S. Kusuoka
A. Yamazaki (Eds.)

Advances in MATHEMATICAL ECONOMICS

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Advances in MATHEMATICAL ECONOMICS

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Some variational convergence results for a class of evolution inclusions of second order using Young measures

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Summary. This paper has two main parts. In the first part, we discuss the existence and uniqueness of the $W_E^{2,1}$-solution $u_{\mu,\nu}$ of a second order differential equation with two boundary points conditions in a finite dimensional space, governed by controls μ, ν which are measures on a compact metric space. We also discuss the dependence on the controls and the variational properties of the value function $V_h(t,\mu) := \sup_{\nu\in\mathcal{R}} h(u_{\mu,\nu}(t))$, associated with a bounded lower semicontinuous function h. In the second main part, we discuss the limiting behaviour of a sequence of dynamics governed by second order evolution inclusions with two boundary points conditions. We prove that (up to extracted sequences) the solutions stably converge to a Young measure ν and we show that the limit measure ν satisfies a Fatou-type lemma in Mathematical Economics with variational-type inclusion property.

Key words: Fatou lemma, value function, second order differential equation, second order differential inclusion, Young measure, fiber product.

1. Introduction

The study of the value function and of Fatou-type lemmas occurs in Mathematical Economics. In the present paper, we discuss in a first part (Section 3) the existence and uniqueness of the $W_E^{2,1}$-solution $u_{\mu,\nu}$ of the second order differential equation (ODE) with two boundary points conditions of the form

$$\ddot{u}(t) = g(t, u_{\mu,\nu}, \mu_t, \nu_t),\ t \in [0,1]; \quad u_{\mu,\nu}(0) = u_{\mu,\nu}(1) = 0$$

in a finite dimentional space E, where g is a Carathéodory mapping defined on $[0,1] \times E \times \mathcal{M}_+^1(S) \times \mathcal{M}_+^1(Z)$ with values in E, S and Z are two compact metrizable spaces, $\mathcal{M}_+^1(S)$ (resp. $\mathcal{M}_+^1(Z)$) is the compact metrizable (for the vague topology) space of all probability Radon measures on S (resp. Z), and the controls $t \mapsto \mu_t$ (resp. $t \mapsto \nu_t$) are Lebesgue-measurable mappings from $[0,1]$ to $\mathcal{M}_+^1(S)$ and $\mathcal{M}_+^1(Z)$ respectively. We study the dependence of the solution $u_{\mu,\nu}$ with respect to the controls μ, ν where μ belongs to a compact subset $\mathcal{H}$ for the convergence in probability of Lebesgue-measurable mappings from $[0,1]$ to $\mathcal{M}_+^1(S)$ and ν belongs to a compact subset $\mathcal{R}$ for the stable topology [4, 18] of Lebesgue-measurable control mappings from $[0,1]$ to $\mathcal{M}_+^1(Z)$, and we discuss the variational properties of the value function

$$V_h(t,\mu) := \sup_{\nu \in \mathcal{R}} h(u_{\mu,\nu}(t)),$$

associated with a bounded lower semicontinuous real valued function h defined on E. In the second part (Section 4), we discuss the limiting behaviour of a sequence of dynamics governed by second order evolution inclusions (EI) with two boundary points conditions of the form

$$-\ddot{u}_n(t) \in \partial f_n(t, u_n(t)) + H(t, u_n(t), \dot{u}_n(t)), \quad u_n(0) = u_n(1).$$

Here (∂f_n) is a sequence of subdifferential operators associated with a sequence of nonnegative normal convex integrands (f_n) defined on $[0,1] \times E, u_n$ is a $W_E^{2,1}$-solution of the preceding second order evolution inclusion, H is a Carathéodory mapping defined on $[0,1] \times E \times E$ with values in E. Provided that (i) H satisfies some growth condition, (ii) $(\ddot{u}_n)$ is bounded in $L_E^1([0,1])$, (iii) there exists $\overline{v} \in L_E^\infty([0,1])$ such that $\sup_n \int_0^1 f_n(t, \overline{v}(t))\, dt < +\infty$, and (iv) (f_n) is *integrably dominated* by a nonnegative normal convex integrand f_∞, that is, $\limsup_n \int_A f_n(t, v(t))\, dt \leq \int_A f_\infty(t, v(t))\, dt$ for every Lebesgue-measurable set $A \subset [0,1]$ and for every $v \in L_E^\infty([0,1])$, and (f_n) lower epiconverges to f_∞, we prove (Proposition 4.8) that (up to extracted sequences)

