reason for the revival of generalized continua is that some effects of the mechanical behavior of solids and fluids could not be explained by the available classical models. Examples of this are the turbulence of a fluid or the behavior of solids with a significant and very complex microstructure. Since the suggested models satisfy all requirements from Continuum Thermomechanics (the balance laws were formulated and the general representations of the constitutive equations were suggested) the scientific community accepted for a while but missed real applicative developments.

Indeed, for practical applications the developed models were not useful. The reason for this was a gap between the formulated constitutive equations and the possibilities to identify the material parameters. As often the case one had much more parameters compared to classical models.

During the last ten years the situation has drastically changed. More and more researches emerged, being kindled by the partly for-

gotten models since now one has available much more computational possibilities and very complex problems can be simulated numerically. In addition, with the increased attention paid to a large number of materials with complex microstructure and a deeper understanding of the meaning of the material parameters (scale effects) the identification becomes much more well founded. We have thus contributions describing the micro- and macrobehaviors, new existence and uniqueness theorems, the formulation of multi-scale problems, etc, and now it is time to ponder again the state of matter and to discuss new trends and applications. In addition, generalized continua models are not included in the actual BSc or MSc programs.

At present the attention of the scientists in this field is focussed on the most recent research items

- new models,
- application of well-known models to new problems,
- micro-macro aspects,
- computational effort, and
- possibilities to identify the constitutive equations

The new research directions were discussed during the course from the point of view of modeling and simulation, identification, and numerical methods. The following lectures were presented:

- On the Roots of Continuum Mechanics in Differential Geometry - A Review - by Paul Steinmann
- Cosserat Media by Holm Altenbach & Victor A. Eremeyev
- Cosserat-type Shells by Holm Altenbach & Victor A. Eremeyev
- Cosserat-type Rods by Holm Altenbach, Mircea Bîrsan & Victor A. Eremeyev
- Micromorphic Media by Samuel Forest
- Electromagnetism and Generalized Continua by Gérard A. Maugin
- Computational Methods for Generalised Continua by René de Borst

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Holm Altenbach and Victor A. Eremeyev

CONTENTS

On the Roots of Continuum Mechanics in Differential Geometry – A Review –

Paul Steinmann
Chair of Applied Mechanics, University of Erlangen-Nuremberg, Germany
E-mail: Paul.Steinmann@ltm.uni-erlangen.de

Abstract The aim of this contribution is to illustrate the roots of the geometrically nonlinear kinematics of (generalized) continuum mechanics in differential geometry. Firstly several relevant concepts from differential geometry, such as connection, parallel transport, torsion, curvature, and metric (in index notation) for holonomic and anholonomic coordinate transformations are reiterated. The notation and the selection of these topics are essentially motivated by their relation to the geometrically nonlinear kinematics of continuum mechanics. Then, secondly, the kinematics are considered from the point of view of nonlinear coordinate transformations and nonlinear point transformations, respectively. Together with the discussion on the integrability conditions for the (first-order) distortions, the concept of dislocation density tensors is introduced. After touching on the possible interpretations of nonlinear elasticity using concepts from differential geometry, a detailed discussion of the kinematics of multiplicative elastoplasticity is given. The discussion culminates in a comprehensive set of twelve different types of dislocation density tensors. Potentially, these can be used to model densities of geometrically necessary dislocations and the accompanying hardening in crystalline materials. Continuum elastoplasticity formulations of this kind fall into the class of generalized (gradient-type) plasticity models.

1 Introduction

The kinematics of geometrically nonlinear continuum mechanics is deeply rooted in differential geometry. An appreciation thereof is thus particularly illuminating. This is especially true for some generalized models of continuum mechanics, for example, gradient crystal plasticity. Here, the

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amount of accumulated dislocations (point defects in an otherwise perfect crystalline lattice) is typically deemed responsible for the state of hardening that the crystalline material displays. Thereby, the total amount of arrested dislocations is decomposed into *statistically stored dislocations* (SSD) and *geometrically necessary dislocations* (GND). The former are then assumed responsible for isotropic hardening. The latter are necessary to support the plastic part of the deformation and form an (additional) obstacle to further dislocation flow. Geometrically necessary dislocations may be subdivided further into dislocations responsible for a macroscopically stress free curvature of the crystal lattice, and dislocations responsible for macroscopic residual stresses, both after the removal of external loads. The illuminating relation between the stress free curvature of the crystal lattice and the part of the dislocation density that is geometrically necessary to support this curvature was establishes by Nye (1953). Both contributions to the geometrically necessary dislocations, i.e. those resulting in a stress free curvature of the crystal lattice and those resulting in residual stresses, constitute additional contributions to the hardening of the crystalline material. Thus, geometrically necessary dislocations obviously have to be taken into account when modelling of plasticity to describe the hardening behaviour more realistically and thus more accurately.

A consideration of the continuum version of geometrically necessary dislocations, i.e. the *dislocation density tensor*, in a thermodynamically consistent modelling framework inevitably results in a form of gradient crystal plasticity, see Steinmann (1996), Menzel and Steinmann (2000). The dislocation density tensor, however, is intimately related to one of the key concepts in non-Riemann differential geometry, i.e. the third-order torsion tensor as introduced by Cartan (1922). For anholonomic coordinates, as in the case of crystal plasticity, the Cartan torsion coincides moreover with the so-called anholonomic object of differential geometry. The important relation between the continuum description of dislocation density and a non-Riemann geometry was discovered by Kondo (1952) and Bilby et al. (1955); Bilby and Smith (1956). Prior to this, differential geometry was instrumental in the development of general relativity and the theory of gravitation, see Misner et al. (1998). Important contributions to the elaboration of differential geometry in this context have been made by Schouten (1954, 1989).

Kröner (1958) proposed a geometrically linear continuum theory of residual stresses based on the concept of dislocation densities. Motivated by insights into differential geometry, the corresponding extension to the geometrically nonlinear case was developed by Kröner and Seeger (1959) and Kröner (1960). It turned out that the interplay between continuum mechanics and differential geometry is extremely helpful: firstly, rather involved relations of the geometrically nonlinear kinematics of continuum mechanics such as the connection between the dislocation density and the St. Venant compatibility conditions for the strains could be clarified; and secondly, generalized continuum formulations that consider more general (point) defects besides dislocations, such as the distribution of quasi-dislocations caused, e.g., by inhomogeneous temperature distributions, electric or magnetic fields, vacancies, interstitial atoms and the like in the crystal lattice, are motivated by the existence of other, more involved, types of differential geometries, see e.g. the contributions by Anthony (1970a,b, 1971). A comprehensive account of the geometrically linearized version of the continuum theory of general defects in crystal lattices is found in de Wit (1981). Further interesting contributions to the continuum theory of dislocations are e.g. by Kondo (1964), Noll (1967), and Kröner (1981).

After the prolific developments in the 1950's to 1970's the topic became somewhat dormant, but since the 1990's there has been a renewed interest. This had to do with, on the one hand, the intense research on possibilities to overcome the pathological dependencies on the discretization that computational solutions, mainly based on the finite element method, displayed for the simulation of inelastic materials with a softening response. The incorporation (in one way or another) of gradients of the inelastic variables into the modelling has a regularizing effect that results in discretization-independent simulations, see e.g. Liebe and Steinmann (2001). On the other hand, the continuing trend towards miniaturization made clear that the inelastic response of a material especially is length scale (size) dependent. Again, size dependence can be included into the modelling by incorporating gradients of the inelastic variables. For an overview of a variety of possibilities to arrive at a generalized model of plasticity see, e.g., Hirschberger and Steinmann (2009); the micromorphic approach has recently been advocated strongly by Forest (2009) and Grammenoudis and Tsakmakis (2010).

However, purely phenomenological approaches for generalized models of continuum mechanics are somewhat unsatisfying if a clear link to the underlying physics is lacking. The plasticity of crystalline materials is a notable exception, as the mechanisms of plasticity are well understood to depend on the concepts of dislocations and dislocation flow. The flow of dislocations causes the plastic deformation process while the accumulating arrest of single dislocations represents an obstacle that has to be overcome if ongoing flow of dislocations is to occur. To better capture the underlying physics of crystalline material was the main motivation for the proposal in Steinmann (1996) to include the dislocation density tensor as an additional argument in the free energy density. As a consequence a gradient-type crystal plasticity formulation emerges. Subsequently, many more or less related formulations considering specific versions of dislocation density tensors were pursued, among them the important contributions by, e.g., Le and Stumpf (1996), Acharya and Bassani (2000), Cermelli and Gurtin (2001), Gurtin (2002), Svendsen (2002), Becker (2006), Reddy et al. (2008), Clayton et al. (2006) (and many more). Other aspects such as the gauge theory of dislocations as treated, e.g., by Lazar and Hehl (2010) or nonsingular stress and strain fields of dislocations and disclinations embedded in gradient elasticity, see Lazar and Maugin (2005) are exciting topics of current research activities.

It is the aim of this contribution to review and highlight the roots of the kinematics of this type of generalized crystal plasticity using relevant concepts of differential geometry. A comprehensive and as clear as possible exposition of relevant concepts from differential geometry, alone an interesting field in itself, serves as a strong guide for the sound formulation of physically based continuum theories. The interplay between materials science and mathematical underpinning results in a very powerful and fruitful approach. It is the hope that in this fashion the way may be paved to more complex continuum models that take into account, e.g., disclinations and further types of distributed (point) defects.

This contribution is decomposed into two major sections: In Sect. 2, the essential concepts from *differential geometry* are reviewed. Thereby, in the spirit of deduction, the present exposition is clearly in reverse to the historical developments in which, starting with the idea of an Euclidean space, the complexity was increased step by step resulting eventually in the treatment of general affine spaces. Thus, after giving an overview of various geometries of spaces in Sect. 2.1, some aspects of general manifolds are treated in Sect. 2.2. The linear connection, the concept of parallel transport and the torsion are then touched upon in Sects. 2.3, 2.4 and 2.5. Section 2.6 highlights the general concept of curvature. The previous concepts are equipped with more structure by introducing the metric in Sect. 2.7. The implications of the metric on the curvature are considered in Sect. 2.8. Section 3 applies the previously outlined concepts from differential geometry to the kinematics of *continuum mechanics*. To this end, Sect. 3.1 recalls the underlying ideas of continuum kinematics while Sect. 3.2 investigates the formulation of the distortion. In Sect. 3.3 the integrability of the distortion into a compatible vector field is analysed. The kinematics of elasticity are studied subsequently in terms of concepts from differential geometry in Sect. 3.4. Finally, in Sect. 3.5, as the main outcome of this review, the case of (crystal) elastoplasticity is treated along the same lines, and, in particular, a set of twelve different dislocation density tensors is proposed.