(u_n) converges uniformly to a $W_E^{2,1}$-function u, $(\dot u_n)$ converges pointwisely to $\dot u$, $(\ddot u_n)$ stably converges to a Young measure ν with barycenter $t \mapsto \mathrm{bar}(\nu_t) \in L^1_E([0,1])$, and the following variational-type inclusion holds:
(a)

$$-\mathrm{bar}(\nu_t) \in \partial f_\infty(t, u(t)) + H(t, u(t), \dot u(t)) \text{ a.e. on } [0,1].$$

(b) Further the following Fatou-type lemma holds:

$$\liminf_n \int_0^1 \langle -\ddot u_n(t) - H(t, u_n(t), \dot u_n(t)), v(t) - u_n(t)\rangle \, dt \\ \geq \int_0^1 \langle -\mathrm{bar}(\nu_t) - H(t, u(t), \dot u(t)), v(t) - u(t)\rangle \, dt,$$

provided that $(f_n(.,u_n(.)))$ is uniformly integrable, and for each $v \in L^\infty_E([0,1])$ for which $(f_n(.,v(.)))$ is uniformly integrable. So (a) and (b) show that the limit measure ν satisfies a Fatou-type lemma in Mathematical Economics with variational-type inclusion property. For more on Fatou type-lemma in Mathematical Economics, see [3, 5, 7, 11, 13, 19] and the references therein. The present work is essentially a continuation of [17, 15, 16] dealing with control problems where the dynamics are given by ordinary differential equations [17] and evolution inclusions governed by nonconvex sweeping process and m-accretive operators [15, 16] via the fiber product of Young measures [17]. Here we derive from [2, 17] new results of variational convergence for both ODE of second order and EI of second order governed by the subdifferential of convex lower semicontinuous functions. Our results shed a new light on the use of the fiber product of Young measures developed in [17] in the study of the variational limits in the problems under consideration. In Section 3 we present, for simplicity a Bolza-type problem associated with the second order ordinary differential equation with two points-boundary conditions where the controls are two Young measures and in particular we give some variational properties of the value function associated with a bounded real valued upper semicontinuous function. We refer to [25] for the pioneering work on the problem of two points-boundary conditions for ODE, and to [2] for a recent study of the problem of three points-boundary conditions for second order differential inclusions in Banach spaces.

In Section 4 we present some variational limits for a class of second order evolution inclusions governed by a family of subdifferentials of convex lower semicontinuous functions via the lower epiconvergence of *normal integrands* and the fiber product of Young measures. In particular, we discuss a Fatou-type property which occurs therein. We refer to [1, 9, 8, 30] for other related results regarding second order evolution problem. We refer to [6, 21, 22, 23, 29, 39, 40] for control problems governed by first order ODE.

2. Notations, definitions, preliminaries

Throughout, $(\Omega, \mathcal{S}, P)$ is a complete probability space, S and T are two Polish spaces, $E = R^d$ is a finite dimensional space (unless otherwise specified), $\mathcal{L}([0,1])$ is the σ-algebra of Lebesgue-measurable sets of $[0,1]$, and $\lambda = dt$ is the Lebesgue measure on $[0,1]$. By $L^1_E([0,1], dt)$ we denote the space of all Lebesgue-Bochner integrable E-valued functions defined on $[0,1]$. Let $\mathcal{C}_E([0,1])$ be the Banach space of all continuous functions $u : [0,1] \to E$ equipped with the sup-norm. By $W^{1,1}_E([0,1])$ we denote the space of all continuous functions $u \in \mathcal{C}_E([0,1])$ such that their first derivatives are absolutely continuous. For the sake of completeness, we summarize some useful facts concerning Young measures. Let X be a Polish space and let $\mathcal{C}^b(X)$ be the space of all bounded continuous functions defined on X. Let $\mathcal{M}^1_+(X)$ be the set of all Borel probability measures on X equipped with the narrow topology. A Young measure $\lambda : \Omega \to \mathcal{M}^1_+(X)$ is, by definition, a *scalarly measurable* mapping from Ω into $\mathcal{M}^1_+(X)$, that is, for every $f \in \mathcal{C}^b(X)$, the mapping $\omega \mapsto \langle f, \lambda_\omega \rangle := \int_X f(x)\, d\lambda_\omega(x)$ is $\mathcal{S}$-measurable. A sequence (λ^n) in the space of Young measures $\mathcal{Y}(\Omega, \mathcal{S}, P; X)$ *stably converges* to a Young measure $\lambda \in \mathcal{Y}(\Omega, \mathcal{S}, P; X)$ if the following holds:

$$\lim_n \int_A [\int_X f(x)\, d\lambda^n_\omega(x)]\, dP(\omega) = \int_A [\int_X f(x)\, d\lambda_\omega(x)]\, dP(\omega)$$

for every $A \in \mathcal{S}$ and for every $f \in \mathcal{C}^b(X)$. If X and Y are Polish spaces and if $\lambda \in \mathcal{Y}(\Omega, \mathcal{S}, P; X)$ and $\mu \in \mathcal{Y}(\Omega, \mathcal{S}, P; Y)$, the *fiber product* of λ and μ is the Young measure $\lambda \underline{\otimes} \mu \in \mathcal{Y}(\Omega, \mathcal{S}, P; X \times Y)$ defined by

$$(\lambda \underline{\otimes} \mu)_\omega = \lambda_\omega \otimes \mu_\omega$$

for all $\omega \in \Omega$. We recall the following result concerning the fiber product lemma of Young measures, see [17, Theorem 2.3.1] (or [18, Theorem 3.3.1]). For more on Young measures, see [4, 37, 38, 18] and the references therein.

Proposition 2.1. *Assume that S and T are Polish spaces. Let (μ^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{S}, P; S)$ and let (ν^n) be a sequence in $\mathcal{Y}(\Omega, \mathcal{S}, P; T)$. Assume that*
(i) (μ^n) converges in probability to $\mu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; S)$,

(ii) (ν^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; T)$.
Then $(\mu^n \underline{\otimes} \nu^n)$ stably converges to $\mu^\infty \underline{\otimes} \nu^\infty$.

For the sake of completeness, let us also mention a general result of convergence for Young measures in [17], which we need in the statement of next results.

Proposition 2.2. *Assume that S and T are Polish spaces. Let (u^n) be sequence of $\mathcal{S}$-measurable mappings from Ω into S such that (u^n) converges in probability to a $\mathcal{S}$-measurable mapping u^∞ from Ω into S and (v^n) be a sequence of $\mathcal{S}$-measurable mappings from Ω into T such that (v^n) stably converges to $\nu^\infty \in \mathcal{Y}(\Omega, \mathcal{S}, P; T)$. Let $h : \Omega \times S \times T \to R$ be a Carathéodory integrand such that the sequence $(h(., u_n(.), v_n(.))$ is uniformly integrable. Then the following holds:*

$$\lim_{n\to\infty} \int_\Omega h(\omega, u^n(\omega), v^n(\omega))\, dP(\omega) = \int_\Omega [\int_T h(\omega, u^\infty(\omega), t)\, d\nu_\omega^\infty(t)]\, dP(\omega).$$

3. Control problem governed by a second order ODE with measures

In the remainder S and Z are two compact metric spaces. Let $\mathcal{H}$ be a subset in $\mathcal{Y}([0,1], S)$ equipped with the convergence in probability. By $W_E^{2,1}([0,1])$ we denote the set of all continuous functions in $\mathcal{C}_E([0,1])$ such that their first derivatives are continuous and their second derivatives belong to $L_E^1([0,1])$. Let us consider a mapping $f : [0,1] \times E \times E \times S \times Z \to E$ satisfying:
(i) For every $t \in [0,1], f(t, ., ., ., .)$ is continuous on $E \times E \times S \times Z$.
(ii) For every $(x, y, s, z) \in E \times E \times S \times Z, f(., x, y, s, z)$ is Lebesgue-measurable on $[0,1]$.
(iii) There is a constant $c > 0$ such that $f(t, x, y, s, z) \in c(1 + ||x|| + ||y||)\bar{B}_E(0,1)$ for all $(t, x, y, s, z) \in [0,1] \times E \times E \times S \times Z$.
(iv) There exist Lipschitz constants λ_1, λ_2 satisfying $\lambda_1 + \lambda_2 < 1/2$ such that