2 Differential Geometry

This section is intended to give a concise but self contained exposition of the here relevant basics of differential geometry as needed in the following section to discuss the kinematics of (generalized) continuum mechanics.

2.1 Overview

Differential geometry deals with the geometry of spaces, which may be characterized essentially in terms of only a few fundamental objects that will be discussed in detail in the following sections, i.e.:

- Connection ${\mathcal{L}^I}_{JK}$
  $\rightarrow$ Torsion ${\mathcal{T}^I}_{JK} = {\mathcal{L}^I}_{[JK]}$
- Curvature ${\mathcal{R}^I}_{JKL}$
- Metric $\mathcal{M}_{IJ}$

These objects then allow for the classification of (affine) geometries as outlined in Table 1. A geometry with vanishing *torsion* is called *symmetric*, a geometry with vanishing *curvature* is called *flat* (or equivalently a geometry with *teleparallelism*), and a geometry with vanishing covariant derivative of the metric with respect to the connection is called *metric*, it thus possesses a *metric connection.*

An *Euclidean space* is defined as a symmetric, flat and metric geometry; a symmetric, non-flat but metric geometry defines a *Riemann space*; a non-symmetric but flat and metric geometry defines a *Cartan space*, a *Riemann-Cartan space* is defined as a non-symmetric and non-flat but metric geometry; finally a *general affine space* may be defined as a non-symmetric, non-flat and non-metric geometry.

It is interesting to note that all of these geometries have corresponding counterparts in the kinematics of various continuum theories: the kinematics of elasticity may be considered a flat Riemann geometry, i.e. simply an Euclidean geometry; a Riemann geometry describes, e.g., the kinematics of (a somewhat exotic) continuum-disclination-based elastoplasticity; a Cartan geometry describes, e.g., the kinematics of (well-accepted) continuum-dislocation-based elastoplasticity; the kinematics of (an again exotic) continuum-disclination- and continuum-dislocation-based elastoplasticity may be regarded a Riemann-Cartan geometry; and the kinematics of the continuum version of even more general (point) defects such as the distribution of quasi-dislocations caused, e.g., by inhomogeneous temperature distributions, electric or magnetic fields, vacancies, interstitial atoms and the like may finally be considered within a general affine geometry that is

Table 1. Classification of affine geometries of spaces based on three fundamental attributes (symmetric, flat, and metric) from differential geometry.

	Symmetric	Flat	Metric
Euclid	yes	yes	yes
Riemann	yes	no	yes
Cartan	no	yes	yes
Riemann & Cartan	no	no	yes
General Affine	no	no	no

essentially characterized by a non-metric connection, see Anthony (1971) and more recently Clayton (2011).

2.2 Manifolds

Central to the following discussions is the notion of a manifold. Thereby, the key idea of a manifold is to allow for general coordinate systems and corresponding transformations between these coordinate systems, see e.g. the discussion in Marsden and Hughes (1994). Correspondingly and more formal is the following

Definition:
A smooth n_{dm}-dimensional manifold is a set $\mathcal{M}$ such that for each point $\mathcal{P} \in \mathcal{M}$ there is a subset $\mathcal{U}$ of $\mathcal{M}$ containing $\mathcal{P}$, and a one-to-one mapping called chart (coordinate system) $\{\mathcal{X}^I\}$ from $\mathcal{U}$ onto an open set in $\mathbb{R}^{n_{dm}}$. Multiple charts may be needed to cover the manifold. Coordinate transformations $\{\mathcal{X}^I\} \to \{\chi^i\}$ (on a region of $\mathcal{M}$) are infinitely differentiable, i.e. C^∞. A collection of charts covering $\mathcal{M}$ is called an atlas. □

As a simple example for a manifold consider either a circle or a sphere that can only be covered by at least two charts. Thus the corresponding atlas also consists of at least two charts. Abstracting from of our usual idea of a (Euclidean) space a manifold may also be considered as a generalized space. Thus less formal is the alternative

Definition:
A system that is assigned to n_{dm} variables $\mathcal{X}^1, \mathcal{X}^2, \cdots, \mathcal{X}^{n_{dm}}$ is a point $\mathcal{P}$ of an n_{dm}-dimensional manifold $\mathcal{M}$. The n_{dm} numbers $\mathcal{X}^1, \mathcal{X}^2, \cdots, \mathcal{X}^{n_{dm}}$ are the coordinates of the point $\mathcal{P}$. The set of all points $\mathcal{P}$ then defines the manifold $\mathcal{M}$. □

To illuminate this viewpoint consider as specific examples: (i) a mechanical system with n_{dm} generalized coordinates $\mathcal{X}^1, \mathcal{X}^2, \cdots, \mathcal{X}^{n_{dm}}$, (ii) the set of ellipsoids with $n_{dm} = 3$ half-axes $\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3$, or (iii) as the most basic case simply the ordinary n_{dm}-dimensional Euclidean space.

Differentials Let a chart (coordinate system) consist of n_{dm} coordinates

$$\mathcal{X}^1, \mathcal{X}^2, \cdots, \mathcal{X}^{n_{dm}} := \{\mathcal{X}^I\}. \tag{1}$$

Then a coordinate transformation from the n_{dm} coordinates $\{\mathcal{X}^I\}$ to a new n_{dm}-dimensional set of coordinates $\{\chi^i\}$ is given by the (one-to-one) mapping

$$\chi^i = y^i(\{\mathcal{X}^J\}) \quad \text{with} \quad \mathcal{X}^J = \mathcal{Y}^J(\{\chi^i\}). \tag{2}$$

Consequently the chain rule allows to work out the transformation behaviour of coordinate differentials simply as

$$\mathrm{d}\chi^i = \frac{\partial y^i}{\partial \mathcal{X}^J}\,\mathrm{d}\mathcal{X}^J =: \mathcal{F}^i{}_J\,\mathrm{d}\mathcal{X}^J \quad \text{and} \quad \mathrm{d}\mathcal{X}^J = \frac{\partial \mathcal{Y}^J}{\partial \chi^i}\,\mathrm{d}\chi^i =: f^J{}_i\,\mathrm{d}\chi^i. \tag{3}$$

Please note that it is by purpose that the notation for coordinate mappings and their Jacobians resembles notation typically used in the kinematics of continuum mechanics, see Sect. 3.1. Thus to unify terminology, coordinates $\mathcal{X}^I$ and χ^i will also be addressed as material and spatial coordinates, respectively.

Gradients Consider next a (scalar-valued) field that depends on either of the n_{dm}-dimensional coordinate systems

$$\vartheta = \Theta(\{\mathcal{X}^J\}) = \theta(\{\chi^i\}) \circ y^i(\{\mathcal{X}^J\}). \tag{4}$$

Then the total differential involves the gradient of the field with respect to the coordinates

$$\mathrm{d}\vartheta = \frac{\partial \Theta}{\partial \mathcal{X}^J}\,\mathrm{d}\mathcal{X}^J = \frac{\partial \theta}{\partial \chi^i}\,\mathrm{d}\chi^i. \tag{5}$$

Thus by either using the chain rule or by incorporating the coordinate differentials as derived in Eq. (3) the transformation of gradients follows as

$$\begin{aligned} \frac{\partial \Theta}{\partial \mathcal{X}^J} &= \frac{\partial \theta}{\partial \chi^i}\frac{\partial y^i}{\partial \mathcal{X}^J} = \frac{\partial \theta}{\partial \chi^i}\mathcal{F}^i{}_J, \\ \frac{\partial \theta}{\partial \chi^i} &= \frac{\partial \Theta}{\partial \mathcal{X}^J}\frac{\partial \mathcal{Y}^J}{\partial \chi^i} = \frac{\partial \Theta}{\partial \mathcal{X}^J} f^J{}_i. \end{aligned} \tag{6}$$

In conclusion it shall be recognized carefully that differentials and gradients obey different transformation behaviours upon a change of coordinates.[1]

Co- and Contravariant Transformations We may next attach n_{dm}-dimensional tupel $\mathcal{V}^J(\{\mathcal{X}^K\})$ and $\mathcal{V}_J(\{\mathcal{X}^K\})$ to each point $\mathcal{P}$ of $\mathcal{M}$. $\mathcal{V}^J$ are denoted the contravariant coefficients (of a vector) while $\mathcal{V}_J$ are the covariant coefficients (of a covector), both evaluated at point $\mathcal{P}$ with coordinates $\{\mathcal{X}^K\}$. Obviously these have to be distinguished by their transformation behavior upon a change of coordinates:

Contravariant coefficients (of a vector) transform like differentials

$$v^i = \mathcal{F}^i{}_J \mathcal{V}^J \quad \text{and} \quad \mathcal{V}^J = f^J{}_i v^i, \tag{7}$$

whereas covariant coefficients (of a covector) transform like gradients

$$\mathcal{V}_J = v_i \mathcal{F}^i{}_J \quad \text{and} \quad v_i = \mathcal{V}_J f^J{}_i. \tag{8}$$

Tensors Sloppily speaking tensors are objects with multiple indices that respect the following

Definition:
Coefficients of tensors change in a 'proper way' with coordinate transformations. □

As an example the previously introduced vectors and covectors may be regarded as first-order tensors with transformation properties

$$u^i = \mathcal{F}^i{}_J \mathcal{U}^J \quad \text{and} \quad u_i = f^J{}_i \mathcal{U}_J. \tag{9}$$

Consequently four different types[2] of (simple) second-order tensors may be constructed from dyadic products of first-order tensors and may be distin-

[1] Recall that the coordinate basis in a manifold corresponding to the coordinate system $\{\mathcal{X}^I\}$ is denoted by $\boldsymbol{\partial}_{\mathcal{X}^I}$, whereas the dual basis is denoted by $\mathbf{d}\mathcal{X}^I$, see Marsden and Hughes (1994). Then the coordinate representation of a vector reads $\boldsymbol{V}^\sharp = \mathcal{V}^I(\boldsymbol{\partial}_{\mathcal{X}^I})$, the coordinate representation of a covector (one-form) correspondingly follows as $\boldsymbol{V}^\flat = \mathcal{V}_I \mathbf{d}\mathcal{X}^I$. It is only in an Euclidean space parameterized by curvilinear coordinates $\{\mathcal{X}^I\}$ that the coordinate and dual basis coincide with the co- and contravariant base vectors $\boldsymbol{G}_I = \boldsymbol{\partial}_{\mathcal{X}^I}$ and $\boldsymbol{G}^I = \mathbf{d}\mathcal{X}^I$ that in turn may be related to the orthonormal Cartesian base vectors $\boldsymbol{E}^A$ and $\boldsymbol{E}_A$, respectively, see Sect. 3.1.