$$||f(t, x_1, y_1, s, z) - f(t, x_2, y_2, s, z)|| \le \lambda_1 ||x_1 - x_2|| + \lambda_2 ||y_1 - y_2||$$

for all $(t, x_1, y_1, s, z), (t, x_2, y_2, s, z) \in [0,1] \times E \times E \times S \times Z$.
We are given a measurable multifunction Γ defined on $[0,1]$ with nonempty compact values in Z. We consider the $W_E^{2,1}([0,1])$-solutions set of the two following second order differential equations:

$$(\mathcal{D}_{\mathcal{O},\mathcal{H}}) \begin{cases} \ddot{u}_{\mu,\zeta}(t) = \int_S f(t, u_{\mu,\zeta}(t), \dot{u}_{\mu,\zeta}(t), s, \zeta(t))\mu_t(ds) \text{ a.e. } t \in [0,1], \\ u_{\mu,\zeta}(0) = u_{\mu,\zeta}(1) = 0, \end{cases}$$

where $\mu \in \mathcal{H}$ and ζ belongs to the set S_Γ of all original controls, which means that ζ is a Lebesgue-measurable mapping from $[0,1]$ into Z with $\zeta(t) \in \Gamma(t)$ for a.e. $t \in [0,1]$, and

$$(\mathcal{D}_{\mathcal{O},\mathcal{R}}) \begin{cases} \ddot{u}_{\nu,\mu}(t) = \\ \int_{\Gamma(t)} [\int_S f(t, u_{\mu,\nu}(t), \dot{u}_{\mu,\nu}(t), s, z)\, \mu_t(ds)]\, \nu_t(dz) \text{ a.e. } t \in [0,1], \\ u_{\mu,\nu}(0) = u_{\mu,\nu}(1) = 0, \end{cases}$$

where ν belongs to the set $\mathcal{R}$ of all relaxed controls, which means that ν is a Lebesgue-measurable selection of the multifunction Σ defined by

$$\Sigma(t) := \{\sigma \in \mathcal{M}^1_+(Z) : \sigma(\Gamma(t)) = 1\}$$

for all $t \in [0,1]$, and $\mu \in \mathcal{H}$. Note that the existence of $W^{2,1}_E([0,1])$-solutions for the preceding equations follows from [2, Theorem 1.4] dealing with the problem of three points boundary conditions for the same dynamic f which can be applied in the particular case of two points-boundary conditions that we present below. For the sake of completeness, we recall some results developed in [25, 2] and summarize some facts.

Proposition 3.1. *([2, Lemma 1.1 and Proposition 1.4]) Let* $G : [0,1] \times [0,1] \to [-1,+1]$ *be the function defined by*

$$G(t,s) = s(t-1) \text{ if } 0 \le s \le t \le 1,$$

and

$$G(t,s) = (s-1)t \text{ if } 0 \le t \le s \le 1.$$

1) If $u \in W^{2,1}_E([0,1])$ *with* $u(0) = u(1) = 0$*, then*

$$u(t) = \int_0^1 G(t,s)\ddot{u}(s)\, ds, \ \forall t \in [0,1].$$

In fact, u *is given explicitely by*

$$u(t) = (t-1)\int_0^t s\ddot{u}(s)\, ds + t\int_t^1 (s-1)\ddot{u}(s)\, ds.$$

2) Let $f \in L^1_E([0,1])$*, then the function* u_f *defined by*

$$u_f(t) = \int_0^1 G(t,s) f(s)\, ds, \ \ \forall t \in [0,1]$$

is the unique $W^{2,1}_E([0,1])$*-solution of the second order ODE*

$$\ddot{u}(t) = f(t),\ t \in [0,1],\ u(0) = u(1) = 0$$

and satisfies

$$\dot{u}_f(t) = \int_0^1 \frac{\partial G}{\partial t}(t,s) f(s)\, ds,\ \ \forall t \in [0,1].$$

where $\frac{\partial G}{\partial t}(.,.)$ *is Borel with* $|\frac{\partial G}{\partial t}(t,s)| \leq 1$ *for all* $t, s, \in [0,1]$.
3) For each $(\mu, \nu) \in \mathcal{H} \times \mathcal{R}$, *the second order ODE*

$$\ddot{u}(t) = \int_{\Gamma(t)} [\int_S f(t, u(t), \dot{u}(t), s, z)\, \mu_t(ds)]\, \nu_t(dz)$$

with $u(0) = u(1) = 0$, *has a unique solution* $u \in W_E^{2,1}([0,1])$. *Further, for some constant* $m > 0$ *which depends only on* c, λ_1, λ_2, *one has* $||\ddot{u}(t)|| \leq m$ *for almost all* $t \in [0,1]$.

Now comes a Bolza-type optimal control problem associated with the preceding second order ODE where the controls are Young measures.

Theorem 3.2. *Assume that* E *is a finite dimensional space and* $\mathcal{H}$ *is compact for the convergence in probability. Let* $I : [0,1] \times E \times E \times S \times Z \to \mathbb{R}$ *be an* L^1*-bounded Carathéodory integrand, (that is,* $I(t,.,.,.,.)$ *is continuous on* $E \times E \times S \times Z$ *for every* $t \in [0,1]$ *and* $I(.,x,y,s,z)$ *is Lebesgue-measurable on* $[0,1]$, *for every* $(x,y,s,z) \in E \times E \times S \times Z$*) which satisfies the condition: there is a positive Lebesgue- integrable function* h *such that* $|I(t,x,y,s,z)| \leq h(t)$ *for all* $(t,x,y,s,z) \in [0,1] \times E \times E \times S \times Z$. *Let us consider the control problems*

$$(P_{\mathcal{H},\mathcal{O}}) : \inf_{(\mu,\zeta)\in \mathcal{H}\times S_\Gamma} \int_0^1 [\int_S I(t, u_{\mu,\zeta}(t), \dot{u}_{\mu,\zeta}(t), s, \zeta(t)) \mu_t(ds)]\, dt$$

and

$$(P_{\mathcal{H},\mathcal{R}}) : \inf_{(\mu,\nu)\in \mathcal{H}\times \mathcal{R}} \int_0^1 [\int_Z [\int_S I(t, u_{\mu,\nu}(t), \dot{u}_{\mu,\nu}(t), s, z) \mu_t(ds)]\, \nu_t(dz)]\, dt$$

where $u_{\mu,\zeta}$ *(resp.* $u_{\mu,\nu}$*) is the unique* $W_E^{2,1}([0,1])$*-solution associated with* (μ,ζ) *(resp.* (μ,ν)*) to the second order differential equation* $(\mathcal{D}_{\mathcal{H},\mathcal{O}})$ *(resp.* $(\mathcal{D}_{\mathcal{H},\mathcal{R}})$*). Then one has* $\inf(P_{\mathcal{H},\mathcal{O}}) = \inf(P_{\mathcal{H},\mathcal{R}})$.

Proof. Claim 1: The graph of the mapping $(\mu,\nu) \mapsto u_{\mu,\nu}$ defined on the compact space $\mathcal{H} \times \mathcal{R}$ with value in the Banach space $\mathcal{C}_E([0,1])$ of all continuous mappings from $[0,1]$ into E endowed with the sup norm is

compact.
Let (μ^n) be a sequence in $\mathcal{H}$ which converges in probability to $\mu^\infty \in \mathcal{H}$. Let (ν^n) be a sequence in $\mathcal{R} := \mathcal{S}_\Sigma$ which stably converges to $\nu^\infty \in \mathcal{R}$, and, for each $n \in \mathbb{N}\cup\{\infty\}$, let u_{μ^n,ν^n} be the unique $W_E^{2,1}([0,1])$-solution of