[2] Fully contravariant, fully covariant, contra-covariant, and co-contravariant, the latter two collectively being referred to as mixedvariant, types of second-order tensors may be distinguished.

guished by their transformation behaviour

$$
\begin{array}{llllll}
t^{ij} & := & u^i v^j & = & \mathcal{F}^i{}_K \mathcal{U}^K \mathcal{V}^L \mathcal{F}^j{}_L & =: & \mathcal{F}^i{}_K \mathcal{T}^{KL} \mathcal{F}^j{}_L, \\
t_{ij} & := & u_i v_j & = & f^K{}_i \mathcal{U}_K \mathcal{V}_L f^L{}_j & =: & f^K{}_i \mathcal{T}_{KL} f^L{}_j, \\
t^i{}_j & := & u^i v_j & = & \mathcal{F}^i{}_K \mathcal{U}^K \mathcal{V}_L f^L{}_j & =: & \mathcal{F}^i{}_K \mathcal{T}^K{}_L f^L{}_j, \\
t_i{}^j & := & u_i v^j & = & f^K{}_i \mathcal{U}_K \mathcal{V}^L \mathcal{F}^j{}_L & =: & f^K{}_i \mathcal{T}_K{}^L \mathcal{F}^j{}_L.
\end{array}
\tag{10}
$$

Clearly these transformations do also hold for general second-order tensors that are constructed from a sum of simple second-order tensors. The extension to higher-order tensors follows the same pattern and is thus straightforward.

Affine Tangent Space In general no vectors are defined in a manifold $\mathcal{M}$. However a n_{dm}-dimensional vector space (the tangent space $T_{\mathcal{P}}\mathcal{M}$), satisfying the axioms of an affine vector space[3], may be attached to each point $\mathcal{P}$ of an n_{dm}-dimensional manifold $\mathcal{M}$. It thus follows from the

Definition:
The tangent space $T_{\mathcal{P}}\mathcal{M}$ consists of all vectors $\mathcal{V}^I$ emanating from $\mathcal{P}$. □

Moreover at each point $\mathcal{P}$ a (covariant) basis of the affine tangent space denoted by $\boldsymbol{\partial}_{X^I}$ with $I = 1 \cdots n_{dm}$ may be introduced.

As elementary but already specialized examples consider 1-dimensional curves and 2-dimensional surfaces embedded into the Euclidean ambient space: Then for a parameter curve $X^I = X^I(t)$ the 1-dimensional tangent space follows from the assignment $\mathrm{d}X^I \leftrightarrow \mathrm{d}\boldsymbol{X} = \mathrm{d}X^I \boldsymbol{G}_I$. Likewise the 2-dimensional tangent space of the surface is given by its tangent plane spanned by $\boldsymbol{G}_1$ and $\boldsymbol{G}_2$. However, in general a manifold and its tangent space do not necessitate the concept of an embedding Euclidean space.

2.3 Connection

Partial Derivatives Based on the transformation rule for contravariant first-order tensors and the chain rule the partial derivatives (PD) of vectors with respect to the coordinates are computed as

$$
\begin{array}{lllllllll}
v^i & = & \mathcal{F}^i{}_J \mathcal{V}^J & \rightarrow & v^i{}_{,k} & = & \mathcal{F}^i{}_J \mathcal{V}^J{}_{,L} f^L{}_k & + & \underline{\mathcal{F}^i{}_{M,L} \mathcal{V}^M f^L{}_k}, \\
\mathcal{V}^I & = & f^I{}_j v^j & \rightarrow & \mathcal{V}^I{}_{,K} & = & f^I{}_j v^j{}_{,l} \mathcal{F}^l{}_K & + & \underline{f^I{}_{m,l} v^m \mathcal{F}^l{}_K}.
\end{array}
\tag{11}
$$

[3]In an affine vector space addition of vectors and multiplication of vectors with scalars are defined.

Likewise, based on the transformation rule for covariant first-order tensors and the chain rule the partial derivatives of covectors with respect to the coordinates follow as

$$\begin{aligned} v_i &= f^J{}_i \mathcal{V}_J &\rightarrow&& v_{i,k} &= f^J{}_i \mathcal{V}_{J,L} f^L{}_k &+& \underline{\mathcal{V}_M f^M{}_{i,k}}, \\ \mathcal{V}_I &= \mathcal{F}^j{}_I v_j &\rightarrow&& \mathcal{V}_{I,K} &= \mathcal{F}^j{}_I v_{j,l} \mathcal{F}^l{}_K &+& \underline{v_m \mathcal{F}^m{}_{I,K}}. \end{aligned} \tag{12}$$

It is obvious from the discussion in the preceding section and the representation in Eq. (10) that the underlined terms conflict with the transformation rules for second-order tensors. As a result it may be stated that the partial derivative of a vector or a covector does not result in a second-order tensor.

Covariant Derivatives Thus the challenge is to find a correction to the partial derivative of a vector or a covector so as to reinstall the transformation behavior of second-order tensors. As a result an alternative derivative with respect to the coordinates (indicated by a vertical bar |) is sought for vectors that transforms as

$$v^i{}_{|k} \doteq \mathcal{F}^i{}_J \mathcal{V}^J{}_{|L} f^L{}_k \quad \text{and} \quad \mathcal{V}^I{}_{|K} \doteq f^I{}_j v^j{}_{|l} \mathcal{F}^l{}_K. \tag{13}$$

Likewise a corresponding derivative for covectors is sought with the following transformation behaviour

$$v_{i|k} \doteq f^J{}_i \mathcal{V}_{J|L} f^L{}_k \quad \text{and} \quad \mathcal{V}_{I|K} \doteq \mathcal{F}^j{}_I v_{j|l} \mathcal{F}^l{}_K. \tag{14}$$

If such derivatives may be found the resulting operation shall be called *covariant derivative* (CD). A suited ansatz to solve the above problem is to introduce third-order objects $\mathcal{L}^I{}_{KL}$ and $\ell^i{}_{jk}$, the so-called linear (or affine) *connection*. Then the connection allows to reinstall the transformation behavior of the covariant derivatives of vectors and covectors provided the connection satisfies the following non tensorial transformation properties[4]

$$\begin{aligned} \mathcal{F}^i{}_{M,L} f^L{}_k &= \mathcal{F}^i{}_J \mathcal{L}^J{}_{ML} f^L{}_k &-& \ell^i{}_{nk} \mathcal{F}^n{}_M, \\ f^I{}_{m,l} \mathcal{F}^l{}_K &= f^I{}_j \ell^j{}_{ml} \mathcal{F}^l{}_K &-& \mathcal{L}^I{}_{NK} f^N{}_m. \end{aligned} \tag{15}$$

[4] A special case occurs whenever the covariant and the partial derivative coincide for a particular coordinate system, which is only possible for a Cartesian coordinate system in Euclidean space, i.e. in a flat manifold. Then the connection in the Cartesian coordinates vanishes identically and the connection in the transformed coordinates consequently reads

$$\ell^i{}_{jk} = F^i{}_A f^A{}_{j,k} \quad \text{and} \quad \mathcal{L}^I{}_{JK} = f^I{}_a F^a{}_{J,K}.$$

Connections of this type are also denoted as *integrable connections*, the reason for this terminology becoming clear only after the concept of curvature has been introduced.

By rearrangement these non tensorial transformation properties of the connection may also be stated equivalently as

$$\begin{aligned} f^M{}_{i,k} &= f^M{}_n \ell^n{}_{ik} - \mathcal{L}^M{}_{JL} f^J{}_i f^L{}_k, \\ \mathcal{F}^m{}_{I,K} &= \mathcal{F}^m{}_N \mathcal{L}^N{}_{IK} - \ell^m{}_{jl} \mathcal{F}^j{}_I \mathcal{F}^l{}_K. \end{aligned} \tag{16}$$

By combining the transformation behaviour of the connection in Eq. (15) with that of the partial derivative of a vector in Eq. (11) the covariant derivative of a vector is eventually given by

$$v^i{}_{|j} = v^i{}_{,j} + \ell^i{}_{mj} v^m \quad \text{and} \quad \mathcal{V}^I{}_{|J} = \mathcal{V}^I{}_{,J} + \mathcal{L}^I{}_{MJ} \mathcal{V}^M. \tag{17}$$

Please observe that the position for the running index m or M, repectively, and thus the precise arrangement of indices in all later expressions that involve the connection varies in the literature, however once defined as in the above it only matters to consequently stick to this convention in the sequel. Likewise the covariant derivative of a covector follows from inserting the transformation in Eq. (16) into Eq. (12) to render

$$v_{i|j} = v_{i,j} - v_m \ell^m{}_{ij} \quad \text{and} \quad \mathcal{V}_{I|J} = \mathcal{V}_{I,J} - \mathcal{V}_M \mathcal{L}^M{}_{IJ}. \tag{18}$$

Then the covariant derivatives of the four types of (simple) second-order tensors follow from the product rule applied to their dyadic representation

$$\begin{aligned} \mathcal{T}^{IJ}{}_{|K} &= \mathcal{T}^{IJ}{}_{,K} + \mathcal{L}^I{}_{MK} \mathcal{T}^{MJ} + \mathcal{L}^J{}_{MK} \mathcal{T}^{IM}, \\ \mathcal{T}^I{}_{J|K} &= \mathcal{T}^I{}_{J,K} + \mathcal{L}^I{}_{MK} \mathcal{T}^M{}_J - \mathcal{L}^M{}_{JK} \mathcal{T}^I{}_M, \\ \mathcal{T}_{IJ|K} &= \mathcal{T}_{IJ,K} - \mathcal{L}^M{}_{IK} \mathcal{T}_{MJ} - \mathcal{L}^M{}_{JK} \mathcal{T}_{IM}, \\ \mathcal{T}_I{}^J{}_{|K} &= \mathcal{T}_I{}^J{}_{,K} - \mathcal{L}^M{}_{IK} \mathcal{T}_M{}^J + \mathcal{L}^J{}_{MK} \mathcal{T}_I{}^M. \end{aligned} \tag{19}$$

Again these expressions do also hold for general second-order tensors that are constructed from a sum of simple second-order tensors. The covariant derivatives of higher-order tensors (and objects) follow likewise, e.g. for third-order objects as occurring in the sequel one finds

$$\begin{aligned} \mathcal{T}^I{}_{JL|K} &= \mathcal{T}^I{}_{JL,K} + \mathcal{L}^I{}_{MK} \mathcal{T}^M{}_{JL} - \mathcal{L}^M{}_{JK} \mathcal{T}^I{}_{ML} - \mathcal{L}^M{}_{LK} \mathcal{T}^I{}_{JM}, \\ \mathcal{T}_{IJL|K} &= \mathcal{T}_{IJL,K} - \mathcal{L}^M{}_{IK} \mathcal{T}_{MJL} - \mathcal{L}^M{}_{JK} \mathcal{T}_{IML} - \mathcal{L}^M{}_{LK} \mathcal{T}_{IJM}. \end{aligned} \tag{20}$$

Based on its definition the covariant derivative obeys a number of important rules, for example:

- The CD of scalars coincides with the PD of scalars,

- The CD obeys the distribution rule,
- The CD obeys the Leibniz (product) rule.

Proof:
The first and second rule are obvious, the proof of the last rule is based on the application of the partial derivative to the contraction of a vector and a covector into a scalar (whereby opposite but otherwise identical indices follow the Einstein summation rule)

$$[\mathcal{V}^I \mathcal{V}_I]_{,J} = \mathcal{V}^I{}_{,J} \mathcal{V}_I + \mathcal{V}^I \mathcal{V}_{I,J}. \tag{21}$$

Since based on the first rule the partial and the covariant derivatives of scalars coincide it also holds that

$$[\mathcal{V}^I \mathcal{V}_I]_{|J} = [\mathcal{V}^I{}_{,J} + \mathcal{L}^I{}_{MJ} \mathcal{V}^M] \mathcal{V}_I + \mathcal{V}^I [\mathcal{V}_{I,J} - \mathcal{V}_M \mathcal{L}^M{}_{IJ}] \doteq [V^I V_I]_{,J}. \tag{22}$$

Comparing the two results in Eqs. (21) and (22) and noting that $\mathcal{L}^I{}_{MJ} \mathcal{V}^M \mathcal{V}_I \equiv \mathcal{V}^I \mathcal{V}_M \mathcal{L}^M{}_{IJ}$ concludes the proof. □

2.4 Parallel Transport

It shall be observed that in general tangent spaces $T_{\mathcal{P}}\mathcal{M}$ and cotangent spaces $T^*_{\mathcal{P}}\mathcal{M}$ at different points $\mathcal{P}$ of a manifold $\mathcal{M}$ are not connected. However, if a covariant derivative of vectors $\mathcal{V}^I|_K = \mathcal{V}^I{}_{,K} + \mathcal{L}^I{}_{JK} \mathcal{V}^J$ and covectors $\mathcal{V}_J|_K = \mathcal{V}_{J,K} - \mathcal{V}_I \mathcal{L}^I{}_{JK}$ based on a linear connection $\mathcal{L}^I{}_{JK}$ is introduced, the notion of *parallel transport* may be defined. Thus the bundle $T\mathcal{M}$ of tangent spaces $T_{\mathcal{P}}\mathcal{M}$ and the bundle $T^*\mathcal{M}$ of cotangent spaces $T^*_{\mathcal{P}}\mathcal{M}$ constitute affinely connected spaces.

Thereby the motivation for the notion of parallel transport $\mathrm{p}\mathcal{V}^I$ of a vector $\mathcal{V}^I$ is as follows: The comparison of two vectors $\mathcal{V}^I(\{\mathcal{X}^J + \mathrm{d}\mathcal{X}^J\})$ and $\mathcal{V}^I(\{\mathcal{X}^J\})$ in two different (infinitesimal close) tangent spaces attached to $\{\mathcal{X}^J\}$ and $\{\mathcal{X}^J + \mathrm{d}\mathcal{X}^J\}$ necessitates first a parallel (back) transport of $\mathcal{V}^I(\{\mathcal{X}^J + \mathrm{d}\mathcal{X}^J\})$ to $\{\mathcal{X}^J\}$. Thereby this parallel transport is assumed proportional to $\mathcal{V}^K$ and $\mathrm{d}\mathcal{X}^J$, i.e. $\mathrm{p}\mathcal{V}^I := -\mathcal{L}^I{}_{KJ} \mathcal{V}^K \,\mathrm{d}\mathcal{X}^J$, the minus sign (and the sequence of indices) being convention. The argument holds likewise for covectors. From these considerations we may derive the

Definition:
The transport along a parameter curve $\mathcal{X}^J(t)$ of a vector $\mathcal{V}^I$ that is attached to a manifold is called parallel if the covariant derivative (or rather the *covariant differential* $\mathrm{D}\mathcal{V}^I$) of $\mathcal{V}^I$ vanishes

$$\mathcal{V}^I{}_{|J} = \mathcal{V}^I{}_{,J} + \mathcal{L}^I{}_{KJ} \mathcal{V}^K = 0 \quad \text{with} \quad \mathrm{D}\mathcal{V}^I := \mathcal{V}^I{}_{|J} \,\mathrm{d}\mathcal{X}^J = 0. \tag{23}$$

Thus for a parallel transport the change of the vector in the direction of the parameter curve (directional derivative) follows as

$$\mathrm{d}\mathcal{V}^I := \mathcal{V}^I{}_{,J}\,\mathrm{d}\mathcal{X}^J \equiv -\mathcal{L}^I{}_{KJ}\mathcal{V}^K\,\mathrm{d}\mathcal{X}^J =: \mathrm{p}\mathcal{V}^I. \tag{24}$$

The notation $\mathrm{p}\mathcal{V}^I$ for the parallel transport of $\mathcal{V}^I$ is motivated by simply rotating the common notation for a differential $\mathrm{d}\mathcal{V}^I$ upside down. □

As a conclusion it may be stated that for a covariant derivative the change of a vector $\mathcal{V}^I$ due to its partial derivative with respect to the coordinates, i.e.

$$\mathrm{d}\mathcal{V}^I := \mathcal{V}^I{}_{,J}\,\mathrm{d}\mathcal{X}^J = \mathcal{V}^I(\{\mathcal{X}^J + \mathrm{d}\mathcal{X}^J\}) - \mathcal{V}^I(\{\mathcal{X}^J\}) \tag{25}$$

is corrected by the contribution of the parallel transport

$$\mathrm{p}\mathcal{V}^I := -\mathcal{L}^I{}_{KJ}\mathcal{V}^K\,\mathrm{d}\mathcal{X}^J \tag{26}$$

to render the covariant differential

$$\mathrm{D}\mathcal{V}^I := \mathrm{d}\mathcal{V}^I - \mathrm{p}\mathcal{V}^I = \left[\mathcal{V}^I{}_{,J} + \mathcal{L}^I{}_{KJ}\mathcal{V}^K\right]\mathrm{d}\mathcal{X}^J. \tag{27}$$

It shall be noted that the same arguments hold likewise for covectors to render eventually

$$\mathrm{D}\mathcal{V}_J := \mathrm{d}\mathcal{V}_J - \mathrm{p}\mathcal{V}_J = \left[\mathcal{V}_{J,K} - \mathcal{V}_I\mathcal{L}^I{}_{JK}\right]\mathrm{d}\mathcal{X}^K. \tag{28}$$

2.5 Torsion

Transformation of Connection As a motivation for the introduction of the *torsion* remember that the linear (or affine) connections $\mathcal{L}^I{}_{KL}$ and ℓ^i_{jk} do not transform like a tensor, but according to Eqs. (15) and (16) transforms rather like

$$\begin{aligned}
\ell^i_{jk} &= \mathcal{F}^i_I\ \mathcal{L}^I{}_{JK}\ f^J_j\ f^K{}_k + \mathcal{F}^i_I\ f^I{}_{j,k},\\
\mathcal{L}^I{}_{JK} &= f^I_i\ \ell^i_{jk}\ \mathcal{F}^j{}_J\ \mathcal{F}^k{}_K + f^I_i\ \mathcal{F}^i{}_{J,K}.
\end{aligned} \tag{29}$$

Observe that it is the second term in each line that conflicts with a tensorial transformation behaviour. By resorting to the following easy to proof relations for the partial derivatives of the tangent maps

$$\mathcal{F}^i{}_I f^I{}_{j,k} = -\mathcal{F}^i{}_{J,K} f^J{}_j f^K{}_k \quad \text{and} \quad f^I{}_i \mathcal{F}^i{}_{J,K} = -f^I{}_{j,k}\mathcal{F}^j{}_J\mathcal{F}^k{}_K \tag{30}$$

it is useful in the sequel to express the transformation of the connections also alternatively as

$$\begin{aligned} \ell^i_{jk} &= \mathcal{F}^i_I \quad \mathcal{L}^I_{JK} \quad f^J_j \quad f^K_k \quad - \quad \mathcal{F}^i_{J,K} \quad f^J_j \quad f^K_k, \\ \mathcal{L}^I_{JK} &= f^I_i \quad \ell^i_{jk} \quad \mathcal{F}^j_J \quad \mathcal{F}^k_K \quad - \quad f^I_{j,k} \quad \mathcal{F}^j_J \quad \mathcal{F}^k_K. \end{aligned} \tag{31}$$

It shall be observed carefully that these transformations of the connections are valid for *holonomic* as well as *anholonomic* coordinate transformations. Here holonomic and anholonomic refers to the *integrability* and *non-integrability* of the tangent map $\mathcal{F}^i_I$ (or likewise f^I_i) into a map $\chi^i = y^i(\mathcal{X}^I)$ (or likewise $\mathcal{X}^I = \mathcal{Y}^I(\chi^i)$).