$$\ddot{u}_{\mu^n,\nu^n}(t) = \int_{\Gamma(t)} [\int_S f(t, u_{\mu^n,\nu^n}(t), \dot{u}_{\mu^n,\nu^n}(t), s, z)\mu_t^n(ds)]\, \nu_t^n(dz),$$

for $t \in [0,1]$ and $u_{\mu^n,\nu^n}(0) = u_{\mu^n,\nu^n}(1) = 0$. Then we claim that $(u_{\mu^n,\nu^n}(.))$ converges uniformly to $u_{\mu^\infty,\nu^\infty}(.)$. Fix $\beta \in]0,1[$ such that $\lambda_1 + \lambda_2 < (1-\beta)/2$. Using the estimation in [2, Theorem 1.4 and Lemma 1.1] involving the use of Hartmann function G given in Proposition 3.1, we may suppose, by extracting subsequences, that $(u_{\mu^n,\nu^n}(.))$ converges uniformly to a $W_E^{2,1}([0,1])$-function $u^\infty(.)$ and $(\dot{u}_{\mu^n,,\nu^n}(.))$ converges pointwisely to $\dot{u}^\infty(.)$ and there exists a positive constant m such that $||u_{\mu^n,\nu^n}(.)|| \le m$ and $||\dot{u}_{\mu^n,,\nu^n}(.)|| \le m$ for all $n \in \mathbb{N}$. By Proposition 3.1 (or [2, Lemma 1.1]), for each $t \in [0,1]$ and for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&u_{\mu^\infty,\nu^\infty}(t) - u_{\nu^n,\nu^n}(t)\\
&= \int_0^1 G(t,\tau)[\int_Z [\int_S f(\tau, u_{\mu^\infty,\nu^\infty}(\tau), \dot{u}_{\mu^\infty,\nu^\infty}(\tau), s, z)\, \mu_\tau^\infty(ds)]\, \nu_\tau^\infty(dz)]\, d\tau\\
&- \int_0^1 G(t,\tau)[\int_Z [\int_S f(\tau, u_{\mu^\infty,\nu^\infty}(\tau), \dot{u}_{\mu^\infty,\nu^\infty}(\tau), s, z)\, \mu_\tau^n(ds)]\, \nu_\tau^n(dz)]\, d\tau\\
&+ \int_0^1 G(t,\tau)[\int_Z [\int_S f(\tau, u_{\mu^\infty,\nu^\infty}(\tau), \dot{u}_{\mu^\infty,\nu^\infty}(\tau), s, z)\, \mu_\tau^n(ds)]\, \nu_\tau^n(dz)]\, d\tau\\
&- \int_0^1 G(t,\tau)[\int_Z [\int_S f(\tau, u_{\mu^n,\nu^n}(\tau), \dot{u}_{\mu^n,\nu^n}(\tau), s, z)\, \mu_\tau^n(ds)]\, \nu_\tau^n(dz)]\, d\tau,
\end{aligned}$$

where G is a continuous mapping from $[0,1]$ into $[-1,1]$. By hypothesis, we have

$$\begin{aligned}
&||f(\tau, u_{\mu^\infty,\nu^\infty}(\tau), \dot{u}_{\mu^\infty,\nu^\infty}(\tau), s, z) - f(\tau, u_{\mu^n,\nu^n}(\tau), \dot{u}_{\mu^n,\nu^n}(s), s, z)||\\
&\quad \le \lambda_1 ||u_{\mu^\infty,\nu^\infty}(\tau) - u_{\mu^n,\nu^n}(\tau)|| + \lambda_2 ||\dot{u}_{\mu^\infty,\nu^\infty}(\tau) - \dot{u}_{\mu^n,\nu^n}(\tau)||\\
&\quad \le (\lambda_1 + \lambda_2)(||u_{\mu^\infty,\nu^\infty}(\tau) - u_{\mu^n,\nu^n}(\tau)|| + ||\dot{u}_{\mu^\infty,\nu^\infty}(\tau) - \dot{u}_{\mu^n,\nu^n}(\tau)||)\\
&\quad < \frac{1-\beta}{2}(||u_{\mu^\infty,\nu^\infty}(\tau) - u_{\mu^n,\nu^n}(\tau)|| + ||\dot{u}_{\mu^\infty,\nu^\infty}(\tau) - \dot{u}_{\mu^n,\nu^n}(\tau)||)
\end{aligned}$$