Holonomic Transformation It is obvious from the previous discussion that the connections $\mathcal{L}^I_{KL}$ and ℓ^i_{jk} do not transform like third-order tensors. Under a holonomic change of coordinates, however, due to the symmetry of the second partial derivatives contained in the second terms of Eq. (29) its (right) skew symmetric contribution does

$$\begin{aligned} \mathcal{F}^i_{[J,K]} &= 0 \quad \rightarrow \quad \ell^i_{[jk]} = \mathcal{F}^i_I \quad \mathcal{L}^I_{[JK]} \quad f^J_j \quad f^K_k, \\ f^I_{[j,k]} &= 0 \quad \rightarrow \quad \mathcal{L}^I_{[JK]} = f^I_i \quad \ell^i_{[jk]} \quad \mathcal{F}^j_J \quad \mathcal{F}^k_K. \end{aligned} \tag{32}$$

Here, skew symmetry in an index pair is denoted by square brackets, i.e. for example $\mathcal{F}^i_{[J,K]} := [\mathcal{F}^i_{J,K} - \mathcal{F}^i_{K,J}]/2$.

Now as a new object, the skew symmetric part of the connection is called the (Cartan) *torsion* or rather the *torsion tensor*

$$\mathcal{T}^I_{JK} := \mathcal{L}^I_{[JK]} \quad \text{and} \quad t^i_{jk} := \ell^i_{[jk]}. \tag{33}$$

The meaning of the torsion can be highlighted by considering the situation sketched in Fig. 1, compare also to Schouten (1989). The parallel transport of two coordinate differentials $\mathrm{d}\mathcal{X}^I$ and $\mathrm{d}\mathcal{Y}^I$ along each other results in a pentagon formed by $\mathrm{d}\mathcal{X}^I$ and $\mathrm{d}\mathcal{Y}^I$ together with the parallel transported coordinate differentials

$$\mathrm{d}\mathcal{X}^I_{(\circ)\rightarrow(\circ\circ)} = \mathrm{d}\mathcal{X}^I - \mathcal{L}^I_{JK}\,\mathrm{d}\mathcal{X}^J\,\mathrm{d}\mathcal{Y}^K \tag{34}$$

and

$$\mathrm{d}\mathcal{Y}^I_{(\bullet)\rightarrow(\bullet\bullet)} = \mathrm{d}\mathcal{Y}^I - \mathcal{L}^I_{JK}\,\mathrm{d}\mathcal{Y}^J\,\mathrm{d}\mathcal{X}^K. \tag{35}$$

From the situation sketched in Fig. 1 it is thus clear that torsion measures the closure gap

$$\mathrm{d}\mathcal{X}^I + \mathrm{d}\mathcal{Y}^I_{(\bullet)\rightarrow(\bullet\bullet)} - \mathrm{d}\mathcal{Y}^I - \mathrm{d}\mathcal{X}^I_{(\circ)\rightarrow(\circ\circ)} = \tag{36}$$

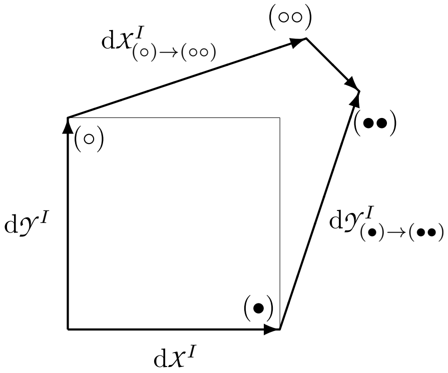

Figure 1. In a space with torsion parallel transport of coordinate differentials along each other results in a pentagon.

$$\mathcal{L}^I{}_{JK}\,\mathrm{d}\mathcal{X}^J\,\mathrm{d}\mathcal{Y}^K - \mathcal{L}^I{}_{JK}\,\mathrm{d}\mathcal{Y}^J\,\mathrm{d}\mathcal{X}^K = 2\mathcal{T}^I{}_{JK}\,\mathrm{d}\mathcal{X}^J\,\mathrm{d}\mathcal{Y}^K.$$

As a result infinitesimal parallelograms constructed from coordinate differentials do only exist in spaces with vanishing torsion.

Anholonomic Transformation Recall that the connections $\mathcal{L}^I{}_{KL}$ and $\ell^i{}_{jk}$ do not transform like third-order tensors. Under an anholonomic change of coordinates its (right) skew symmetric contribution thus transforms as

$$\begin{aligned} \mathcal{F}^i{}_{[J,K]} &\neq 0 \to \ell^i{}_{[jk]} + a^i{}_{jk} = \mathcal{F}^i{}_I\ \mathcal{T}^I{}_{JK}\ f^J{}_j\ f^K{}_k, \\ f^I{}_{[j,k]} &\neq 0 \to \mathcal{L}^I{}_{[JK]} + \mathcal{A}^I{}_{JK} = f^I{}_i\ t^i{}_{jk}\ \mathcal{F}^j{}_J\ \mathcal{F}^k{}_K, \end{aligned} \tag{37}$$

whereby the torsion in the holonomic coordinates follows the standard definition

$$\mathcal{T}^I{}_{JK} = \mathcal{L}^I{}_{[JK]} \quad \text{and} \quad t^i{}_{jk} = \ell^i{}_{[jk]}. \tag{38}$$

The additional contribution appearing in the transformation due to the lack of integrability is called the *anholonomic object*:

$$a^i{}_{jk} := \mathcal{F}^i{}_{[J,K]} f^J{}_j f^K{}_k \quad \text{and} \quad \mathcal{A}^I{}_{JK} := f^I{}_{[j,k]} \mathcal{F}^j{}_J \mathcal{F}^k{}_K. \tag{39}$$

It shall be noted that in the above expressions either the coordinates χ^i in the first row of Eq. (37) or the coordinates $\mathcal{X}^I$ in the second row of Eq. (37) are anholonomic. Based on the anholonomic object and the representation in Eq. (37) the torsion in a space that is equipped with anholonomic coordinates follows from the

Definition:
The torsion in an anholonomic space with either anholonomic coordinate χ^i or anholonomic coordinates $\mathcal{X}^I$, respectively, is given as

$$t^i{}_{jk} := \ell^i{}_{[jk]} + a^i{}_{jk} \quad \text{and} \quad \mathcal{T}^I{}_{JK} := \mathcal{L}^I{}_{[JK]} + \mathcal{A}^I{}_{JK}. \tag{40}$$

The situation is highlighted in Fig. 2. □

It will be shown in the sequel, that the anholonomic objects may be associated with *dislocation density tensors*. Thereby, quite like in the definition of the various stress measures in nonlinear continuum mechanics, Piola-type anholonomic objects corresponding to two-point description dislocation density tensors together with Cauchy-type anholonomic objects follow from the

Definition:
The Piola-type anholonomic object corresponding to the two-point description dislocation density tensor is given by

$$\mathcal{D}^i{}_{JK} := \mathcal{F}^i{}_{[J,K]} \quad \text{and} \quad d^I{}_{jk} := f^I{}_{[j,k]}. \tag{41}$$

Consequently the anholonomic object introduced previously is of Cauchy-type and results from a convection (push-forward/pull-back) by the corre-

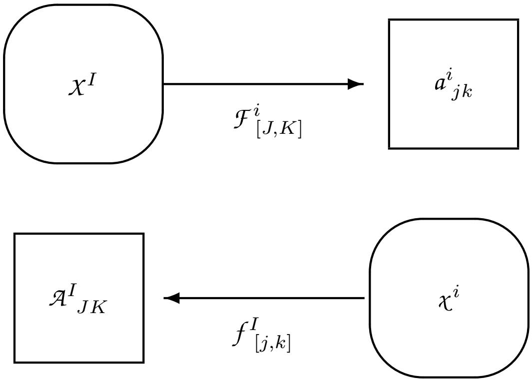

Figure 2. The anholonomic object characterizes the non-integrability of the tangent map, i.e. the transformation of coordinate differentials. In the top figure χ^i are anholonomic while $\mathcal{X}^I$ are holonomic; in the bottom figure the situation is reversed, i.e. $\mathcal{X}^I$ are anholonomic and χ^i are holonomic.

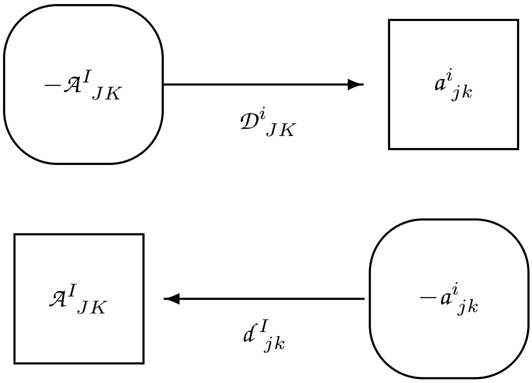

Figure 3. The anholonomic object characterizes the non-integrability of the tangent map, i.e. the transformation of coordinate differentials. In the top figure χ^i are anholonomic while X^I are holonomic; in the bottom figure the situation is reversed, i.e. X^I are anholonomic and χ^i are holonomic.

sponding tangent map

$$a^i{}_{jk} := \mathcal{D}^i{}_{JK} f^J{}_j f^K{}_k \quad \text{and} \quad \mathcal{A}^I{}_{JK} := d^I{}_{jk} \mathcal{F}^j{}_J \mathcal{F}^k{}_K. \tag{42}$$

The situation is highlighted in Fig. 3. □

Finally, Piola-Kirchhoff-type anholonomic objects may be regarded either as the pull-back/push-forward of the Piola-type or the Cauchy-type anholonomic objects due to the

Definition:
The Piola-Kirchhoff-type anholonomic object follows from the convection (pull-back/push-forward) of the Piola-type anholonomic object by the corresponding tangent map

$$-\mathcal{A}^I{}_{JK} = f^I{}_i \mathcal{D}^i{}_{JK} \quad \text{and} \quad -a^i{}_{jk} = \mathcal{F}^i{}_I d^I{}_{jk}. \tag{43}$$

It coincides with the definition of the previously introduced Cauchy-type anholonomic object if the following *anholonomic partial derivatives* are defined

$$f^I{}_i \mathcal{F}^i{}_{[J,K]} =: -f^I{}_{[j,k]} \mathcal{F}^j{}_J \mathcal{F}^k{}_K \quad \text{and} \quad \mathcal{F}^i{}_I f^I{}_{[j,k]} =: -\mathcal{F}^i{}_{[J,K]} f^J{}_j f^K{}_k. \tag{44}$$

Recall that in the above expressions either the coordinates χ^i or the coordinates X^I are anholonomic. The situation is highlighted in Fig. 3. □

From the above definitions it is clear that the terminology *Cauchy-type* and *Piola-Kirchhoff-type* is used interchangeably if instead of the tangent map $\mathcal{F}^i_{\ I} : \mathrm{d}\mathcal{X}^I \mapsto \mathrm{d}\chi^i$ the (inverse) tangent map $f^I_{\ i} : \mathrm{d}\chi^i \mapsto \mathrm{d}\mathcal{X}^I$ is considered.

2.6 Curvature

The notion of *curvature* or rather the *curvature tensor* is central to the differential geometry of manifolds. Formally the curvature tensor is introduced by the following

Definition:
Based on the linear connection a fourth-order object, the curvature tensor, is defined as:

$$\mathcal{R}^I_{\ JKL} := \mathcal{L}^I_{\ JL,K} - \mathcal{L}^I_{\ JK,L} + \mathcal{L}^I_{\ MK}\mathcal{L}^M_{\ JL} - \mathcal{L}^I_{\ ML}\mathcal{L}^M_{\ JK}. \tag{45}$$

The tensorial transformation properties of the curvature tensor will be demonstrated later. □

From its definition the curvature tensor obeys the following skew symmetries:

$$\mathcal{R}^I_{\ JKL} = 2\mathcal{L}^I_{\ J[L,K]} + 2\mathcal{L}^I_{\ M[K}\mathcal{L}^M_{\ JL]} = \mathcal{R}^I_{\ J[KL]}. \tag{46}$$

Note carefully that here, in contrast to most of the literature on differential geometry, the notation for skew symmetrization of the two indices in the term quadratic in the connection is used in the following format

$$2\mathcal{L}^I_{\ M[K}\mathcal{L}^M_{\ JL]} := \mathcal{L}^I_{\ MK}\mathcal{L}^M_{\ JL} - \mathcal{L}^I_{\ ML}\mathcal{L}^M_{\ JK}. \tag{47}$$

This somewhat less heavy notation is here preferred over the traditional $\mathcal{L}^I_{\ M[K}\mathcal{L}^M_{\ |J|L]}$. Less formal and more operational is the alternative

Definition:
The curvature tensor determines the change of a vector $\mathcal{V}^I$ for a parallel transport along infinitesimal closed curves as

$$\Delta\mathcal{V}^I = \mathcal{R}^I_{\ JKL}\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K\,\mathrm{d}\mathcal{Y}^L. \tag{48}$$

Thus, upon transporting a vector $\mathcal{V}^I$ parallel along infinitesimal closed curves if suffers a change $\Delta\mathcal{V}^I$ that depends on the curvature tensor, i.e. on the curvature of the manifold.

Moreover, the curvature tensor also determines the skew symmetric contribution to the second covariant derivatives of a vector

$$2\mathcal{V}^I_{\ |[KL]} = -\mathcal{R}^I_{\ JKL}\mathcal{V}^J - 2\mathcal{V}^I_{\ |M}\mathcal{T}^M_{\ KL}. \tag{49}$$

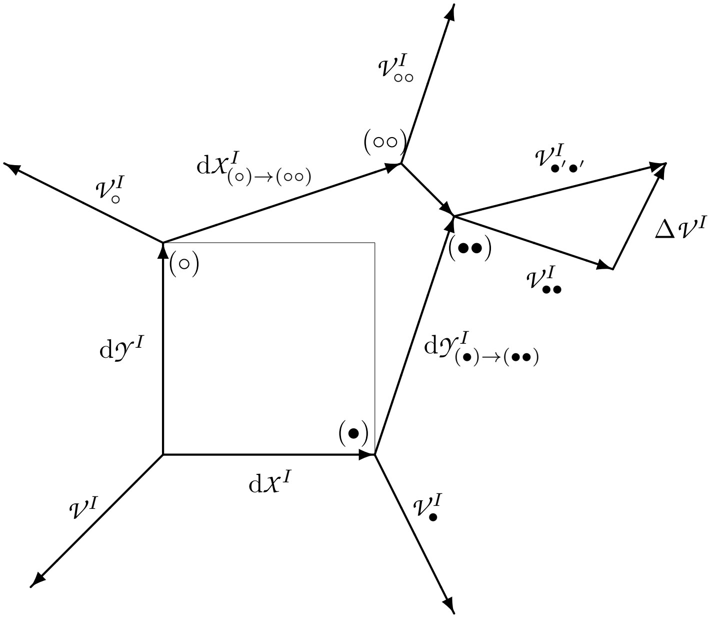

Figure 4. Parallel transport of a vector $\mathcal{V}^I$ along an infinitesimal closed curve. Due to the curvature $\mathcal{V}^I$ suffers a change $\Delta\mathcal{V}^I$. The closed curve consists of the coordinate differentials $\mathrm{d}\mathcal{X}^I$ and $\mathrm{d}\mathcal{Y}^I$ together with their parallel transport along each other $\mathrm{d}\mathcal{Y}^I_{(\bullet)\to(\bullet\bullet)}$ and $\mathrm{d}\mathcal{X}^I_{(\circ)\to(\circ\circ)}$ and the resulting closure gap, compare Fig. 1.

Observe that the second covariant derivatives of a vector involves in particular the torsion. □

Note that similar operational definitions of the curvature tensor hold in terms of covectors. The concrete representation and a proof are however left to the reader.

In the sequel both, the change of a vector $\mathcal{V}^I$ for a *parallel transport along infinitesimal closed curves* and the *skew symmetric contribution to the second covariant derivatives* of a vector shall be investigated.

Parallel Transport Along Infinitesimal Closed Curves By referring to Fig. 4 the proof of $\Delta\mathcal{V}^I = \mathcal{R}^I{}_{JKL}\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K\,\mathrm{d}\mathcal{Y}^L$ may be sketched in nine steps:

1. Parallel transport of $\mathcal{V}^I$ to $(\bullet)$ and connection $\mathcal{L}^I{}_{JK\bullet}$ at $(\bullet)$,
2. Parallel transport of $\mathcal{V}^I_\bullet$ to $(\bullet\bullet)$,

3. Retain terms up to quadratic order,
4. Parallel transport of $\mathcal{V}^I$ to ($\circ$) and connection $\mathcal{L}^I{}_{JK\circ}$ at ($\circ$),
5. Parallel transport of $\mathcal{V}^I_\circ$ to ($\circ\circ$),
6. Retain terms up to quadratic order,
7. Parallel transport from ($\circ\circ$) to ($\bullet\bullet$),
   retain terms up to quadratic order,
8. Substract,
9. Express Result in terms of the curvature tensor.

Proof:
These steps shall now be outlined in more detail:

1. Parallel transport of $\mathcal{V}^I$ to ($\bullet$) and connection $\mathcal{L}^I{}_{JK\bullet}$ at ($\bullet$):

$$\mathcal{V}^I_\bullet = \mathcal{V}^I - \mathcal{L}^I{}_{JK}\mathcal{V}^J \,\mathrm{d}\mathcal{X}^K \quad \text{and} \quad \mathcal{L}^I{}_{JK\bullet} = \mathcal{L}^I{}_{JK} + \mathcal{L}^I{}_{JK,L}\,\mathrm{d}\mathcal{X}^L.$$

2. Parallel transport of $\mathcal{V}^I_\bullet$ to ($\bullet\bullet$):

$$\mathcal{V}^I_{\bullet\bullet} = \mathcal{V}^I_\bullet - \mathcal{L}^I{}_{JK\bullet}\mathcal{V}^J_\bullet\left[\mathrm{d}\mathcal{Y}^K - \mathcal{L}^K{}_{OP}\,\mathrm{d}\mathcal{Y}^O\,\mathrm{d}\mathcal{X}^P\right] =$$

$$\left[\mathcal{V}^I - \mathcal{L}^I{}_{JK}\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K\right] - \left[\mathcal{L}^I{}_{JK} + \mathcal{L}^I{}_{JK,L}\,\mathrm{d}\mathcal{X}^L\right]\times$$

$$\left[\mathcal{V}^J - \mathcal{L}^J{}_{MN}\mathcal{V}^M\,\mathrm{d}\mathcal{X}^N\right]\times\left[\mathrm{d}\mathcal{Y}^K - \mathcal{L}^K{}_{OP}\,\mathrm{d}\mathcal{Y}^O\,\mathrm{d}\mathcal{X}^P\right].$$

3. Retain terms up to quadratic order in the coordinate differentials:

$$\mathcal{V}^I_{\bullet\bullet} = \mathcal{V}^I - \mathcal{L}^I{}_{JK}\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K - \mathcal{L}^I{}_{JK}\mathcal{V}^J\,\mathrm{d}\mathcal{Y}^K - \mathcal{L}^I{}_{JK,L}\,\mathrm{d}\mathcal{X}^L\mathcal{V}^J\,\mathrm{d}\mathcal{Y}^K$$

$$+\mathcal{L}^I{}_{JK}\mathcal{L}^J{}_{MN}\mathcal{V}^M\,\mathrm{d}\mathcal{X}^N\,\mathrm{d}\mathcal{Y}^K + \mathcal{L}^I{}_{JK}\mathcal{L}^K{}_{OP}\mathcal{V}^J\,\mathrm{d}\mathcal{Y}^O\,\mathrm{d}\mathcal{X}^P.$$

4. Parallel transport of $\mathcal{V}^I$ to ($\circ$) and connection $\mathcal{L}^I{}_{JK\circ}$ at ($\circ$):

$$\mathcal{V}^I_\circ = \mathcal{V}^I - \mathcal{L}^I{}_{JK}\mathcal{V}^J\,\mathrm{d}\mathcal{Y}^K \quad \text{and} \quad \mathcal{L}^I{}_{JK\circ} = \mathcal{L}^I{}_{JK} + \mathcal{L}^I{}_{JK,L}\,\mathrm{d}\mathcal{Y}^L.$$