for all $\tau \in [0,1]$ and for all $s, z \in S \times Z$. For simplicity, for each $t \in [0,1]$ and for each $n \in \mathbb{N}$, let us set

$$v^n(t) = \int_0^1 [\int_Z [\int_S G(t,\tau) f(\tau, u_{\mu^\infty,\nu^\infty}(\tau), \dot{u}_{\mu^\infty,\nu^\infty}(\tau), s, z)\, \mu^n_\tau(ds)]\, \nu^n_\tau(dz)]\, d\tau,$$

and

$$v^\infty(t) = \int_0^1 [\int_Z [\int_S G(t,\tau) f(\tau, u_{\mu^\infty,\nu^\infty}(\tau), \dot{u}_{\mu^\infty,\nu^\infty}(\tau), s, z)\, \mu^\infty_\tau(ds)]\, \nu^\infty_\tau(dz)]\, d\tau.$$

Note that the Carathéodory integrand defined on $[0,1] \times S \times Z$ by

$$\varphi_t : (\tau, s, z) \mapsto G(t,\tau) f(\tau, u_{\mu^\infty,\nu^\infty}(\tau), \dot{u}_{\mu^\infty,\nu^\infty}(\tau), s, z)$$

is L^1-bounded because $|G(t,\tau)| \leq 1$ for all $t, \tau \in [0,1]$ and by our assumption, there is a positive constant $M = c(2m+1)$ such that $||f(\tau,x,y,s,z)|| \leq M$ for all $(\tau,x,y,s,z) \in [0,1] \times \overline{B}_E(0,m)] \times \overline{B}_E(0,m) \times S \times Z$. Since (ν^n) stably converges to ν^∞ and μ^n narrowly converges in probability to μ^∞, $\mu^n \underline{\otimes} \nu^n$ stably converges to $\mu^\infty \underline{\otimes} \nu^\infty$. Using Propositions 2.1-2.2, we get

$$\lim_{n\to\infty} v^n(t) = \lim_{n\to\infty} \int_0^1 \langle \varphi_t, \mu^n_\tau \otimes \nu^n_\tau \rangle\, d\tau = \int_0^1 \langle \varphi_t, \mu^\infty_\tau \otimes \nu^\infty_\tau \rangle\, d\tau = v^\infty(t)$$

for every $t \in [0,1]$. Therefore, for each $t \in [0,1]$, we have the estimate

$$\begin{aligned} &||u_{\mu^\infty,\nu^\infty}(t) - u_{\mu^n,\nu^n}(t)|| \\ &\quad < ||v^\infty(t) - v^n(t)|| + \frac{1-\beta}{2} \int_0^1 [||u_{\mu^\infty,\nu^\infty}(s) - u_{\mu^n,\nu^n}(s)|| \\ &\quad + ||\dot{u}_{\mu^\infty,\nu^\infty}(s) - \dot{u}_{\mu^n,\nu^n}(s)||]\, ds, \end{aligned}$$

with $v^\infty(t) - v^n(t)$ tending to 0 when n goes to $+\infty$. Since, for all $t \in [0,1]$,

$$\dot{u}_{\mu^\infty,\nu^\infty}(t) = \int_0^1 [\int_Z [\int_S \frac{\partial G}{\partial t}(t,\tau) f(\tau, u_{\mu^\infty,\nu^\infty}(\tau), \dot{u}_{\mu^\infty,\nu^\infty}(\tau), s, z)\, \mu^\infty_\tau(ds)]\, \nu^\infty_\tau(dz)]\, d\tau$$

and

$$\dot{u}_{\mu^n,\nu^n}(t) = \int_0^1 [\int_Z [\int_S \frac{\partial G}{\partial t}(t,\tau) f(\tau, u_{\mu^n,\nu^n}(\tau), \dot{u}_{\mu^n,\nu^n}(\tau), s, z)\, \mu^n_\tau(ds)]\, \nu^n_\tau(dz)]\, d\tau,$$