5. Parallel transport of $\mathcal{V}^I_\circ$ to ($\circ\circ$):

$$\mathcal{V}^I_{\circ\circ} = \mathcal{V}^I_\circ - \mathcal{L}^I{}_{JK\circ}\mathcal{V}^J_\circ\left[\mathrm{d}\mathcal{X}^K - \mathcal{L}^K{}_{OP}\,\mathrm{d}\mathcal{X}^O\,\mathrm{d}\mathcal{Y}^P\right] =$$

$$\left[\mathcal{V}^I - \mathcal{L}^I{}_{JK}\mathcal{V}^J\,\mathrm{d}\mathcal{Y}^K\right] - \left[\mathcal{L}^I{}_{JK} + \mathcal{L}^I{}_{JK,L}\,\mathrm{d}\mathcal{Y}^L\right]\times$$

$$\left[\mathcal{V}^J - \mathcal{L}^J{}_{MN}\mathcal{V}^M\,\mathrm{d}\mathcal{Y}^N\right]\times\left[\mathrm{d}\mathcal{X}^K - \mathcal{L}^K{}_{OP}\,\mathrm{d}\mathcal{X}^O\,\mathrm{d}\mathcal{Y}^P\right].$$

6. Retain terms up to quadratic order in the coordinate differentials:

$$\mathcal{V}^I_{\circ\circ} = \mathcal{V}^I - \mathcal{L}^I{}_{JK}\mathcal{V}^J\,\mathrm{d}\mathcal{Y}^K - \mathcal{L}^I{}_{JK}\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K - \mathcal{L}^I{}_{JK,L}\,\mathrm{d}\mathcal{Y}^L\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K$$

$$+\mathcal{L}^I{}_{JK}\mathcal{L}^J{}_{MN}\mathcal{V}^M\,\mathrm{d}\mathcal{Y}^N\,\mathrm{d}\mathcal{X}^K+\mathcal{L}^I{}_{JK}\mathcal{L}^K{}_{OP}\mathcal{V}^J\,\mathrm{d}\mathcal{X}^O\,\mathrm{d}\mathcal{Y}^P.$$

7. Subtract:

$$\mathcal{V}^I_{\circ\circ}-\mathcal{V}^I_{\bullet\bullet}=-2\mathcal{L}^I{}_{J[K,L]}\,\mathrm{d}\mathcal{Y}^L\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K$$

$$+2\mathcal{L}^I{}_{J[K}\mathcal{L}^J{}_{MN]}\mathcal{V}^M\,\mathrm{d}\mathcal{Y}^N\,\mathrm{d}\mathcal{X}^K+2\mathcal{L}^I{}_{JK}\mathcal{L}^K{}_{[OP]}\mathcal{V}^J\,\mathrm{d}\mathcal{X}^O\,\mathrm{d}\mathcal{Y}^P.$$

8. Parallel transport from $(\circ\circ)$ to $(\bullet\bullet)$, retain terms up to quadratic order in the coordinate differentials:

$$\mathcal{V}^I_{\bullet'\bullet'}=\mathcal{V}^I_{\circ\circ}-\mathcal{L}^I{}_{JK}\mathcal{V}^J\left[2\mathcal{L}^K{}_{[OP]}\,\mathrm{d}\mathcal{X}^O\,\mathrm{d}\mathcal{Y}^P\right].$$

9. Subtract:

$$\mathcal{V}^I_{\bullet'\bullet'}-\mathcal{V}^I_{\bullet\bullet}=-2\mathcal{L}^I{}_{J[K,L]}\,\mathrm{d}\mathcal{Y}^L\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K+2\mathcal{L}^I{}_{J[K}\mathcal{L}^J{}_{MN]}\mathcal{V}^M\,\mathrm{d}\mathcal{Y}^N\,\mathrm{d}\mathcal{X}^K.$$

Thus, in summary the change of the vector $\mathcal{V}^I$ may be expressed in terms of the curvature tensor as defined in Eq. (46)

$$\Delta\mathcal{V}^I:=\mathcal{V}^I_{\bullet'\bullet'}-\mathcal{V}^I_{\bullet\bullet}=:\mathcal{R}^I{}_{JKL}\mathcal{V}^J\,\mathrm{d}\mathcal{X}^K\,\mathrm{d}\mathcal{Y}^L. \tag{50}$$

Clearly, from the above derivation and in accordance with the definition in Eq. (46) the curvature tensor is eventually recognized as

$$\frac{1}{2}\mathcal{R}^I{}_{JKL}=\mathcal{L}^I{}_{J[L,K]}+\mathcal{L}^I{}_{M[K}\mathcal{L}^M{}_{JL]}. \tag{51}$$

This concludes the proof. □

Skew Symmetric Contribution to Second Covariant Derivatives The second covariant derivative of a vector is computed as the covariant derivative of the (mixedvariant) second-order tensor represented by $\mathcal{V}^I{}_{|J}$, see the definition in Eq. (19).2

$$\mathcal{V}^I{}_{|JK}=\mathcal{V}^I{}_{|J,K}+\mathcal{L}^I{}_{MK}\mathcal{V}^M{}_{|J}-\mathcal{L}^M{}_{JK}\mathcal{V}^I{}_{|M}. \tag{52}$$

Involving next the covariant derivative of a vector in Eq. (17) inflates the above expression to

$$\begin{aligned}\mathcal{V}^I{}_{|JK}=\mathcal{V}^I{}_{,JK}\quad&+\quad\mathcal{L}^I{}_{MJ}\mathcal{V}^M{}_{,K}+\mathcal{L}^I{}_{MJ,K}\mathcal{V}^M\\&+\quad\mathcal{L}^I{}_{MK}\mathcal{V}^M{}_{,J}+\mathcal{L}^I{}_{MK}\mathcal{L}^M{}_{NJ}\mathcal{V}^N\\&-\quad\mathcal{L}^M{}_{JK}\mathcal{V}^I{}_{|M}.\end{aligned} \tag{53}$$

Likewise, changing the sequence of the indices JK renders the corresponding result

$$\begin{aligned} \mathcal{V}^I{}_{|KJ} = \mathcal{V}^I{}_{,KJ} &+ \mathcal{L}^I{}_{MK}\mathcal{V}^M{}_{,J} + \mathcal{L}^I{}_{MK,J}\mathcal{V}^M \\ &+ \mathcal{L}^I{}_{MJ}\mathcal{V}^M{}_{,K} + \mathcal{L}^I{}_{MJ}\mathcal{L}^M{}_{NK}\mathcal{V}^N \\ &- \mathcal{L}^M{}_{KJ}\mathcal{V}^I{}_{|M}. \end{aligned} \tag{54}$$

Finally, substracting the two results in Eqs. (53) and (54) and taking into account the symmetry of the second partial derivatives renders the skew symmetric contribution to the second covariant derivative of a vector in terms of the curvature and the torsion

$$\mathcal{V}^I{}_{|[JK]} = \underbrace{\left[\mathcal{L}^I{}_{N[J,K]} + \mathcal{L}^I{}_{M[K}\mathcal{L}^M{}_{NJ]}\right]}_{-\mathcal{R}^I{}_{NJK}/2} \mathcal{V}^N - \mathcal{T}^M{}_{JK}\mathcal{V}^I{}_{|M}. \tag{55}$$

A similar result may be derived for the second covariant derivative of a covector.

Transformation of the Curvature Tensor Due to the tensor property of the curvature the following convection or rather pull-back ($\mathcal{Y}$)/push-forward (y) relations hold in the case of holonomic coordinate transformations:

$$\boxed{\text{Curvature}(\mathcal{Y}(\text{connection}))=\mathcal{Y}(\text{curvature}(\text{connection}))}$$

$$\boxed{\text{curvature}(y(\text{Connection}))=y(\text{Curvature}(\text{Connection}))}$$

As an example the pull-back of the spatial curvature expressed in terms of the spatial connection equals the material curvature expressed in terms of the pull-back of the spatial connection. The corresponding relation holds if spatial and material objects are exchanged. However, in the case of anholonomic coordinate transformations extra contributions in terms of the anholonomic object arise.

The tensorial transformation of the curvature tensor upon changing the coordinate system between holonomic coordinates $\mathcal{X}^I$ and anholonomic coordinates χ^i is stated as

$$\mathcal{R}^I{}_{JKL} = f^I{}_i r^i{}_{jkl}\mathcal{F}^j{}_J\mathcal{F}^k{}_K\mathcal{F}^l{}_L. \tag{56}$$

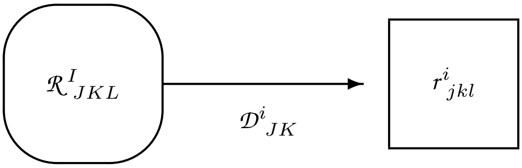

Figure 5. Transformation of the curvature tensor for the case of holonomic $\mathcal{X}^I$ and anholonomic χ^i.

Thereby, for holonomic $\mathcal{X}^I$ and anholonomic χ^i the curvature tensor $\mathcal{R}^I{}_{JKL}$ follows the standard definition in Eqs. (45) and (46) whereas the curvature tensor $r^i{}_{jkl}$ involves extra contributions in terms of the connection and the anholonomic object

$$r^i{}_{jkl} = 2\ell^i{}_{j[l,k]} + 2\ell^i{}_{m[k}\ell^m{}_{jl]} + 2\ell^i{}_{jm}a^m{}_{lk}. \tag{57}$$

The situation is depicted in Fig. 5.

Likewise, the tensorial transformation of the curvature tensor upon changing the coordinate system between holonomic coordinates χ^i and anholonomic coordinates $\mathcal{X}^I$ is stated as

$$r^i{}_{jkl} = \mathcal{F}^i{}_I \mathcal{R}^I{}_{JKL} f^J{}_j f^K{}_k f^L{}_l. \tag{58}$$

Then, for holonomic χ^i and anholonomic $\mathcal{X}^I$ the curvature tensor $r^i{}_{jkl}$ follows the standard definition corresponding to Eqs. (45) and (46) whereas the curvature tensor $\mathcal{R}^I{}_{JKL}$ involves extra contributions in terms of the connection and the anholonomic object

$$\mathcal{R}^I{}_{JKL} = 2\mathcal{L}^I{}_{J[L,K]} + 2\mathcal{L}^I{}_{M[K}\mathcal{L}^M{}_{JL]} + 2\mathcal{L}^I{}_{JM}\mathcal{A}^M{}_{LK}. \tag{59}$$

The situation is depicted in Fig. 6.

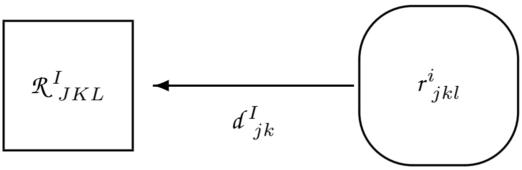

Figure 6. Transformation of the curvature tensor for the case of holonomic χ^i and anholonomic $\mathcal{X}^I$.

Proof:
To proof the above assertions in Eqs. (57) and (59) the transformation of the connection according to Eq. (29) has to be inserted into the standard definition of the curvature tensor in Eq. (45) or (46).

As an example the case of holonomic $\mathcal{X}^I$ and anholonomic χ^i shall be considered in detail. To start with, the transformation of the connection reads as

$$\mathcal{L}^I{}_{JL} = f^I{}_i \ell^i{}_{jl} \mathcal{F}^j{}_J \mathcal{F}^l{}_L + f^I{}_i \mathcal{F}^i{}_{J,L}.$$

Computing the partial derivative of the connection as needed in the definition of the curvature renders the lengthy expression

$$\begin{aligned}
\mathcal{L}^I{}_{JL,K} &= f^I{}_{i,K} \ell^i{}_{jl} \mathcal{F}^j{}_J \mathcal{F}^l{}_L \\
&+ f^I{}_i \ell^i{}_{jl,k} \mathcal{F}^j{}_J \mathcal{F}^k{}_K \mathcal{F}^l{}_L \\
&+ f^I{}_i \ell^i{}_{jl} \mathcal{F}^j{}_{J,K} \mathcal{F}^l{}_L \\
&+ f^I{}_i \ell^i{}_{jl} \mathcal{F}^j{}_J \mathcal{F}^l{}_{L,K} \\
&+ f^I{}_{i,K} \mathcal{F}^i{}_{J,L} \\
&+ f^I{}_i \mathcal{F}^i{}_{J,LK}.
\end{aligned}$$

Unfortunately, upon skew symmetrization in L and K only one term drops out so far

$$\begin{aligned}
\mathcal{L}^I{}_{J[L,K]} &= f^I{}_{i,[K} \ell^i{}_{jl} \mathcal{F}^j{}_J \mathcal{F}^l{}_{L]} \\
&+ f^I{}_i \underline{\ell^i{}_{j[l,k]}} \mathcal{F}^j{}_J \mathcal{F}^k{}_K \mathcal{F}^l{}_L \\
&+ f^I{}_i \ell^i{}_{jl} \mathcal{F}^j{}_{J,[K} \mathcal{F}^l{}_{L]} \\
&+ f^I{}_i \ell^i{}_{jl} \mathcal{F}^j{}_J \mathcal{F}^l{}_{[L,K]} \\
&+ f^I{}_{i,[K} \mathcal{F}^i{}_{J,L]}.
\end{aligned}$$

Observe that the underlined term is already part of the sought for curvature tensor $r^i{}_{jkl}$. Next the term of the curvature quadratic in the connection shall be computed. To this end the transformation of the connection is recalled once again with the right set of indices

$$\begin{aligned}
\mathcal{L}^I{}_{MK} &= f^I{}_i \ell^i{}_{mk} \mathcal{F}^m{}_M \mathcal{F}^k{}_K + f^I{}_i \mathcal{F}^i{}_{M,K}, \\
\mathcal{L}^M{}_{JL} &= f^M{}_m \ell^m{}_{jl} \mathcal{F}^j{}_J \mathcal{F}^l{}_L + f^M{}_m \mathcal{F}^m{}_{J,L}.
\end{aligned}$$

Then multiplication of the two representations of the connection in the above results in the multi-term expression

$$\begin{aligned} & \mathcal{L}^I{}_{MK}\mathcal{L}^M{}_{JL} \\ = \;& f^I_i \ell^i_{mk}\mathcal{F}^m{}_M\mathcal{F}^k{}_K f^M{}_n \ell^n{}_{jl}\mathcal{F}^j{}_J\mathcal{F}^l{}_L \\ + \;& f^I_i \ell^i_{mk}\mathcal{F}^m{}_M\mathcal{F}^k{}_K f^M{}_n \mathcal{F}^n{}_{J,L} \\ + \;& f^I_i \mathcal{F}^i_{M,K} f^M{}_m \ell^m{}_{jl}\mathcal{F}^j{}_J\mathcal{F}^l{}_L \\ + \;& f^I_i \mathcal{F}^i_{M,K} f^M{}_m \mathcal{F}^m{}_{J,L}. \end{aligned}$$

Here many terms may be simplified by taking out multiplications of the tangent map by its inverse and by substituting partial derivatives of the tangent map by those of its inverse in the spirit of Eq. (44):

$$\begin{aligned} & \mathcal{L}^I{}_{MK}\mathcal{L}^M{}_{JL} \\ = \;& f^I_i \underline{\ell^i_{mk}\ell^m{}_{jl}}\mathcal{F}^j{}_J\mathcal{F}^k{}_K\mathcal{F}^l{}_L \\ + \;& f^I_i \ell^i_{jl}\mathcal{F}^j{}_{J,L}\mathcal{F}^l{}_K \\ - \;& f^I{}_{i,K} \ell^i_{jl}\mathcal{F}^j{}_J\mathcal{F}^l{}_L \\ - \;& f^I{}_{i,K} \mathcal{F}^i{}_{J,L}. \end{aligned}$$

Observe that the underlined term will be another part of the sought for curvature tensor $r^i{}_{jkl}$. Upon skew symmetrization in L and K no term drops out:

$$\begin{aligned} \mathcal{L}^I{}_{M[K}\mathcal{L}^M{}_{JL]} \;=\;& f^I_i \underline{\ell^i_{m[k}\ell^m{}_{jl]}}\mathcal{F}^j{}_J\mathcal{F}^k{}_K\mathcal{F}^l{}_L \\ -\;& f^I_i \ell^i_{jl}\mathcal{F}^j{}_{J,[K}\mathcal{F}^l{}_{L]} \\ -\;& f^I{}_{i,[K} \ell^i_{jl}\mathcal{F}^j{}_J\mathcal{F}^l{}_{L]} \\ -\;& f^I{}_{i,[K} \mathcal{F}^i{}_{J,L]}. \end{aligned}$$

However, if we combine the above results so as to produce the curvature tensor $\mathcal{R}^I{}_{JKL}$ many terms drop out and the resulting expression reads as

$$\frac{1}{2}\mathcal{R}^I{}_{JKL} := \mathcal{L}^I{}_{J[L,K]} + \mathcal{L}^I{}_{M[K}\mathcal{L}^M{}_{JL]}$$

$$= f^I_{\ i} \left[\ell^i_{\ j[l,k]} + \ell^i_{\ m[k} \ell^m_{\ jl]} + \ell^i_{\ jm} \mathcal{F}^m_{\ [L,K]} f^L_{\ l} f^K_{\ k} \right] \mathcal{F}^j_{\ J} \mathcal{F}^k_{\ K} \mathcal{F}^l_{\ L}.$$

Inserting finally the definition of the anholonomic object

$$a^m_{\ lk} := \mathcal{F}^m_{\ [L,K]} f^L_{\ l} f^K_{\ k}$$

into the above result concludes the proof. □

2.7 Metric

The *metric* is an important object that introduces more structure into a (differential) manifold as may be seen from the

Definition:
If a n_{dm}-dimensional (differentiable) manifold $\mathcal{M}$ is equipped with a symmetric field of metric coefficients $\mathcal{M}_{IJ}(\mathcal{X}^1, \mathcal{X}^2, \cdots, \mathcal{X}^{n_{dm}})$ such that the arc-length of a parameter curve $\mathcal{X}^I = \mathcal{X}^I(t)$ between parameter values t_a and t_b is given by

$$S(t_b) - S(t_a) = \int_{t_a}^{t_b} \sqrt{\dot{\mathcal{X}}^I \mathcal{M}_{IJ} \dot{\mathcal{X}}^J} \, \mathrm{d}t \qquad (60)$$

the manifold $\mathcal{M}$ is a metric space. Its tangent space $T_{\mathcal{P}}\mathcal{M}$ at $\mathcal{P}$ is an Euclidean (tangent) space. □

Thereby the metric shall obey the following properties:

- $\mathcal{M}_{IJ} = \mathcal{M}_{JI} = \mathcal{M}_{(IJ)}$ with $\mathcal{M}_{[IJ]} = 0$ symmetric
- $\mathcal{V}^I \mathcal{M}_{IJ} \mathcal{V}^J > 0 \quad \forall \{\mathcal{V}^K\} \neq \{0\}$ positive definite
- $\mathcal{M}_{IJ}$ transforms as 2nd-order tensor, i.e. $m_{kl} = f^I_{\ k} \mathcal{M}_{IJ} f^J_{\ l}$

The first property is obvious since any skew symmetric contributions would not contribute to a quadratic form as needed for the determination of the length. The second property is specific to the later application to (three-dimensional) continuum mechanics, relativity and general relativity allow also for indefinite metrics, see, e.g., Misner et al. (1998). Finally the proof of the third property is straightforward from the transformation behaviour of the coordinate differentials, see Eq. (3):

$$\mathrm{d}S^2 = \mathrm{d}\mathcal{X}^I \mathcal{M}_{IJ} \, \mathrm{d}\mathcal{X}^J = \mathrm{d}\chi^k f^I_{\ k} \mathcal{M}_{IJ} f^J_{\ l} \, \mathrm{d}\chi^l = \mathrm{d}\chi^k m_{kl} \, \mathrm{d}\chi^l. \qquad (61)$$

Metric Connection As an immediate consequence the introduction of a metric allows to formulate the